Presmoothed Kaplan-Meier and Nelson-Aalen Estimators

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Abstract

In this paper a modification of the Kaplan-Meier and Nelson-Aalen estimators in the right random censorship model is studied. The new estimators are obtained by replacing the censoring indicator variables in the classical definitions by values of a nonparametric regression estimator. Asymptotic normality is obtained and it is shown that this presmoothing idea leads to a gain in asymptotic mean squared error. A local plug-in bandwidth selector is introduced and the problem of optimal pilot bandwidth selection for this estimator is studied. The gain of the presmoothed estimators with automatic plug-in bandwidth selector is demonstrated in a simulation study.

Abbreviated Title: Presmoothed Kaplan-Meier estimators.
Key words and phrases: Bandwidth selection, censored data, kernel smoothing, plug-in bandwidth.

1 Presmoothed estimators for censored data

Let $Y_1, \ldots, Y_n$ be independent, identically distributed (iid) positive random variables (survival times or failure times) with unknown continuous distribution function (df) $F(\cdot)$. In the right random censorship model these survival times are censored to the right by positive iid random variables $C_1, \ldots, C_n$ with unknown continuous df $G(\cdot)$. For each $i = 1, \ldots, n$ we observe $(T_i, \delta_i)$ where $T_i = Y_i \wedge C_i$.
and $\delta_i = 1\{Y_i \leq C_i\}$. We assume the $Y_i$’s independent of the $C_i$’s. Therefore the df $H(\cdot)$ of $T_i$ satisfies 

$$1 - H(t) = (1 - F(t))(1 - G(t)).$$

The variable $\delta_i$ indicates whether $Y_i$ is censored ($\delta_i = 0$) or uncensored ($\delta_i = 1$). The success probability for this binary outcome is 

$$\gamma = P(\delta_1 = 1) = E(\delta_1) = P(Y_1 \leq C_1) = \int_0^\infty (1 - G(t))dF(t) = H^u(+\infty)$$

where $H^u(t) = P(T_1 \leq t, \delta_1 = 1)$ is the subdistribution function of the uncensored observations. Note that 

$$H^u(t) = \int_0^t p(s)dH(s)$$

where 

$$p(t) = P(\delta_1 = 1 \mid T_1 = t) = E(\delta_1 \mid T_1 = t).$$

The function $p(\cdot)$ is the conditional probability that the observation is non-censored given that $T_1 = t$. The importance of the function $p(\cdot)$ is clear from the following relations (see also Dikta (1998)):

$$\Lambda_F(t) = \int_0^t \frac{1}{1 - H(s)}dH^u(s) = \int_0^t p(s)d\Lambda_H(s) \quad (1)$$

with $\Lambda_F(\cdot)$ and $\Lambda_H(\cdot)$ the cumulative hazard functions corresponding to $F(\cdot)$ and $H(\cdot)$.

From (1) we easily obtain 

$$1 - F(t) = \exp(-\Lambda_F(t)) = \exp\left(-\int_0^t p(s)d\Lambda_H(s)\right) \quad (2)$$

and 

$$\lambda_F(t) = p(t)\lambda_H(t) \quad (3)$$

with $\lambda_F(\cdot)$ and $\lambda_H(\cdot)$ the hazard functions.

Note that $p(t) \equiv 1$ in case of no censoring. If $\delta_1$ is independent of $T_1$, i.e., 

$$p(t) = E(\delta_1 = 1 \mid T_1 = t) = E(\delta_1) = \gamma$$

then (1) implies that we have the Koziol-Green proportional hazards model (see Koziol and Green (1976)):

$$1 - F(t) = \exp(-\Lambda_F(t)) = \exp(-\gamma\Lambda_H(t)) = (1 - H(t))^{\gamma}$$
or equivalently $1 - G(t) = (1 - F(t))^\beta$ with $\beta = (1 - \gamma)/\gamma$.

The classical estimator for $\Lambda_F(t)$ is the Nelson-Aalen estimator (see Nelson (1972) and Aalen (1978))

$$\Lambda_n^{NA}(t) = \sum_{T_{i(o)} \leq t} \frac{\delta_{[i]}}{n - i + 1}$$

(4)

where $T_{(1)} \leq \ldots \leq T_{(n)}$ are the ordered $T_i$’s and the $\delta_{[i]}$’s are the concomitants. The intuitive idea behind the Nelson-Aalen estimator is to consider the purely empirical version of $\Lambda_F(t) = \int_0^t \frac{p(s)}{1 - H(s^-)} dH(s)$, i.e., at the $i$-th ordered jump (of size $n^{-1}$) of $H_n(s) = n^{-1} \sum_{i=1}^n \mathbb{1}\{T_i \leq s\}$ we estimate $\frac{p(s)}{1 - H(s^-)}$ by the purely empirical value $\frac{1}{1 - (i - 1)/n} = \frac{n \delta_{[i]}}{n - i + 1}$. It is however clear that in this expression we can replace $\delta_{[i]}$ by a more smooth (parametric or nonparametric) estimator of $p(\cdot)$. An appealing intuitive idea is to estimate $p(T_{(i)})$ by

$$\hat{p}(T_{(i)})$$

where

$$\hat{p}(t) = \frac{(nb)^{-1} \sum_{i=1}^n K\left(\frac{t - T_i}{b}\right) \delta_i}{(nb)^{-1} \sum_{i=1}^n K\left(\frac{t - T_i}{b}\right)} = \frac{1}{n} \sum_{i=1}^n K_b(t - T_i) \delta_i$$

(5)

with $K(\cdot)$ a kernel, $K_b(u) = \frac{1}{b} K(u/b)$ and $b \equiv b_n$, $n = 1, 2, \ldots$, a bandwidth sequence $(\hat{p}(\cdot))$ is the Nadaraya-Watson kernel estimator for $p(\cdot)$ based on the binary responses $\delta_i$ with covariates $T_i$, $i = 1, \ldots, n$. This yields the pre-smoothed estimator

$$\Lambda_n^P(t) = \sum_{T_{i(o)} \leq t} \frac{\hat{p}(T_{(i)})}{n - i + 1}.$$  

(5)

It should be noted that the parametric version of this idea appeared in Dikta (1998). He proposed working with a parametric estimator for $p(\cdot)$ The unpublished work by Ziegler (1995) also deals with some modification of the nonparametric presmoothed estimators presented here.

It is immediate from (3) that $\lambda_F(t)$ can be estimated by means of a presmoothed hazard function estimator

$$\lambda_n^P(t) = \hat{p}(t) \hat{\lambda}(t)$$

(6)

where $\hat{\lambda}(t)$ is an estimator of $\lambda_H(t)$ (e.g. the Watson and Leadbetter (1964a,b) kernel estimator). Note that the estimator like the one in (6) is simply the product of two estimators based on the iid observations at hand. Indeed $\hat{p}(t)$
is based on \((T_i, \delta_i), i = 1, \ldots, n\) and \(\hat{X}(t)\) is based on \(T_i, i = 1, \ldots, n\). Any possible non-parametric estimators for \(p(t)\) and for \(\lambda_H(t)\) can be used to make a product estimator like in (6). The asymptotic properties of such estimators are easily derived from the properties of the two factors, see Subsection 2.3 for some guidelines.

Some good features of the presmoothed Nelson-Aalen estimator, defined by (5), are:

(i) \(A_n^P(\cdot)\) has a jump at any of the observations, so that from a graphical point of view \(A_n^P(\cdot)\) provides more information on the local behaviour than the classical Nelson-Aalen estimator.

(ii) Using a binary regression smoother to estimate \(p(\cdot)\) means that we can extrapolate the available information to better describe the tail behaviour. This feature should be clear from the graphical performance of this estimator.

(iii) The presmoothed Nelson-Aalen estimator has a smaller asymptotic variance than the Nelson-Aalen estimator. In this paper it will be shown that this results in a better mean squared error performance, showing that presmoothing is beneficial.

Using the first equation in (2), expressions (4) and (5) and the approximation \(e^{-x} \simeq 1 - x\) for \(x\) close to 0, we easily obtain the following two estimators for \(1 - F(t) = \exp(-\Lambda_F(t))\), the survival function at \(t\):

\[
1 - F_n^{KM}(t) = \prod_{T_i \leq t} \left( 1 - \frac{\delta_i}{n - i + 1} \right)
\]

and

\[
1 - F_n^{P}(t) = \prod_{T_i \leq t} \left( 1 - \frac{\hat{p}(T_i)}{n - i + 1} \right)
\]

The estimator in (7) is the classical Kaplan-Meier estimator (see Kaplan and Meier (1958)) while (8) gives the new presmoothed estimator proposed in this paper. It is straightforward to show that, for \(t\) such that \(H(t) < 1\),

\[
1 - F_n^{KM}(t) = \exp(-\Lambda_n^{NA}(t)) + O_P(n^{-1})
\]

and also that

\[
1 - F_n^{P}(t) = \exp(-\Lambda_n^{P}(t)) + O_P(n^{-1})
\]

Remark 1 Instead of using the approximation

\[
\exp \left( -\frac{\hat{p}(T_i)}{n - i + 1} \right) \simeq 1 - \frac{\hat{p}(T_i)}{n - i + 1}
\]
that leads to (8), we can consider

$$\exp\left(-\frac{\hat{\beta}(T_{ij})}{n-i+1}\right) = \exp\left(-\frac{1}{n-i+1}\right) \hat{\beta}(T_{ij}) \approx \left(1 - \frac{1}{n-i+1}\right) \hat{\beta}(T_{ij})$$

that would lead to a different version of the presmoothed Kaplan-Meier estimator:

$$1 - F_n^{p(2)}(t) = \prod_{T_{ij} \leq t} \left(1 - \frac{1}{n-i+1}\right) \hat{\beta}(T_{ij}).$$

This estimator is also asymptotically equivalent to that in (8) since it also satisfies:

$$1 - F_n^{p(2)}(t) = \exp(-\Lambda_n(t)) + O_P(n^{-1}).$$

**Remark 2** Rather than estimating \(p(\cdot)\) in a nonparametric way, one can assume that the conditional expectation \(p(\cdot)\) belongs to a parametric family, i.e., \(p(x) \equiv p(x; \theta)\) with \(p(\cdot; \cdot)\) a known continuous function and \(\theta \in \Theta\) an unknown parameter. Given the data \((T_i, \delta_i)\) one can estimate \(\theta\). This approach has been followed by Dikta (1998). He studied the semiparametric estimators

$$\Lambda_n^{D}(t) = \sum_{T_{ij} \leq t} \frac{p(T_{ij}, \hat{\theta}_n)}{n-i+1}$$

and

$$1 - F_n^{D}(t) = \prod_{T_{ij} \leq t} \left(1 - \frac{1}{n-i+1}\right) p(T_{ij}, \hat{\theta}_n)$$

with \(\hat{\theta}_n\) the MLE under random censoring.

In this paper the following items on the presmoothed Nelson-Aalen estimator and the presmoothed Kaplan-Meier estimator will be covered. In Section 2 we give representations for \(\Lambda_n^{p}(t) - \Lambda_F(t)\) and for \(F_n^{p}(t) - F(t)\) and, as an application, we study their asymptotic distributional behaviour. We also include a short discussion on the asymptotic properties of hazard function estimators. The beneficial effect of presmoothing is demonstrated in Section 3. There we look at the mean squared error of the dominant part of the presmoothed Nelson-Aalen estimator and the presmoothed Kaplan-Meier estimator, and we show
that these MSE’s are smaller than the corresponding expressions for the Nelson-Aalen and the Kaplan-Meier estimator. Section 4 includes some proposal of a plug-in bandwidth selector in this setup. The asymptotic results presented there for this selector show the rate of convergence of this data driven bandwidth to its populational counterpart. A small simulation study is included, in Section 5, to show the quality of the presmoothed estimators with plug-in bandwidth. Finally, Section 6 contains many auxiliary lemmas and the proofs.

2 Asymptotic representations

Our results will require the following conditions.

On the kernel function, $K$:

(K.1) $K$ is a non negative, symmetric, twice differentiable function of bounded variation, with bounded second derivative. It also satisfies $\int_{-L}^{L} K(x)dx = 1$, $K$ has support in the interval $[-L, L]$, for some $L > 0$ and $K(L) = K'(L) = K''(L) = 0$.

On the conditional probability of uncensoring, $p$:

(P.1) $p$ is five times differentiable in $[0, \infty)$, with continuous fifth derivative.
(P.2) $p(0) = 1$ and $\varepsilon = \sup\{t : p(x) = 1, \forall x \in [0, t]\} > 0$.

On the distribution function $H$:

(H.1) There exists some $t_0$ such that $\varepsilon < t_0$ and $H(t_0) < 1$, $H$ is five times differentiable in $[0, t_0]$, with fifth continuous derivative and there exists some $\delta > 0$ such that $H''(t) = h(t) > \delta$, $\forall t \in [\varepsilon/2, t_0]$.

Conditions (K.1), (P.1) and (H.1) are standard regularity conditions. The degree of differentiability in (P.1) and (H.1) could be relaxed for the asymptotic representations in this section. However, this is not the case for the rates of the plug-in bandwidth that will be presented in Section 4. Condition (P.2) is a technical one and may look rather surprising. It essentially states that a lifetime cannot be censored by an arbitrary small number. There should exist some positive lower bound for censoring times. This does not seem to be a restrictive condition for real data applications.

2.1 Presmoothed Nelson-Aalen estimator

With $H_n(s) = n^{-1} \sum_{i=1}^{n} 1(T_i \leq s)$, let $\hat{\Lambda}_H(t)$ be the empirical estimator of $\Lambda_H(t)$:

$$\hat{\Lambda}_H(t) = \int_0^t \frac{dH_n(s)}{1 - H_n(s-)}.$$


We then have that
\[
\Lambda_P^n(t) - \Lambda_F(t) = \int_0^t p(s)d(\Lambda_H(s) - \Lambda_H(s)) + \int_0^t (\tilde{p}(s) - p(s))d\Lambda_H(s) + \int_0^t (\tilde{p}(s) - p(s))d(\Lambda_H(s) - \Lambda_H(s)) = (I) + (II) + (III).
\]

**Theorem 3** Assume (K.1), (H.1) and \( b = c_0n^{-\alpha} + o(n^{-\alpha}) \) for some \( 1/4 < \alpha < 1/2 \) and some \( c_0 > 0 \). Then
\[
\Lambda_P^n(t) - \Lambda_F(t) = \Lambda_P^n(t) - \Lambda_F(t) + o_P(n^{-1/2})
\]
with
\[
\Lambda_P^n(t) = \Lambda_F(t) + \frac{1}{n} \sum_{i=1}^n (g_1(T_i) - g_2(T_i) + g_3(T, \delta_i))
\]
where
\[
g_1(T_i) = \frac{p(t)}{1 - H(t)}(1(T_i \leq t) - H(t))
\]
\[
g_2(T_i) = \int_0^t \frac{1(1 \leq s) - H(s)}{1 - H(s)}p'(s)ds
\]
\[
g_3(T_i, \delta_i) = \int_0^t \frac{K_0(s - T_i)(\delta_i - p(s))}{1 - H(s)}ds.
\]

The asymptotic normality of \( \Lambda_P^n(t) - \Lambda_F(t) \) is an easy consequence of the asymptotic representation in Theorem 3. For an explicit result we need a closed formula for the asymptotic variance. Based on moment calculations, collected in Lemma 4, we obtain in Theorem 5, a nice expression for the asymptotic variance. We need the following notations:
\[
d_K = \int_{-L}^L v^2 K(v)dv \quad K(v) = \int_{-L}^v K(s)ds \quad e_K = \int_{-L}^L vK(v)K(v)dv
\]
\[
\alpha(t) = \int_0^t \frac{1}{2} \left\{ \frac{p''(s)h(s) + p'(s)h'(s)}{1 - H(s)} \right\} ds
\]
\[
= \frac{1}{2} \left\{ \frac{p'(t)h(t)}{1 - H(t)} + \int_0^t p'(s) \left\{ \frac{h'(s)}{1 - H(s)} - \frac{h^2(s)}{(1 - H(s))^2} \right\} ds \right\}
\]
\[
q(t) = \frac{p(t)(1 - p(t))h(t)}{(1 - H(t))^2}.
\]
Lemma 4 Assume (K.1) and (C.1). Then,
\[
E[g_3(T_1, \delta_1)] = d_K \alpha(t) b^2 + o(b^2)
\] (17)
\[
\text{Var}[g_1(T_1)] = \frac{p^2(t) H(t)}{1 - H(t)}
\] (18)
\[
\text{Var}[g_2(T_1)] = 2 \int_0^t (p(t) - p(s)) \frac{H(s)}{1 - H(s)} p'(s) ds
\] (19)
\[
\text{Var}[g_3(T_1, \delta_1)] = \int_0^t q(v) dv - 2bq(t) e_K + O(b^2)
\] (20)
\[
E[g_1(T_1)g_2(T_1)] = p(t) \int_0^t \frac{H(s)}{1 - H(s)} p'(s) ds
\] (21)
\[
E[g_1(T_1)g_3(T_1, \delta_1)] = O(b^2)
\] (22)
\[
E[g_2(T_1)g_3(T_1, \delta_1)] = O(b^2).
\] (23)

Theorem 5 Assume (K.1) and (C.1). Then
\[
\text{Var}(\Lambda_n^p(t) - \Lambda_F(t)) = n^{-1}[\gamma(t) - 2e_K bq(t) + O(b^2)].
\]
where
\[
\gamma(t) = \int_0^t \frac{dH^a(s)}{(1 - H(s))^2}.
\] (24)

2.2 Presmoothed Kaplan-Meier estimator

To obtain an iid representation for the presmoothed Kaplan-Meier estimator, we rely on the relation $1 - F_n^p(t) = \exp(-\Lambda_n^p(t)) + O_P(n^{-1})$. Therefore a second order Taylor expansion yields
\[
F_n^p(t) - F(t) = \exp(-\Lambda_F(t)) - \exp(-\Lambda_n^p(t)) + O_P(n^{-1})
\]
\[
= (1 - F(t))(\Lambda_n^p(t) - \Lambda_F(t))
\]
\[
- \frac{1}{2}(\Lambda_n^p(t) - \Lambda_F(t))^2 \exp(-\eta_n(t)) + O_P(n^{-1})
\]
with $\eta_n(t)$ a stochastic intermediate value between $\Lambda_n^p(t)$ and $\Lambda_F(t)$. Moreover it follows from Theorems 2.1 and 2.3 that $\Lambda_n^p(t) - \Lambda_F(t) = O_P(n^{-1/2})$, so that
\[
F_n^p(t) - F(t) = (1 - F(t))(\Lambda_n^p - \Lambda_F(t)) + O_P(n^{-1}).
\]

We therefore have the following result.

Theorem 6 Assume (K.1), (H.1) and $b = c_0 n^{-\alpha} + o(n^{-\alpha})$, for some $1/4 < \alpha < 1/2$ and some $c_0 > 0$. Then,
\[
F_n^p(t) - F(t) = \overline{T_n^p(t)} - F(t) + o_P(n^{-1/2})
\]
with
\[
\overline{F}_n(t) = F(t) + (1 - F(t)) \frac{1}{n} \sum_{i=1}^{n} (g_1(T_i) - g_2(T_i) + g_3(T_i, \delta_i))
\]
and
\[
\text{Var}(\overline{F}_n(t) - F(t)) = n^{-1}(1 - F(t))^2[\gamma(t) - 2eKbq(t) + O(b^2)].
\]

2.3 Presmoothed hazard rate estimator

In Section 1 we noted in (3) that \(\lambda_F(t) = p(t)\lambda_H(t)\) and in (6) that a presmoothed hazard rate estimator can take the form \(\lambda_P(t) = b_{p}(t)\lambda_H(t)\). From these expressions the usefulness of the following lemma, a direct application of the delta method, is immediate.

**Lemma 7** Let \(\hat{\lambda}(t)\) and \(\hat{p}(t)\) be estimators for \(\lambda_H(t)\) and \(p(t)\). Assume that, for some normalizing sequence \(a_n\) we have that
\[ a_n(\hat{\lambda}(t) - \lambda_H(t), \hat{p}(t) - p(t))' \overset{d}{\to} N_2(b(t), V(t)) \]
where \(b(t) = (b_1(t), b_2(t))'\) and
\[
V(t) = \begin{pmatrix}
  v_{11}(t) & v_{12}(t) \\
  v_{21}(t) & v_{22}(t)
\end{pmatrix}.
\]

Then,
\[ a_n(\lambda_P(t) - \lambda_F(t)) \overset{d}{\to} N(\mu(t), \sigma^2(t)) \]
with \(\mu(t) = p(t)b_1(t) + \lambda_H(t)b_2(t)\) and \(\sigma^2(t) = p^2(t)v_{11}(t) + \lambda_H^2(t)v_{22}(t) + 2p(t)\lambda_H(t)v_{12}(t)\).

**Remark 8** We will not elaborate on this result in an explicit way, but we include some possible further directions. For kernel estimators, the typical normalizing sequence is \(a_n = (nb)^{-1/2}\) if \(b\) is the common smoothing parameter in both \(\hat{\lambda}(t)\) and \(\hat{p}(t)\). For different bandwidths of the same order any sequence of this order is a possible choice for \(a_n\). For bandwidths of different order some of the terms in the asymptotic bias and variance expressions will vanish.

**Remark 9** The estimators \(\hat{\lambda}(t)\) and \(\hat{p}(t)\) typically have the form
\[
\hat{\lambda}(t) = \frac{\hat{\lambda}'(t)}{1 - \hat{H}(t)}
\]
and
\[
\hat{p}(t) = \frac{\hat{p}(t)}{\hat{H}'(t)}
\]
where \( \hat{h}^{(1)}(t) \) and \( \hat{h}^{(2)}(t) \) are estimators for the density \( h(t) \), \( \hat{H}(t) \) is an estimator for \( H(t) \) and \( \hat{\psi}(t) \) is an estimator for \( \psi(t)h(t) \). Examples include the Watson and Leadbetter (1964) estimator for \( \lambda_H(t) \) and the Nadaraya-Watson estimator for \( p(t) \) (see e.g. Härdle (1990)). If we take \( \hat{h}^{(1)}(t) = \hat{h}^{(2)}(t) \), the product estimator reduces to

\[
\hat{\lambda}_n(t) = \frac{\hat{\psi}(t)}{1 - \hat{H}(t)}
\]

and the limit distribution is easily obtained from the limit distribution of \((\hat{\psi}(t), \hat{H}(t))\).

For example, for the combination of the Watson and Leadbetter and the Nadaraya-Watson estimators we get that \( \hat{\lambda}_n(t) \) is the estimator already studied by Blum and Susarla (1980).

### 3 Mean squared error: beneficial effect of presmoothing

In this section we show the beneficial effect of presmoothing. In fact we show that the asymptotic mean squared errors (AMSE) of \( \hat{\Lambda}_n(t) \) and \( \hat{\Phi}_n(t) \) are smaller than the mean squared errors of the Nelson-Aalen estimator and of the Kaplan-Meier estimator.

We illustrate this for the AMSE of \( \hat{\Lambda}_n(t) \). This AMSE is defined as follows

\[
\text{AMSE}(\hat{\Lambda}_n(t)) = \text{Var}(\hat{\Lambda}_n(t)) + (\text{Bias}(\hat{\Lambda}_n(t)))^2
\]

where \( \text{Var} \) is the sum of the first two order terms of the variance and \( \text{Bias} \) is the dominant term of the bias of the linear approximation in the asymptotic representation of Theorem 3. From Theorem 5 we have that

\[
\text{Var}(\hat{\Lambda}_n(t)) = n^{-1} \gamma(t) - 2e_K q(t)n^{-1}b
\]

where \( q(t) \), \( e_K \) and \( \gamma(t) \) are given in (16), (14) and (24).

For the asymptotic bias we have from (17) in Lemma 4 that

\[
\text{Bias}(\hat{\Lambda}_n(t)) = d_K \alpha(t)b^2
\]

where \( \alpha(t) \) is given in (15).

We therefore have that

\[
\text{AMSE}(\hat{\Lambda}_n(t)) = n^{-1} \gamma(t) - 2e_K q(t)n^{-1}b + d_K^2 \alpha^2(t)b^4.
\]

Thus, the asymptotic optimal bandwidth, \( b_{OPT}(t) \), is

\[
b_{OPT}(t) = \left( \frac{e_K q(t)}{2d_K^2 \alpha^2(t)n} \right)^{1/3}.
\]
For the asymptotically optimal bandwidth (26), $AMSE\left(\bar{\Lambda}_n^P(t)\right)$ becomes

$$n^{-1}\gamma(t) - \frac{3}{24^{4/3}} \left(\frac{e_k^4 q^4(t)}{d_k^2 \alpha^2(t)}\right)^{1/3} n^{-4/3}.$$  

This expression shows that the version $\bar{\Lambda}_n^P(t)$ of the presmoothed Nelson-Aalen estimator is more efficient than the classical Nelson-Aalen estimator. Indeed, for the latter we have that

$$\text{Var}(\Lambda_{NA}^n(t)) = n^{-1}\gamma(t) + O(n^{-3/2}).$$

This makes clear that the second order term of the variance of the Nelson-Aalen estimator is negligible with respect to the order $n^{-4/3}$. Hence the second order efficiency of $\bar{\Lambda}_n^P(t)$ with respect to $\Lambda_{NA}^n(t)$ together with the order of the moments of the remainder term in there (see e.g. Lo, Mack and Wang (1989) or Gijbels and Wang (1993)).

Similar properties for $\Lambda_n^P(t)$ are not easy to derive since the $o_p(n^{-1/2})$ term in (9) is not negligible enough to obtain such a result from our previous discussion and Theorem 3. A deeper analysis of this term would be a rather complicated task. However, the simulations results in Section 5 indicate that this second order efficiency is also present for $\Lambda_n^P(t)$.

4 Plug-in bandwidth selection

Using the asymptotic expression (25),

$$AMSE(\bar{\Lambda}_n^P(t)) = n^{-1}\gamma(t) - 2e_k q(t)n^{-1}b + d_k^2 \alpha^2(t)b^4$$

where $\gamma(t)$, $q(t)$ and $\alpha(t)$ are given in (24), (16) and (15), it is easy to obtain the following expression for the weighted asymptotic mean integrated squared error:

$$AMISE_w(b) = n^{-1} \int_0^\infty \gamma(t)w(t)dt - 2e_k n^{-1}b \int_0^\infty q(t)w(t)dt + d_k^2 b^4 \int_0^\infty \alpha^2(t)w(t)dt,$$

where $w$ is a positive weight function. Therefore, the optimal bandwidth, in the sense of $AMISE_w(b)$, is

$$b_{OPT} = \arg\min_{b>0} AMISE_w(b) = \left(\frac{e_k Q}{2d_k^2 n A}\right)^{1/3}$$

(27)
where

\[ Q = \int_0^\infty q(t)w(t)dt, \]
\[ A = \int_0^\infty \alpha^2(t)w(t)dt. \]

From now on, we will consider the following plug-in bandwidth selector of \( b \)

\[ \hat{b} = \left( \frac{e_k \bar{Q}}{2d_k^2 n A} \right)^{1/3} \]  

(28)

where

\[ \bar{Q} = \frac{1}{n} \sum_{i=1}^n \left( 1 - H_n(T_i) + \frac{1}{n} \right)^{-2} \hat{p}(T_i) (1 - \hat{p}(T_i)) w(T_i), \]  

(29)

\[ \bar{A} = \int_0^\infty \alpha^2(t)w(t)dt, \]  

(30)

\[ \tilde{\alpha}(t) = \int_0^t \left( 1 - H_n(s) + \frac{1}{n} \right)^{-1} \left( \hat{p}''(s)\hat{h}(s)/2 + \hat{p}'(s)\hat{h}'(s) \right) ds, \]  

(31)

and \( \hat{p}, \hat{p}' \) and \( \hat{p}'' \) are the Nadaraya-Watson estimators of \( p \) and its first and second derivatives,

\[ \hat{p}(t) = \frac{1}{n} \sum K_g(t - T_i) \delta_i \]
\[ \hat{p}'(t) = \frac{\hat{p}'(t)\hat{h}(t) - \hat{p}(t)\hat{h}'(t)}{\hat{h}(t)^2}, \]

\[ \hat{p}''(t) = \frac{\hat{p}''(t)\hat{h}(t)^2 - \hat{p}'(t)\hat{h}''(t)\hat{h}(t) - 2\hat{p}'(t)\hat{h}'(t)\hat{h}(t) + 2\hat{p}(t)\hat{h}'(t)^2}{\hat{h}(t)^3} \]  

(34)

where \( f^{(k)} \) denotes the \( k \)-th derivative of \( f \),

\[ \hat{\psi}^{(k)}(t) = \frac{1}{n} \sum K_g^{(k)}(t - T_i) \delta_i, \]
\[ \hat{h}^{(k)}(t) = \frac{1}{n} \sum K_g^{(k)}(t - T_i), \]
\[ K_g^{(k)}(t) = \frac{1}{g^{k+1}} K^{(k)} \left( \frac{t}{g} \right), \]
and the estimator $\hat{b}^{(k)}(t), k = 1, 2, \ldots$, is the $k$-th derivative of $\hat{b}(t)$. The functions $\hat{h}$ and $\hat{h}'$ are the Parzen-Rosenblatt kernel estimators of the density $h$ and its first derivative and $H_n$ is the empirical distribution function of the $T_i$.

It is worth mentioning that there exist similar expressions to (32), (33) and (34) but for populational quantities. Indeed,

$$p(t) = \frac{\psi(t)}{h(t)},$$

$$p'(t) = \frac{\psi'(t)h(t) - \psi(t)h'(t)}{h(t)^2},$$

$$p''(t) = \frac{\psi''(t)h(t)^2 - \psi(t)h''(t)h(t) - 2\psi'(t)h'(t)h(t) + 2\psi(t)h'(t)^2}{h(t)^3},$$

where $h$ is the density function of the observed lifetime and $\psi = ph$.

Typically, the effective calculation of $\hat{b}$ requires the election of some pilot bandwidths, $g_1$ and $g_2$, for the estimators $\hat{A}$ and $\hat{Q}$, respectively.

As a preliminary step for the choice of the pilot bandwidths some asymptotic expressions for the mean squared errors $E\left((\hat{A} - A)^2\right)$ and $E\left((\hat{Q} - Q)^2\right)$ will be obtained. The criterion to choose the pilot bandwidths $g_1$ and $g_2$ will consist in minimizing the dominant part of those expressions.

In the rest of this paper, we will make use of some further assumptions:

On the weight function $w$:

(W.1) $w$ is non negative, with support within the interval $(\epsilon/2, t_0)$ and twice differentiable in $[\epsilon/2, t_0]$, with continuous second derivative.

On the pilot bandwidths, $g_1$ and $g_2$:

(V.1) $ng_1^3\left(\log \frac{1}{g_1}\right)^{-3} \to \infty$ and $ng_1^6 \to 0$.

(V.2) $ng_2^6 \to \infty$ and $ng_2^4 \to 0$.

Assumption (P.2), stated in Section 2, warranties that for any $t$ in the interval $[0, \epsilon/2]$ there exists some (common) value for the bandwidth used in the estimators $\hat{p}''(s)$ and $\hat{p}'(s)$ in $\hat{\alpha}(t)$ such that for bandwidths smaller than that value, $\hat{\alpha}(t) = 0$, with probability 1. For these reason when studying the asymptotic behaviour of $\hat{A}$ we will consider its asymptotic equivalent term

$$\tilde{A}_{\epsilon/2} = \int_{\epsilon'}^{\infty} \left( \int_{\epsilon'}^{t} (1 - H_n(s) + n^{-1})^{-1} \frac{\hat{p}''(s)\hat{h}(s)/2 + \hat{p}'(s)\hat{h}'(s)}{2} ds \right) dw(t) dt$$

where $\epsilon' = \frac{\epsilon}{2}$. 

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Lemma 10 Under conditions (K.1), (H.1), (W.1) and (V.1),
\[ \tilde{A}_{c/2} = \tilde{A}_1 + o_P(\tilde{A}_1) \]
where
\[ \tilde{A}_1 = \frac{1}{4} \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{t} \int_{\varepsilon'}^{t} (1 - H(r))^{-1}(1 - H(s))^{-1} \times \left( \tilde{\psi}''(r) - \tilde{p}(r)\tilde{h}''(r) \right) \left( \tilde{\psi}''(s) - \tilde{p}(s)\tilde{h}''(s) \right) w(t)drdsdt. \]

The term \( \tilde{A}_1 \) has still to be linearized, since the estimator \( \tilde{p} \) has a random denominator. To obtain such a linearization the expression \( \tilde{\psi}'' - \tilde{p}\tilde{h}'' \) is factorized as follows
\[ \tilde{\psi}'' - \tilde{p}\tilde{h}'' = \psi'' - ph'' + (\tilde{\psi}'' - \psi'') - p(\tilde{h}'' - h'') - (\tilde{\psi} - \psi)h''h^{-1} \]
\[ + p(\tilde{h} - h)h''h^{-1} + (\tilde{p} - p)((\tilde{h} - h)h''h^{-1} - (\tilde{h}'' - h'')) \] \( (38) \)
where we have used the following relation
\[ \tilde{p} - p = (\tilde{\psi} - \psi)h^{-1} - p(\tilde{h} - h)h^{-1} - (\tilde{p} - p)(\tilde{h} - h)h^{-1}. \] \( (39) \)

Substituting (38) in expression (67) we obtain
\[ \tilde{A}_1 = \frac{1}{4} \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{t} \int_{\varepsilon'}^{t} (1 - H(r))^{-1}(1 - H(s))^{-1} \times \left( \psi''(r) - p(r)h''(r) + (\tilde{\psi}'' - \psi'')(r) - p(r)(\tilde{h}''(r) - h''(r)) \right) \]
\[ - (\tilde{\psi}(r) - \psi(r))h''h^{-1} + p(r)(\tilde{h}(r) - h(r))h''h^{-1}(r)h^{-2} \]
\[ + (\tilde{p}(r) - p(r))(\tilde{h}(r) - h(r))h''h^{-1}(r)h^{-2} - (\tilde{h}''(r) - h''(r))) \]
\[ \times \left( \psi''(s) - p(s)h''(s) + (\tilde{\psi}'' - \psi'')(s) - p(s)(\tilde{h}''(s) - h''(s)) \right) \]
\[ - (\tilde{\psi}(s) - \psi(s))h''h^{-1}(s)h^{-1} + p(s)(\tilde{h}(s) - h(s))h''h^{-1}(s)h^{-1} \]
\[ + (\tilde{p}(s) - p(s))(\tilde{h}(s) - h(s))h''h^{-1}(s)h^{-1} - (\tilde{h}''(s) - h''(s))) \right) w(t)drdsdt. \]

Now using (36) and (37),
\[ A = \int_{0}^{\infty} \left( \int_{0}^{t}(1 - H(s))^{-1}(\psi''(s)/2 + p'(s)h'(s))ds \right)^2 w(t)dt \]
\[ = \frac{1}{4} \int_{0}^{\infty} \left( \int_{0}^{t}(1 - H(s))^{-1}(\psi''(s) - p(s)h''(s))ds \right)^2 w(t)dt \]
\[ = \frac{1}{4} \int_{0}^{\infty} \int_{0}^{t} (1 - H(r))^{-1}(1 - H(s))^{-1} \times (\psi''(r) - p(r)h''(r)) (\psi''(s) - p(s)h''(s)) w(t)drdsdt \]

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which imply

\[
\hat{A}_1 = A + \int_{e'}^t \int_{e'}^t (1 - H(r))^{-1} \left( (\hat{\psi}'(r) - \psi'(r)) - p(r)(\hat{h}'(r) - h'(r)) \right)
- (\hat{\psi}(r) - \psi(r))h''(r)h(r)^{-1} + p(r)(\hat{h}(r) - h(r))h''(r)h(r)^{-1}
\times \alpha(t)w(t)drdt
\]
\[
+ \frac{1}{4} \int_{e'}^\infty \int_{e'}^t \int_{e'}^t (1 - H(r))^{-1}(1 - H(s))^{-1}
\times \left( (\hat{\psi}''(r) - \psi''(r)) - p(r)(\hat{h}'(r) - h'(r)) \right)
- (\hat{\psi}(r) - \psi(r))h''(r)h(r)^{-1} + p(r)(\hat{h}(r) - h(r))h''(r)h(r)^{-1}
\times \left( (\hat{\psi}''(s) - \psi''(s)) - p(s)(\hat{h}'(s) - h'(s)) \right)
\times w(t)drdsdt
\]

\[
\]
Repeating the linearization of the second but last summand,
\[
\int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{t} (1 - H(r))^{-1} (\hat{p}(r) - p(r)) \\
\times \left( (\hat{h}(r) - h(r)) h''(r) h(r)^{-1} - (\hat{h}'(r) - h''(r)) \right) \alpha(t) w(t) dr dt
\]
\[= \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{t} (1 - H(r))^{-1} \left( (\hat{\psi}(r) - \psi(r)) h(r)^{-1} - \hat{\psi}'(r) - h''(r) \right) \\
\times \left( (\hat{h}(r) - h(r)) h''(r) h(r)^{-1} - (\hat{h}'(r) - h''(r)) \right) \alpha(t) w(t) dr dt
\]
\[-\int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{t} (1 - H(r))^{-1} (\hat{\psi}(r) - \psi(r)) h(r)^{-1} \\
\times \left( (\hat{h}(r) - h(r)) h''(r) h(r)^{-1} - (\hat{h}'(r) - h''(r)) \right) \alpha(t) w(t) dr dt.
\]
Define
\[
\hat{A}_{11} = \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{1} (1 - H(r))^{-1} \left( (\hat{\psi}'(r) - \psi''(r)) - p(r)(\hat{h}'(r) - h''(r)) \right) \\
\times \left( (\hat{\psi}(r) - \psi(r)) - p(r)(\hat{h}(r) - h(r)) \right) \alpha(t) w(t) dr dt;
\]
\[
\hat{A}_{12} = \frac{1}{4} \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{1} \int_{\varepsilon'}^{t} (1 - H(r))^{-1} \left( 1 - H(s) \right)^{-1} \\
\times \left( (\hat{\psi}'(r) - \psi''(r)) - p(r)(\hat{h}'(r) - h''(r)) \right) \\
\times \left( (\hat{\psi}(r) - \psi(r)) - p(r)(\hat{h}(r) - h(r)) \right) \alpha(t) w(t) dr ds dt;
\]
\[
\hat{A}_{13} = \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{1} (1 - H(r))^{-1} h(r)^{-1} \left( (\hat{\psi}(r) - \psi(r)) - p(r)(\hat{h}(r) - h(r)) \right) \\
\times \left( (\hat{h}(r) - h(r)) h''(r) h(r)^{-1} - (\hat{h}'(r) - h''(r)) \right) \alpha(t) w(t) dr dt.
\]
where and, thus, the bandwidth that minimizes MSE to check that under conditions (K.1), (P.1), (P.2), (H.1), (W.1) and (V.1),

\[
\text{MSE} \left( \hat{A}_1 \right) = \text{AMSE} \left( \hat{A}_1 \right) + O \left( g_1^6 \right) + o \left( n^{-1} g_1^{-1} \right) + o \left( n^{-2} g_1^{-6} \right)
\]

where

\[
\text{AMSE} \left( \hat{A}_1 \right) = \left( C_1 g_1^2 + C_2 n^{-1} g_1^{-3} \right)^2
\]

and, thus, the bandwidth that minimizes AMSE \( \hat{A}_1 \) is

\[
g_{1,\text{AMSE}} = C n^{-\frac{1}{2}}
\]

(43)
where

\[ C = \begin{cases} \left(-\frac{C_1}{C_2}\right)^{\frac{1}{2}}, & \text{if } C_1 < 0 \\ \left(\frac{3C_2}{2C_1}\right)^{\frac{1}{2}}, & \text{if } C_1 > 0 \end{cases} \]

\[ C_1 = \frac{1}{2}K \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} (1 - H(r))^{-1} \left(p^{(4)}(r)h(r) + 4p^{(3)}(r)h'(r) + 5p''(r)h''(r)ight. \]
\[ \left. + 4p'(r)h^{(3)}(r) - 2p'(r)h(r)^{-1}h'(r)h''(r)\right) \alpha_w(t)dt \, dt \quad (46) \]

and

\[ C_2 = \frac{1}{4}K \int_{\varepsilon}^{\infty} (1 - H(x))^{-2} (1 - p(x)) p(x)h(x)w(x)dx. \quad (47) \]

The term \( \hat{A}_{14} \) can be proved to be negligible as stated in the following lemma.

**Lemma 12** Under the conditions (K.1), (P.1), (P.2), (H.1), (W.1) and (V.1), we have

\[ \hat{A}_{14} = o_P \left(n^{-1} g_1^{-3}\right). \]

It remains to study the estimator of \( Q \) proposed in (29). First of all let us state some result that gives its dominant term.

**Lemma 13** Under conditions (K.1), (H.1), (W.1), and (V.2), it holds

\[ \hat{Q} = \hat{Q}_1 + o_P \left(\hat{Q}_1\right) \quad (48) \]

where

\[ \hat{Q}_1 = \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} \hat{p}(T_i) (1 - \hat{p}(T_i)) w(T_i). \]

For the term \( \hat{Q}_1 \), the representation

\[ \hat{p}(1 - \hat{p}) = p(1 - p) + (\hat{p} - p)(1 - 2p) - (\hat{p} - p)^2 \]

gives

\[ \hat{Q}_1 = \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} p(T_i) (1 - p(T_i)) w(T_i) \]
\[ + \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{p}(T_i) - p(T_i)) (1 - 2p(T_i)) w(T_i) \]
\[ - \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{p}(T_i) - p(T_i))^2 w(T_i) \quad (49) \]
Writing the difference \( \hat{p} - p \) in terms of the functions \( \psi, h \) and their pertaining estimators \( \hat{\psi} \) and \( \hat{h} \),

\[
\hat{p} - p = (\hat{\psi} - p\hat{h})h^{-1} - (\hat{p} - p)(\hat{h} - h)h^{-1},
\]

the last two summands of (49) can be easily linearized:

\[
\frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i)) (1 - 2p(T_i)) w(T_i)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i)) \hat{h}(T_i)) (1 - 2p(T_i)) h(T_i)^{-1} w(T_i)
\]

\[
- \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{p}(T_i) - p(T_i)) (\hat{h}(T_i) - h(T_i)) (1 - 2p(T_i)) h(T_i)^{-1} w(T_i)
\]

\[(50)\]  

and

\[
\frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i))^2 w(T_i)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i)) \hat{h}(T_i)\hat{h}(T_i) h(T_i)^{-2} w(T_i)
\]

\[
- \frac{2}{n} \sum_{i=1}^{n} (\hat{\psi}(T_i) - p(T_i)) \hat{h}(T_i)) \hat{p}(T_i) - p(T_i))
\]

\[
\times (\hat{h}(T_i) - h(T_i)) (1 - H(T_i))^{-2} h(T_i)^{-2} w(T_i)
\]

\[
+ \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i))^2 (\hat{h}(T_i) - h(T_i)) h(T_i)^{-2} w(T_i).
\]

\[(51)\]  

The same procedure is repeated for the second summand of (51), in order to obtain a linearized term with two factors of the type \( \hat{\psi} - p\hat{h} \) or \( \hat{h} - h \), as in the first summand of (52),

\[
\frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i)) \hat{h}(T_i) - h(T_i)) (1 - 2p(T_i)) h(T_i)^{-1} w(T_i)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i)) \hat{h}(T_i) - h(T_i)) (1 - 2p(T_i)) h(T_i)^{-2} w(T_i)
\]

\[
- \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{p}(T_i) - p(T_i)) (\hat{h}(T_i) - h(T_i))^2 (1 - 2p(T_i)) h(T_i)^{-2} w(T_i).
\]

Define

\[
Q_{11} = \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} p(T_i) (1 - p(T_i)) w(T_i),
\]

\[19\]
\[ \hat{Q}_{12} = \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i)\hat{h}(T_i)) (1 - 2p(T_i)) h(T_i)^{-1} w(T_i), \]

\[ \hat{Q}_{13} = -\frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i)\hat{h}(T_i))^2 h(T_i)^{-2} w(T_i) \]

\[ -\frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i)\hat{h}(T_i)) (\hat{h}(T_i) - h(T_i)) \]

\[ \times (1 - 2p(T_i)) h(T_i)^{-2} w(T_i) \]

and

\[ \hat{Q}_{14} = \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i))^2 (1 - 2p(T_i)) h(T_i)^{-2} w(T_i) \]

\[ + \frac{2}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i)\hat{h}(T_i)) (\hat{h}(T_i) - h(T_i)) \]

\[ \times (\hat{h}(T_i) - h(T_i)) h(T_i)^{-2} w(T_i) \]

\[ -\frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} (\hat{\psi}(T_i) - p(T_i))^2 (\hat{h}(T_i) - h(T_i))^2 h(T_i)^{-2} w(T_i), \quad (53) \]

to obtain the representation

\[ \hat{Q}_1 - Q = Q_{11} - Q + \hat{Q}_{12} + \hat{Q}_{13} + \hat{Q}_{14}. \]

or, equivalently, defining \( \hat{Q}_1 = \hat{Q}_1 - \hat{Q}_{14}, \)

\[ \hat{Q}_1 - Q = \hat{Q}_1 - Q + \hat{Q}_{14}. \quad (54) \]

Now, the mean squared error of \( \hat{Q}_1 \) is given in the next result.

**Theorem 14** Under conditions (K.1), (P.1), (H.1), (W.1) and (V.2),

\[ \text{MSE} \left( \hat{Q}_1 \right) = \text{AMSE} \left( \hat{Q}_1 \right) + O \left( g_2^5 \right) + o \left( n^{-1} g_2^{-1} \right) + o \left( n^{-2} g_2^{-6} \right) \]

where

\[ \text{AMSE} \left( \hat{Q}_1 \right) = (D_2 g_2^2 + D_2 n^{-1} g_2^{-1})^2 \]

and the smoothing parameter that minimizes \( \text{AMSE} \left( \hat{Q}_1 \right) \) is

\[ g_{2, \text{AMSE}} = D n^{-\frac{1}{2}} \quad (55) \]
where
\[
D = \begin{cases} 
\left( \frac{D_2}{2} \right)^{\frac{1}{2}}, & \text{if } D_2 < 0 \\
\left( -\frac{D_2}{2} \right)^{\frac{1}{2}}, & \text{if } D_2 > 0 
\end{cases}
\]

\[
D_1 = \mu_K \int_{c'}^{\infty} (1 - H(x))^{-2} (1 - 2p(x))(p'(x)h'(x) + \frac{1}{2}p''(x)h(x))w(x)dx
\]

and
\[
D_2 = -c_K \int_{c'}^{\infty} (1 - H(x))^{-2} p(x)(1 - p(x))w(x)dx
\]

Now, the term $\hat{Q}_{14}$ is proved to be negligible in the following result.

**Lemma 15** Under conditions (K.1), (P.1), (H.1), (W.1) and (V.2),
\[
\hat{Q}_{14} = O_P \left( n^{-\frac{2}{15}} \right).
\]

Our next result gives the rate of convergence of the plug-in bandwidth selector when using the asymptotically optimal pilot bandwidths.

**Theorem 16** Under conditions (K.1), (P.1), (P.2), (H.1), (W.1), (V.1) and (V.2) and using the pilot bandwidths in (45) and (55) we have
\[
\hat{b} - b_{OPT} = O_P \left( n^{-\frac{2}{15}} \right). \tag{59}
\]

\[
\frac{\hat{b} - b_{OPT}}{b_{OPT}} = O_P \left( n^{-\frac{4}{15}} \right). \tag{60}
\]

Practical implementation of this plug-in bandwidth needs of selecting the pilot bandwidths $g_1$ and $g_2$. To do this, we used equations (45) and (55) and estimated the underlying functions $H$, $h$ and $p$ in (46), (47), (56) and (57) using a lognormal parametric fit for the first two and a logistic fit for the last one.

5 **Simulations**

Some simulations have been carried out in order to evaluate the practical performance of the presmoothed Nelson-Aalen estimator with plug-in bandwidth selector. To fulfill condition (P.2), some shifted version of a Weibull distribution has been considered for the censoring time. For some $\varepsilon > 0$, we define
which means $C - \varepsilon \overset{\text{d}}{=} W(\alpha_G, \beta_G)$, while $Y \overset{\text{d}}{=} W(\alpha_F, \beta_F)$, where $W(\alpha, \beta)$ denotes the Weibull distribution with shape parameter $\alpha$ and scale parameter $\beta$, with density

$$f(x) = \beta \alpha x^{\alpha-1} \exp(-\beta x^\alpha), \quad x > 0.$$  

Table 1 collects the parameters used for these two distributions in the four models considered here. The cumulative observable distribution function, $H$, and the conditional probability of uncensoring, $p$, pertaining to these models are plotted in Figure 1. The vertical dotted lines in this Figure indicate the left and right endpoints of the support of the weight function, $w$, which has been set to a constant within these limits. These endpoints have been selected to meet the condition (W.1). For comparison reasons the unconditional censoring probability for these models is very similar (between 0.32 and 0.34).

<table>
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<tr>
<th>Model</th>
<th>$\alpha_F$</th>
<th>$\beta_F$</th>
<th>$\alpha_G$</th>
<th>$\beta_G$</th>
<th>$\varepsilon$</th>
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<td>7</td>
<td>0.1</td>
</tr>
<tr>
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<td>10</td>
<td>1</td>
<td>7</td>
<td>7</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 1. Parameters for the distribution of the lifetime ($W(\alpha_F, \beta_F)$) and the censoring variable ($W^\varepsilon(\alpha_G, \beta_G)$).
Figure 1. Functions $H$ (thin line) and $p$ (thick line) for models 1–4.

Using 1000 samples of size $n = 30, 500$ and 1000, the $MISE_w$ ratio:

$$\frac{MISE_w(\Lambda_{n}^{P})}{MISE_w(\Lambda_{n}^{NA})}$$

has been approximated by simulation for a grid in a wide range of possible bandwidths. These functions of the smoothing parameter are plotted in Figures 2–5 for the four models considered here. Values of the $MISE_w$ ratio below 1 indicate that the presmoothed Nelson-Aalen estimator is better than the ordinary Nelson-Aalen estimator for these bandwidths.

Figure 2. $MISE_w$ ratio of $\Lambda_{n}^{P}$ with respect to $\Lambda_{n}^{NA}$ for $n = 30$ (thin line), $n = 500$ (medium line) and $n = 1000$ (thick line) for model 1.
Figure 3. $MISE_w$ ratio of $\Lambda_n^P$ with respect to $\Lambda_{nA}^N$ for $n = 30$ (thin line), $n = 500$ (medium line) and $n = 1000$ (thick line) for model 2.
Figure 4. $MISE_e$ ratio of $\Lambda_{w,n}^P$ with respect to $\Lambda_{w,n}^{NA}$ for $n = 30$ (thin line), $n = 500$ (medium line) and $n = 1000$ (thick line) for model 3.
Figures 2—5 show that there are quite wide ranges of presmoothing parameters for which the new estimator is better than the classical Nelson-Aalen estimator. Most of the time the minimal $MISE_w$ ratio may be about 0.8 to 0.95, depending on the model and the sample size. Typically, those optimal $MISE_w$ ratios get closer to 1 as the sample size increases. Some special case is model 4, for which, for any possible presmoothing factor in an extremely large range, the $MISE_w$ ratio is smaller than 1. Figure 1 shows that, for model 3, the $p$ function is almost constant in the interval $[0.6, 1.2]$ where almost all observed data fall, so the Koziol-Green model nearly holds. This means that the ACL estimator of $\Lambda$ (see Abdushukurov (1987) and Cheng and Lin (1987)), that corresponds to and infinitely large presmoothing parameter, is more efficient than the classical Nelson-Aalen estimator.

In order to investigate the practical performance of the plug-in bandwidth proposed in Section 4 we first obtained this bandwidth selector, $\hat{b}$, for 1000 samples of size $n = 500$ drawn from models 1-4 and computed a Parzen-Rosenblatt kernel density estimation, using the Sheather-Jones bandwidth selector. These curves, together with the optimal $AMISE_w$ bandwidth, $b_{OPT}$, are plotted in Figure 6. Although for models 1 and 3 $\hat{b}$ presents a clear bias, it is a reasonable
selector for $b_{OPT}$. For instance the simulation results indicate that

$$\left| \frac{\hat{b}}{b_{OPT}} - 1 \right| < 0.32$$

for about 90% of the simulated samples in models 1, 2 and 4. This means that the plug-in selector is within 32% of deviation from the optimal bandwidth in 90% of the cases. For model 3 the same statement is only valid within 81% of deviation. This is not surprising since a large fluctuation of the bandwidth around its optimal value gives a small loss in terms of $MISE_w$ (see the flat shape of the functions in Figure 4 around their minima).

![Figure 6](image_url)

Figure 6. Kernel estimation of the density of the plug-in bandwidth selector, $\hat{b}$, for models 1-4 (solid line) and $b_{OPT}$ bandwidth (dotted vertical line).

Similar empirical studies to those performed above for a range of presmoothing parameters have been carried out for the presmoothing Nelson-Aalen estimator with plug-in presmoothing parameter. The $MISE_w$ ratio for the data-driven presmoothed estimator has been computed:

$$\frac{MISE_w(\Lambda_{P_n,b})}{MISE_w(\Lambda_{P_n,A})}$$

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Some Monte Carlo approximation of this quantity based on 1000 samples has been computed for models 1–4 and different sample sizes. These results are collected in Table 2. The figures in this table show that the presmoothed Nelson-Aalen estimator with automatic plug-in bandwidth is about 5% to 10% more efficient than the classical Nelson-Aalen estimator (in terms of $MISE_w$) for models 2–3. The values

<table>
<thead>
<tr>
<th>Model</th>
<th>$n = 30$</th>
<th>$n = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.11</td>
<td>1.03</td>
</tr>
<tr>
<td>2</td>
<td>0.93</td>
<td>0.96</td>
</tr>
<tr>
<td>3</td>
<td>0.88</td>
<td>0.88</td>
</tr>
<tr>
<td>4</td>
<td>0.86</td>
<td>0.88</td>
</tr>
</tbody>
</table>

Table 2. $MISE_w$ ratio of the presmoothed Nelson-Aalen estimator with plug-in bandwidth and the classical Nelson-Aalen estimator.

6 Proofs

6.1 Proofs of the results of Section 2

Proof of Theorem 3
As to term $(I)$ in (9), we have the well known iid representation

$$\hat{\Lambda}_H(s) - \Lambda_H(s) = \frac{1}{n} \sum_{i=1}^{n} \frac{1(T_i \leq s) - H(s)}{1 - H(s)} + r_n(s) \quad (61)$$

where for $t > 0$ such that $H(t) < 1$:

$$\sup_{0 \leq s \leq t} |r_n(s)| = O(n^{-1} \log n) \quad \text{a.s.}$$

as $n \to \infty$. This result follows from the more general representation in the censored data case, due to Lo and Singh (1986). Using integration by parts gives that

$$(I) = \frac{1}{n} \sum_{i=1}^{n} (g_1(T_i) - g_2(T_i)) + O(n^{-1} \log n) \quad \text{a.s.}$$

with $g_1$ and $g_2$ as in (11) and (12).

For the term $(II)$ in (9), we have, with $\hat{h}(s) = n^{-1} \sum_{i=1}^{n} K_b(s - T_i)$:

$$\hat{p}(s) - p(s) = \frac{n^{-1} \sum_{i=1}^{n} K_b(s - T_i)(\delta_i - p(s))}{h(s)} - \frac{(\hat{p}(s) - p(s))(\hat{h}(s) - h(s))}{h(s)} \quad (62)$$
This gives that

\[(II) = \frac{1}{n} \sum_{i=1}^{n} g_3(T_i, \delta_i) + o_p(n^{-1/2})\]  

(63)

with \(g_3\) given by (13). The \(o_p(n^{-1/2})\) remainder term in (63) is obtained as follows:

\[\left| \frac{1}{h(s)} \int_0^t (\hat{p}(s) - p(s))(\hat{h}(s) - h(s)) d\Lambda_H(s) \right| \leq \frac{t}{1 - H(t)} \left\{ \sup_{0 \leq s \leq t} | \hat{p}(s) - p(s) | \right\} \left\{ \sup_{0 \leq s \leq t} | \hat{h}(s) - h(s) | \right\}.\]  

(64)

Under the given conditions, it follows from Lemma 1 and Theorem B in Mack and Silverman (1982) that

\[\sup_{0 \leq s \leq t} | \hat{p}(s) - p(s) | = O_p((nb)^{-1/2}(\log(1/b))^{1/2})\]

and

\[\sup_{0 \leq s \leq t} | \hat{h}(s) - E(h(s)) | = O_p((nb)^{-1/2}(\log(1/b))^{1/2}).\]

By a standard Taylor expansion argument, it also follows that \(\sup_{0 \leq s \leq t} | E(h(s)) - h(s) | = O_p(b^2)\). Therefore, the right hand side in (64) is \(o_p(n^{-1/2})\).

For the term \((III)\) in (9), we first plug in representation (61), which yields

\[(III) = \int_0^t (\hat{p}(s) - p(s)) \frac{d\left( \frac{H_n(s) - H(s)}{1 - H(s)} \right)}{1 - H(s)} + o(n^{-1}\log n) \text{ a.s.}\]

\[= \int_0^t \frac{\hat{p}(s) - p(s)}{1 - H(s)} dH_n(s) - \int_0^t \frac{\hat{p}(s) - p(s)}{1 - H(s)} dH(s)\]

\[+ \int_0^t \frac{(\hat{p}(s) - p(s))(H_n(s) - H(s))}{(1 - H(s))^2} dH(s) + o(n^{-1}\log n).\]  

(65)

The absolute value of the third term in the right hand side of (65) is bounded above by

\[\frac{1}{(1 - H(t))^2} \left\{ \sup_{0 \leq s \leq t} | \hat{p}(s) - p(s) | \right\} \left\{ \sup_{0 \leq s \leq t} | H_n(s) - H(s) | \right\} = o_p(n^{-1/2})\]
since Theorem A in Mack and Silverman (1982) gives that $\sup_{0 \leq s \leq t} | \hat{p}(s) - p(s) | = o_p(1)$ and a classical empirical process result gives that $\sup_{0 \leq s \leq t} | H_n(s) - H(s) | = O_p(n^{-1/2})$. For the second term in the right hand side of (65), we have by (62) that it equals

$$\frac{1}{n} \sum_{i=1}^{n} \int_{0}^{t} \frac{K_b(s - T_i)(\delta_i - p(s))}{1 - H(s)} ds + o_p(n^{-1/2}).$$

The first term in (65) is equal to

$$\frac{1}{n} \sum_{i=1}^{n} \frac{\hat{p}(T_i) - p(T_i)}{1 - H(T_i)} 1(T_i \leq t)$$

$$= \frac{1}{n^2} \sum_{i,j=1}^{n} \frac{K_b(T_i - T_j)(\delta_j - p(T_i))}{(1 - H(T_i))h(T_i)} 1(T_i \leq t) + o_p(n^{-1/2})$$

by using representation (62) and the same bounds for $\sup_{0 \leq s \leq t} | \hat{p}(s) - p(s) |$ and $\sup_{0 \leq s \leq t} | \hat{h}(s) - h(s) |$ as before. Introducing

$$\varphi(T_i, T_j, \delta_j) = \frac{K_b(T_i - T_j)(\delta_j - p(T_i))}{(1 - H(T_i))h(T_i)} 1(T_i \leq t)$$

we have that

$$(III) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi(T_i, T_j, \delta_j)$$

$$- \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{t} \frac{K_b(s - T_i)(\delta_i - p(s))}{1 - H(s)} ds + o_p(n^{-1/2})$$

since $\frac{1}{n^2} \sum_{i=1}^{n} \varphi(T_i, T_i, \delta_i) = O_p\left( \frac{1}{n^{3/2}b} \right) = o_p(n^{-1/2})$, because it is $\frac{1}{n^2 b}$ times a sum of zero mean, iid random variables with finite variance. By symmetrization of the kernel $\varphi$, i.e. by putting

$$\psi(T_i, \delta_i, T_j, \delta_j) = \frac{1}{2}(\varphi(T_i, T_j, \delta_j) + \varphi(T_j, T_i, \delta_i)),$$

we obtain that

$$(III) = U_n - \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{t} \frac{K_b(s - T_i)(\delta_i - p(s))}{1 - H(s)} ds + o_p(n^{-1/2}) \quad (66)$$
where $U_n$ is the $U$-statistic with symmetric kernel $\psi$, i.e.

$$U_n = \left(\frac{n}{2}\right)^{-1} \sum_{1 \leq i < j \leq n} \psi(T_i, \delta_i, T_j, \delta_j).$$

For the Hajek projection of $U_n$, we have

$$G(T_1, \delta_1) = E[\psi(T_1, \delta_1, T_2, \delta_2) \mid T_1, \delta_1]$$

$$= \frac{1}{2} \{E(\varphi(T_1, T_2, \delta_2) \mid T_1) + E(\varphi(T_2, T_1, \delta_1) \mid T_1, \delta_1)\}$$

$$:= \frac{1}{2} \{G_1(T_1) + G_2(T_1, \delta_1)\}.$$  

We calculate

$$G_1(t_1) = E(\varphi(T_1, T_2, \delta_2) \mid T_1 = t_1)$$

$$= E \left[ \frac{K_\psi(t_1 - T_2)|\delta_2 - p(t_1)|}{(1 - H(t_1))h(t_1)} 1\{t_1 \leq t\} \right]$$

$$= \frac{1\{t_1 \leq t\}}{(1 - H(t_1))h(t_1)} \int K_\psi(t_1 - t_2)E[\delta_2 \mid T_2 = t_2 - p(t_1)]h(t_2)dt_2$$

$$= \frac{1\{t_1 \leq t\}}{(1 - H(t_1))h(t_1)} \int K_\psi(t_1 - t_2)[p(t_2) - p(t_1)]h(t_2)dt_2$$

and

$$G_2(t_1, d_1) = E(\varphi(T_2, T_1, \delta_1) \mid T_1 = t_1, \delta_1 = d_1)$$

$$= E \left[ \frac{K_\psi(T_2 - t_1)}{(1 - H(T_2))h(T_2)}[d_1 - p(T_2)] 1\{T_2 \leq t\} \right]$$

$$= \int K_\psi(t_2 - t_1) \frac{d_1 - p(t_2)}{1 - H(t_2)} I(t_2 \leq t)dt_2.$$  

>From $U$-statistic theory, it follows that

$$U_n = -\theta_n + \frac{1}{n} \sum_{i=1}^{n} G_1(T_i) + \frac{1}{n} \sum_{i=1}^{n} G_2(T_i, \delta_i) + o_p(n^{-1/2})$$

where

$$\theta_n = E(\psi(T_1, \delta_1, T_2, \delta_2))$$

and, from (66),

$$(III) = -\theta_n + \frac{1}{n} \sum_{i=1}^{n} G_1(T_i) + o_p(n^{-1/2}).$$

To show that $\theta_n = O(b^2)$, note that by Taylor expansion it is easy to see that $E[G_1(T_1)] = O(b^2)$, so that we only have to show that $E[G_2(T_1, \delta_1)] = O(b^2).$
Use \( p(T_1) = E[\delta_1 \mid T_1] \) to obtain, for \( h \) small enough,

\[
E[G_2(T_1, \delta_1)] = -b \int_0^{t-hL} \frac{p'(t_1)}{1 - H(t_1)} \left[ \int_{-L}^L vK(v)dv \right] dH(t_1) \\
- h \int_{t-hL}^{t+hL} \frac{p'(t_1)}{1 - H(t_1)} \left[ \int_{-L}^{(t-t_1)/b} vK(v)dv \right] dH(t_1) + O(b^2) \\
= O(b^2)
\]

Similarly, it follows that \( E[G_2^2(T_1) = O(b^2) \), so that we can conclude that \((III) = o_p(u^{-1/2}) \) (since \( nb^4 \to 0 \)).

**Proof of Lemma 4**

Using the fact that \( p(u) = E(\delta_1 \mid T_1 = u) \), we have

\[
E[g_3(T_1, \delta_1)] = E \left[ \int_0^t \frac{K(s-T_1)(\delta_1 - p(s))}{1 - H(s)} ds \right] \\
= \int_0^t \int_0^t \frac{K(s-u)(p(u) - p(s))}{1 - H(s)} h(u) du ds \\
= \int_0^t \int_0^t \frac{K(v)(p(s-bv) - p(s))}{1 - H(s)} h(s-bv) dv ds.
\]

Under the smoothness conditions of \((C)\), performing a Taylor expansion of \( p(s-bv) \) and \( h(s-bv) \), and using that \( \int_{-L}^L K(v)dv = 1 \) and \( \int_{-L}^L vK(v)dv = 0 \), we obtain

\[
E[g_3(T_1, \delta_1)] = d_K \alpha(t)b^2 + o(b^2)
\]

with \( d_K \) and \( \alpha(t) \) given in (14) and (15).

The derivation of \((18) \) and \((19) \) is straightforward. For \( \text{Var}[g_3(T_1, \delta_1)] \) we have the following:

\[
E[g_3^2(T_1, \delta_1)] \\
= E \left[ \int \int K(u_1)K(u_2)[p(T_1) - p(T_1 + bu_1)](\delta_1 - p(T_1 + bu_2)) \\
\times \left( 1 - H(T_1 + bu_1) \right) \left( 1 - H(T_1 + bu_2) \right) \\
\times 1\{T_1 \leq (t-bu_1) \land (t-bu_2) \} du_1 du_2 \right] \\
= \int \int \int K(u_1)K(u_2) \left\{ p(t_1) - p(t_1, t_1 + bu_1) + p(t_1 + bu_1) + p(t_1 + bu_1)p(t_1 + bu_2) \right\} \\
\times \left( 1 - H(t_1 + bu_1) \right) \left( 1 - H(t_1 + bu_2) \right) \\
\times 1\{t_1 \leq t - b(u_1 \lor u_2) \} dH(t_1) du_1 du_2
\]

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\[
\begin{align*}
&= \iiint \frac{p(t_1)(1-p(t_1))h(t_1)}{(1-H(t_1))^2} 1\{t_1 \leq t - b(u_1 \vee u_2)\} K(u_1)K(u_2)du_1du_2 + O(b^2) \\
&= \iiint Q(t - b(u_1 \vee u_2))K(u_1)K(u_2)du_1du_2 + O(b^2)
\end{align*}
\]

where \(Q(s) = \int_0^s q(v)dv\). Now
\[
\begin{align*}
&= \iiint Q(t - b(u_1 \vee u_2))K(u_1)K(u_2)du_1du_2 \\
&= 2 \int_{-L}^L \int_{-L}^{u_2} Q(t - bu_2)K(u_2)du_1du_2 \\
&= 2 \int_{-L}^L Q(t - bu_2)K(u_2)du_2 \\
&= 2Q(t) \int_{-L}^L K(u_2)K(u_2)du_2 - 2bq(t) \int_{-L}^L u_2K(u_2)K(u_2)du_2 + O(b^2) \\
&= Q(t)K^2(u_2)\Big|_{-L}^L - 2bq(t)e_K + O(b^2) \\
&= \int_0^t \frac{p(v)(1-p(v))}{(1-H(v))^2}H(v) - 2bq(t)e_K + O(b^2).
\end{align*}
\]

The derivation of (21) is straightforward. For (22), we have
\[
E[g_1(T_1)g_3(T_1, \delta_1)] = p(t) \int_0^t \int_0^\infty \frac{K_h(s-u)(p(u) - p(s))}{1-H(s)}h(u)du ds
\]
and this is \(O(b^2)\) by Taylor expansion arguments. Similarly for (23).

**Proof of Theorem 5.** From expression (10) in Theorem 3 it follows that
\[
\begin{align*}
n\text{Var}(\Lambda_n(t) - \Lambda_F(t)) &= \text{Var}[g_1(T_1)] + \text{Var}[g_2(T_1)] \\
& + \text{Var}[g_3(T_1, \delta_1)] - 2\text{Cov}(g_1(T_1), g_2(T_1)) \\
& + 2\text{Cov}(g_1(T_1), g_3(T_1, \delta_1)) - 2\text{Cov}(g_2(T_1), g_3(T_1, \delta_1)).
\end{align*}
\]

Now use the expressions in Lemma 4 and some extra algebra. \(\blacksquare\)

### 6.2 Auxiliary results for the term \(\hat{A}\)

**Proof of Lemma 10.** Using (33) and (34)
\[
\hat{A}_{t/2} = \frac{1}{4} \int_{c'}^t \left( \int_{c'}^t \left( 1 - H_n(s) + n^{-1} \right)^{-1} \left( \psi''(s) - \tilde{p}(s)\tilde{h}''(s) \right) ds \right)^2 w(t) dt \\
= \frac{1}{4} \int_{c'}^t \int_{c'}^t \int_{c'}^t \left( 1 - H_n(r) + n^{-1} \right)^{-1} \left( 1 - H_n(s) + n^{-1} \right)^{-1} \\
\times \left( \psi''(r) - \tilde{p}(r)\tilde{h}''(r) \right) \left( \psi''(s) - \tilde{p}(s)\tilde{h}''(s) \right) w(t) dr ds dt.
\]

Since the denominator of \(\hat{A}_{t/2}\) is random we decompose it as a sum of two terms, the first of which replaces the empirical distribution by the populational distribution. Thus

\[
\hat{A}_{t/2} = \hat{A}_1 + \hat{A}_2
\]

where

\[
\hat{A}_1 = \frac{1}{4} \int_{c'}^t \int_{c'}^t \int_{c'}^t \left( 1 - H(r) \right)^{-1} \left( 1 - H(s) \right)^{-1} \\
\times \left( \psi''(r) - \tilde{p}(r)\tilde{h}''(r) \right) \left( \psi''(s) - \tilde{p}(s)\tilde{h}''(s) \right) w(t) dr ds dt
\]

and

\[
\hat{A}_2 = \frac{1}{4} \int_{c'}^t \int_{c'}^t \int_{c'}^t \left( \psi''(r) - \tilde{p}(r)\tilde{h}''(r) \right) \left( \psi''(s) - \tilde{p}(s)\tilde{h}''(s) \right) \\
\times \left( 1 - H_n(r) + \frac{1}{n} \right)^{-1} \left( 1 - H_n(s) + \frac{1}{n} \right)^{-1} \left( 1 - H(r) \right)^{-1} \left( 1 - H(s) \right)^{-1} \\
\times w(t) dr ds dt,
\]

which may be alternatively written as

\[
\hat{A}_2 = \frac{1}{4} \int_{c'}^t \left( \int_{c'}^t \left( \psi''(r) - \tilde{p}(r)\tilde{h}''(r) \right) \left( 1 - H_n(r) + \frac{1}{n} \right)^{-1} dr \right)^2 \\
- \left( \int_{c'}^t \left( \psi''(r) - \tilde{p}(r)\tilde{h}''(r) \right) \left( 1 - H(r) \right)^{-1} dr \right)^2 w(t) dt \\
= \frac{1}{4} \int_{c'}^t \int_{c'}^t \left( \psi''(r) - \tilde{p}(r)\tilde{h}''(r) \right) \left( 1 - H(r) \right)^{-1} \\
\times \left( 1 - H_n(r) + \frac{1}{n} \right)^{-1} \left( H_n(r) - H(r) - \frac{1}{n} \right) dr \\
\times \int_{c'}^t \left( \psi''(s) - \tilde{p}(s)\tilde{h}''(s) \right) \\
\times \left( 1 - H(s) \right)^{-1} + \left( 1 - H_n(s) + \frac{1}{n} \right)^{-1} ds w(t) dt.
\]
Using Cauchy-Schwarz inequality,

\[
|\tilde{A}_2| \leq \frac{1}{4} \left( \int_{\varepsilon'}^{t} \left( \int_{\varepsilon}^{t} \left( \tilde{\psi}''(r) - \tilde{\rho}(r)\tilde{h}''(r) \right) \left( 1 - H(r) \right)^{-1} \times \left( 1 - H_n(r) + \frac{1}{n} \right)^{-1} \left( H_n(r) - H(r) - \frac{1}{n} \right) dr \right)^2 w(t) dt \right)^{\frac{1}{2}} \times \left( \int_{\varepsilon'}^{t} \left( \int_{\varepsilon}^{t} \left( \tilde{\psi}''(s) - \tilde{\rho}(s)\tilde{h}''(s) \right) \left( 1 - H(s) \right)^{-1} + \left( 1 - H_n(s) + \frac{1}{n} \right)^{-1} \right) ds \right)^2 w(u) du
\]

Clearly,

\[
\int_{\varepsilon'}^{t} \left( \int_{\varepsilon}^{t} \left( \tilde{\psi}''(s) - \tilde{\rho}(s)\tilde{h}''(s) \right) \left( 1 - H(s) \right)^{-1} + \left( 1 - H_n(s) + \frac{1}{n} \right)^{-1} \right) ds \right)^2 w(u) du
\]

\[
= O_p \left( \tilde{A}_1 + \tilde{A}_{\varepsilon/2} \right).
\]

On the other hand a further application of Cauchy-Schwarz leads to

\[
\int_{\varepsilon'}^{t} \left( \int_{\varepsilon}^{t} \left( \tilde{\psi}''(r) - \tilde{\rho}(r)\tilde{h}''(r) \right) \left( 1 - H(r) \right)^{-1} \times \left( 1 - H_n(r) + \frac{1}{n} \right)^{-1} \left( H_n(r) - H(r) - \frac{1}{n} \right) dr \right)^2 w(t) dt \leq \int_{\varepsilon'}^{t} \int_{\varepsilon}^{t} \left( \left( \tilde{\psi}''(r_1) - \tilde{\rho}(r_1)\tilde{h}''(r_1) \right) \left( 1 - H(r_1) \right)^{-1} \right)^2 dr_1
\]

\[
\times \left( 1 - H_n(r_2) + \frac{1}{n} \right)^{-2} \left( H_n(r_2) - H(r_2) - \frac{1}{n} \right)^2 dr_2 w(t) dt.
\]

Now since, for every \( r_2 \),

\[
1 - H_n(r_2) + \frac{1}{n} = 1 - H(r_2) - \left( H_n(r_2) - H(r_2) \right) + \frac{1}{n}
\]

\[
\geq \frac{1}{n} - H(r_2)
\]

and, using (H.1), there exist \( n_0 \in \mathbb{N} \) and \( \delta > 0 \) such that for any \( n \geq n_0 \) and \( r_2 > \varepsilon' \)

\[
\left| \frac{1}{n} - H(r_2) \right| > \delta.
\]

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is straightforward to check, using (W.1), that
\[
\int_{c'}^{\infty} \int_{c'}^{\infty} \left( \left( \psi''(r_1) - \tilde{\psi}(r_1) \tilde{h}''(r_1) \right) (1 - H(r_1))^{-1} \right)^2 dr_1
\times \int_{c'}^{t} \left( 1 - H_n(r_2) + \frac{1}{n} \right)^{-2} \left( H_n(r_2) - H(r_2) - \frac{1}{n} \right)^2 dr_2 w(t) dt
\leq \frac{2t_0}{\delta^2} \left( \sup_{c' < r_2 < t_0} (H_n(r_2) - H(r_2))^2 + \frac{1}{n^2} \right)
\times \int_{c'}^{\infty} \int_{c'}^{t} \left( \left( \psi''(r_1) - \tilde{\psi}(r_1) \tilde{h}''(r_1) \right) (1 - H(r_1))^{-1} \right)^2 dr_1 w(t) dt.
\]

Taking into account that
\[
\sup_{c' < r_2 < t_0} |H_n(r_2) - H(r_2)| = O_P \left( n^{-\frac{1}{2}} \right)
\]
and condition (V.1), we conclude
\[
\int_{c'}^{\infty} \int_{c'}^{t} \left( \left( \psi''(r_1) - \tilde{\psi}(r_1) \tilde{h}''(r_1) \right) (1 - H(r_1))^{-1} \right)^2 dr_1
\times \int_{c'}^{t} \left( 1 - H_n(r_2) + \frac{1}{n} \right)^{-2} \left( H_n(r_2) - H(r_2) - \frac{1}{n} \right)^2 dr_2 w(t) dt
= O_P \left( n^{-1} \right)
\]
which, going back to (69), leads to
\[
\hat{A}_2 = O_P \left( \left( \left( \hat{A}_1 + \hat{A}_{e/2} \right) n^{-1} \right)^{\frac{1}{2}} \right). \tag{70}
\]

Using \( \frac{1}{nA_1} = o_P(1) \) (which will be proved later) it is straightforward to check
\[
\hat{A}_2 = o_P \left( \hat{A}_1 \right).
\]
Note that, using (70), it is straightforward to get
\[
\hat{A}_2 = O_P \left( n^{-\frac{1}{2}} \right),
\]
which will be used in the proof of Theorem 16. \( \blacksquare \)

We proceed by studying the terms in the right handsie of expression (44). It is worth mentioning that the terms \( \hat{A}_{11}, \hat{A}_{12} \) and \( \hat{A}_{13} \) differ in the number of factors of the form \( (\psi - \psi), (\tilde{\psi} - \psi), (\tilde{\psi}'' - \psi'') \) and \( (\tilde{h}'' - h'') \).

Thus, \( \hat{A}_{11} \) is the only term in which these factors appear only once. They appear twice in \( \hat{A}_{12} \) and in \( \hat{A}_{13} \) as well. However, while in \( \hat{A}_{13} \) the two factors depend on the same variable, in \( \hat{A}_{12} \) each factor depend on a different variable.
To simplify the expressions that will appear below the following definitions are introduced:

\[ c_K = \int_{-L}^{L} K(x)^2 \, dx, \quad \text{and, in general, } c_{K^{(r)}} = \int_{-L}^{L} K^{(r)}(x)^2 \, dx, \text{ for } r = 1, 2, ..., \]

\[ \mu_K = \int_{-L}^{L} K(x)x^2 \, dx, \quad \alpha_w = \alpha w, \quad A_w(x) = \int_{-L}^{\infty} \alpha_w(t) \, dt, \]

\[ z_1 = (1 - H)^{-1}, \quad z_2 = -(1 - H)^{-1} h^{\prime \prime} h^{-1}, \]
\[ z_3 = -(1 - H)^{-1} h^{-2}, \quad z_4 = -(1 - H)^{-1} h^{-1}, \]
\[ z_5 = (1 - H)^{-2} w, \quad z_6 = (1 - H)^{-2} (1 - 2p) \, w, \]

\[ z_{11} = z_1 (1 - p), \quad z_{12} = z_1 p, \]
\[ z_{21} = z_2 (1 - p), \quad z_{22} = z_2 p, \]
\[ z_{31} = z_3 (1 - p), \quad z_{32} = z_3 p, \]
\[ z_{41} = z_4 (1 - p), \quad z_{42} = z_4 p, \]
\[ z_{51} = z_5 h^{-2}, \quad z_{52} = z_5 h^{-1}, \]
\[ z_{61} = z_6 h^{-1}, \quad z_{62} = z_6 h^{-2}, \]
\[ z_{11} = z_{11} \alpha_w, \quad z_{12} = z_{12} \alpha_w, \]
\[ z_{12} = z_{12} \alpha_w, \]
\[ \phi = (1 - p) h, \quad \zeta = \psi^\prime - ph^\prime. \]

**Lemma 17** Assume conditions of Lemma 12. Then, it holds

\[
E \left( \hat{A}_{11} \right) = g_1^{1/2} \frac{1}{\pi^2} \int_{\pi}^{\infty} \int_{\pi}^{t} (1 - H(r))^{-1} \left( p^{(4)}(r) h(r) \right. \\
+ 4p^{(3)}(r) h^\prime(r) + 5p^\prime(r) h^\prime(r) + 4p(r) h^{(3)}(r) - 2p^\prime(r) h(r)^{-1} h^\prime(r)^2 \left. \right) \alpha_w(t) \, dt \, dr + o \left( g_1^2 \right) .
\]

and

\[
Var \left( \hat{A}_{11} \right) = n^{-1} \left( \int_{\infty}^{\infty} (2z_{11}(x) \alpha_w(x) + z_{11}(x) \alpha'_w(x) - A_w(x) z_{11}^\prime(x))^2 \psi(x) \, dx \\
+ \int_{\infty}^{\infty} (2z_{12}(x) \alpha_w(x) + z_{12}(x) \alpha'_w(x) - A_w(x) z_{12}^\prime(x))^2 \phi(x) \, dx \\
- 4 \left( \int_{\infty}^{\infty} \alpha(t)^2 w(t) \, dt \right)^2 \\
+ \int_{\infty}^{\infty} z_2(x)^2 A_w(x)^2 (1 - p(x)) p(x) h(x) \, dx \\
\right) + o \left( n^{-1} \right). 
\]

**Proof.** Let us consider the decomposition

\[
\hat{A}_{11} = \hat{A}_{111} + \hat{A}_{112}
\]
where
\[
\tilde{A}_{111} = \int_{t'}^\infty \int_{t'}^t z_1(r) \left( \dot{\psi}'(r) - p(r) \dot{h}'(r) - \zeta(r) \right) \alpha(t) w(t) \, dr \, dt
\]
and
\[
\tilde{A}_{112} = \int_{t'}^\infty \int_{t'}^t z_2(r) \left( \ddot{\psi}(r) - p(r) \dot{h}(r) \right) \alpha(t) w(t) \, dr \, dt.
\]

The thesis in the lemma is a straightforward consequence of Lemmas ??-22, that will be stated and proved below. \( \blacksquare \)

**Lemma 18** Under the assumptions of Lemma 17,
\[
E \left( \tilde{A}_{111} \right) = \frac{1}{2} g^2 \mu K \int_{t'}^\infty \int_{t'}^t z_1(r) \left( p^{(4)}(r) h(r) + 4 p^{(3)}(r) h'(r) + 6 p''(r) h''(r) + 4 p'(r) h^{(3)}(r) \right) \alpha_w(t) \, dr \, dt + o \left( g_1^2 \right). \tag{72}
\]

**Proof.** First of all observe that
\[
E \left( \tilde{A}_{111} \right) = E \left( \int_{t'}^\infty \int_{t'}^t z_1(r) \frac{1}{n} \sum_{i=1}^n K''_{g_1}(r - T_i) (\delta_i - p(r)) \alpha_w(t) \, dr \, dt \right)
- \int_{t'}^\infty \int_{t'}^t z_1(r) \zeta(r) \alpha_w(t) \, dr \, dt
= E \left( \int_{t'}^\infty \int_{t'}^t z_1(r) K''_{g_1}(r - T_1) (\delta_1 - p(r)) \alpha_w(t) \, dr \, dt \right)
- 2 \int_{t'}^\infty \alpha(t)^2 w(t) \, dt. \tag{73}
\]
Now defining
\[
I_{111}(g_1) = E \left( \int_{t'}^\infty \int_{t'}^t z_1(r) K''_{g_1}(r - T_1) (\delta_1 - p(r)) \alpha_w(t) \, dr \, dt \right),
\]
we have

\[
I_{111}(g_1) = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty z_1(r) K''_g(r - x) (p(x) - p(r)) \alpha_w(t) h(x) dr dt\ dx
\]

\[
= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty z_1(r) K''_g(r - x) (p(x) - p(r)) \alpha_w(t) h(x) dx dr dt
\]

\[
= g_1^{-2} \int_0^\infty \int_{-\infty}^\infty \int_{-L}^{L} z_1(r) K''(x) (p(r - g_1 x) - p(r)) h(r - g_1 x)\ dx dr dt
\]

\[
\times \alpha_w(t) dx_1 dr dt,
\]

where we have applied Fubini’s Theorem and the change of variable \( \frac{r-x}{g_1} = x_1 \) and used that \( K'' \) vanishes outside the interval \([-L, L]\).

It is straightforward that if \( g_1 < \frac{L}{\epsilon_0} \) then \( L \wedge L = L \), which implies

\[
I_{111}(g_1) = g_1^{-2} \int_0^\infty \int_{-\infty}^\infty \int_{-L}^{L} z_1(r) K''(x) (p(r - g_1 x) - p(r)) h(r - g_1 x)\ dx dr dt
\]

\[
\times \alpha_w(t) dx_1 dr dt.
\]

Taylor expansions of \( p(r - g_1 x) - p(r) \) and \( h(r - g_1 x) \) around \( r \) give

\[
p(r - g_1 x) - p(r) = \sum_{i=1}^4 \frac{(-1)^i}{i!} g_1^i x_1^i p^{(i)}(r) - \frac{1}{5!} g_1^5 x_1^5 p^{(5)}(\theta_1)
\]

and

\[
h(r - g_1 x) = \sum_{i=0}^3 \frac{(-1)^i}{i!} g_1^i x_1^i h^{(i)}(r) + \frac{1}{4!} g_1^4 x_1^4 h^{(4)}(\theta_2),
\]

39
where $\theta_1$ and $\theta_2$ are some intermediate points between $r - g_1 x_1$ and $r$. Therefore

$$I_{111}(g_1) = g_1^{-2} \int_{c'}^{\infty} \int_{c'}^{L} z_1(r) K''(x_1) \sum_{i=1}^{4} \frac{(-1)^i}{i!} g_1^i x_1^i p^{(i)}(r)$$

$$\times \sum_{i=0}^{3} \frac{(-1)^i}{i!} g_1^i x_1^i h^{(i)}(r) \alpha_w(t) dx_1 drdt + o\left(g_1^2\right)$$

$$= -g_1^{-1} \int_{-L}^{L} K''(x_1) x_1 dx_1 \int_{c'}^{\infty} \int_{c'}^{L} p'(r) z_1(r) h(r) \alpha_w(t) drdt$$

$$+ \int_{-L}^{L} K''(x_1) x_1^2 dx_1 \int_{c'}^{\infty} \int_{c'}^{L} z_1(r) \left(\frac{1}{2} p''(r) h(r) + h'(r) p'(r)\right) \alpha_w(t) drdt$$

$$- g_1 \int_{-L}^{L} K''(x_1) x_1^3 dx_1$$

$$\times \int_{c'}^{\infty} \int_{c'}^{L} z_1(r) \left(\frac{1}{2} p'(r) h''(r) + \frac{1}{2} p''(r) h'(r) + \frac{1}{3} p^{(3)}(r) h(r)\right)$$

$$\times \alpha_w(t) drdt$$

$$+ g_1^2 \int_{c'}^{\infty} \int_{c'}^{L} \int_{-L}^{L} z_1(r) K''(x_1) x_1^4 dx_1$$

$$\times \left(\frac{1}{3} p''(r) h^{(3)}(r) + \frac{1}{4} p''(r) h''(r) + \frac{1}{3} p^{(3)}(r) h'(r) + \frac{1}{4} p^{(4)}(r) h(r)\right)$$

$$\times \alpha_w(t) dx_1 drdt + o\left(g_1^2\right).$$

The conditions on the kernel function and partial integration can be used to prove

$$\int_{-L}^{L} K''(x_1) x_1 dx_1 = 0, \quad \int_{-L}^{L} K''(x_1) x_1^2 dx_1 = 2, \quad \int_{-L}^{L} K''(x_1) x_1^3 dx_1 = 0, \quad \int_{-L}^{L} K''(x_1) x_1^4 dx_1 = 12 \mu_K,$$

(74) and, consequently

$$I_{111}(g_1) = 2 \int_{c'}^{\infty} \alpha(t)^2 w(t) dt$$

$$+ \frac{1}{2} g_1^2 \mu_K \int_{c'}^{\infty} \int_{c'}^{L} z_1(r) \left(p^{(4)}(r) h(r) + 4 p^{(3)}(r) h'(r) + 6 p''(r) h''(r) + 4 p'(r) h^{(3)}(r)\right) \alpha_w(t) drdt + o\left(g_1^2\right).$$

(75)

Therefore,

$$E \left(\hat{A}_{111}\right) = \frac{1}{2} g_1^2 \mu_K \int_{c'}^{\infty} \int_{c'}^{L} z_1(r) \left(p^{(4)}(r) h(r) + 4 p^{(3)}(r) h'(r) + 6 p''(r) h''(r) + 4 p'(r) h^{(3)}(r)\right) \alpha_w(t) drdt + o\left(g_1^2\right).$$

(75)

Finally (72) can be obtained as a consequence of (P.2).
Lemma 19 Under the conditions of Lemma 17,

\[ E\left(\hat{A}_{112}\right) = g_2^2 \mu_K \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\varepsilon} z_2(r) \left(\frac{1}{2}p''(r)h(r) + p'(r)h'(r)\right) \alpha_w(t) dr dt + o\left(g_2^2\right) \]  

Proof. The proof is similar to that of the previous lemma,

\[ E\left(\hat{A}_{112}\right) = E\left(\int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\varepsilon} z_2(r) \sum_{i=1}^{n} K_{g_1}(r - T_i) (\delta_1 - p(r)) \alpha_w(t) dr dt\right) \]

\[ = E\left(\int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\varepsilon} z_2(r) K_{g_1}(r - x) (p(x) - p(r)) h(x) \alpha_w(t) dr dt dx\right) \]

\[ = \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\varepsilon} \int_{-L}^{L} z_2(r) K(x) (p(r - g_1 x) - p(r)) h(r - g_1 x) \alpha_w(t) dr dt dx \]

where we have applied Fubini’s Theorem and the change of variable \( \frac{r-x}{g_1} = x_1 \).

Arguing as done for \( \hat{A}_{111} \), with \( g_1 < \frac{1}{\varepsilon} \) we have \( \frac{r-x}{g_1} \wedge L = L \), and hence, under these conditions,

\[ E\left(\hat{A}_{112}\right) = \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\varepsilon} \int_{-L}^{L} z_2(r) K(x) (p(r - g_1 x) - p(r)) h(r - g_1 x) \alpha_w(t) dr dt dx \]

\[ = \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\varepsilon} \int_{-L}^{L} z_2(r) K(x_1) x_1 dx_1 \alpha_w(t) dr dt dx_1 \]

\[ + g_2^2 \int_{-L}^{L} K(x) x_1^2 dx_1 \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\varepsilon} z_2(r) \left(\frac{1}{2}p''(r)h(r) + p'(r)h'(r)\right) \alpha_w(t) dr dt dx_1 \]

\[ = g_2^2 \mu_K \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\varepsilon} z_2(r) \left(\frac{1}{2}p''(r)h(r) + p'(r)h'(r)\right) \alpha_w(t) dr dt + o\left(g_2^2\right) \]

where, as in the study of \( \hat{A}_{111} \), some Taylor expansions of \( p(r - g_1 x_1) - p(r) \) and \( h(r - g_1 x_1) \) around \( r \) have been used.

To prove (76) it suffices to use condition (P.2).  

\[ \square \]
Lemma 20 Under the conditions of Lemma 17 it holds

\[
\text{Var} \left( \tilde{A}_{111} \right) = n^{-1} \int_{\varepsilon}^{\infty} \left( 2z_{11}(x)\alpha_w(x) + z_{11}(x)\alpha'_w(x) - A_w(x)z'_{11}(x) \right)^2 \psi(x)dx
\]

\[
+ n^{-1} \int_{\varepsilon}^{\infty} \left( 2z_{12}(x)\alpha_w(x) + z_{12}(x)\alpha'_w(x) - A_w(x)z'_{12}(x) \right)^2 \phi(x)dx
\]

\[-4n^{-1} \left( \int_{\varepsilon}^{\infty} \alpha(t)^2 w(t)dt \right)^2 + o(n^{-1}). \quad (77)
\]

Proof.

\[
\text{Var} \left( \tilde{A}_{111} \right) = \text{Var} \left( \int_{\varepsilon}^{t} \int_{\varepsilon}^{t} z_1(r) \left( \tilde{\psi}'(r) - p(r)\tilde{h}'(r) - \zeta(r) \right) \alpha(t)w(t)drdt \right)
\]

\[
= \text{Var} \left( \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_1(r) \left( \tilde{\psi}'(r) - p(r)\tilde{h}'(r) \right) \alpha_w(t)drdt \right)
\]

\[
= \text{Var} \left( \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_1(r) \frac{1}{n} \sum_{i=1}^{n} K''_{g_1}(r-T_i) (\delta_i - p(r)) \alpha_w(t)drdt \right)
\]

\[
= \frac{1}{n} \text{Var} \left( \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_1(r)K''_{g_1}(r-T_1) (\delta_1 - p(r)) \alpha_w(t)drdt \right)
\]

\[
= \frac{1}{n} E \left( \left( \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_1(r)K''_{g_1}(r-T_1) (\delta_1 - p(r)) \alpha_w(t)drdt \right)^2 \right) - \frac{1}{n} I_{111}(g_1)^2
\]

where \( I_{111}(g_1) \) has been defined in the proof of Lemma ??.

Define

\[
J^{11}(g_1) = E \left( \left( \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_1(r)K''_{g_1}(r-T_1) (\delta_1 - p(r)) \alpha_w(t)drdt \right)^2 \right),
\]

then

\[
J^{11}(g_1) = J^{11}_1(g_1) + J^{11}_2(g_1), \quad (78)
\]

where,

\[
J^{11}_1(g_1) = \int_{0}^{\infty} \left( \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_{11}(r)K''_{g_1}(r-x)\alpha_w(t)drdt \right)^2 \psi(x)dx
\]

\[
J^{11}_2(g_1) = \int_{\varepsilon}^{\infty} \left( \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_{12}(r)K''_{g_1}(r-x)\alpha_w(t)drdt \right)^2 \phi(x)dx.
\]

Note that the integration limits determined in the previous expressions have been computed taking into account that the functions \( z_{11} \) and \( \phi \) vanish in the interval \([0, \varepsilon]\).
We now study both terms in the decomposition (78). Using in $J_{1}^{1,1}(g_1)$ the change of variable $\frac{r-x}{g_1} = r_1$,

$$J_{1}^{1,1}(g_1) = g_1^{-4} \int_0^\infty \left( \int_0^\infty \int_{\frac{r-x}{g_1}}^{r_1} z_{11}(x + g_1 r_1) K''(r_1) \alpha_w(t) dr_1 dt \right)^2 \psi(x) dx$$

and, partial integration and $z_{11}(\varepsilon) = z_{11}'(\varepsilon) = 0$ gives

$$\int_{\frac{r-x}{g_1}}^{r_1} z_{11}(x + g_1 r_1) K''(r_1) dr_1$$

$$= K' \left( \frac{t-x}{g_1} \right) z_{11}(t) - K' \left( \frac{\varepsilon-x}{g_1} \right) z_{11}(\varepsilon)$$

$$- g_1 \left( K \left( \frac{t-x}{g_1} \right) z_{11}'(t) - K \left( \frac{\varepsilon-x}{g_1} \right) z_{11}'(\varepsilon) \right)$$

$$+ g_1^2 \int_{\frac{r-x}{g_1}}^{r_1} K(r_1) z_{11}''(x + g_1 r_1) dr_1$$

$$= K' \left( \frac{t-x}{g_1} \right) z_{11}(t) - g_1 K \left( \frac{t-x}{g_1} \right) z_{11}'(t)$$

$$+ g_1^2 \int_{\frac{r-x}{g_1}}^{r_1} K(r_1) z_{11}''(x + g_1 r_1) dr_1$$

which leads to
\[ J^{11}_1(g_1) = g_1^{-4} \int_0^\infty \left( \int_0^\infty K'(t-x) \int_0^\infty K(t-x) \int_0^\infty z_{11}(t) \alpha \omega(t) dt \right)^2 \psi(x) dx \\
-2g_1^{-3} \int_0^\infty K'(t-x) z_{11}(t) \alpha \omega(t) dt \\
\times \int_0^\infty K\left(\frac{m-x}{g_1}\right) z_{11}(m) \alpha \omega(m) dm \psi(x) dx \\
+g_1^{-2} \int_0^\infty \left( \int_0^\infty K'(t-x) \int_0^\infty K(t-x) \int_0^\infty z_{11}(t) \alpha \omega(t) dt \right)^2 \psi(x) dx \\
+2g_1^{-2} \int_0^\infty \left( \int_0^\infty K'(t-x) \int_0^\infty K(t-x) \int_0^\infty z_{11}(t) \alpha \omega(t) dt \right)^2 \psi(x) dx \\
\times \int_0^\infty K(0) \int_0^\infty K(0) \int_0^\infty \left( \int_0^\infty z_{11}(x) \alpha \omega(x) dx \right) \psi(x) dx \\
-2g_1^{-1} \int_0^\infty \left( \int_0^\infty K'(t-x) \int_0^\infty K(t-x) \int_0^\infty z_{11}(t) \alpha \omega(t) dt \right)^2 \psi(x) dx \\
\times \int_0^\infty K(0) \int_0^\infty K(0) \int_0^\infty \left( \int_0^\infty z_{11}(x) \alpha \omega(x) dx \right) \psi(x) dx \\
+ \int_0^\infty \left( \int_0^\infty \int_0^\infty K'(t) \int_0^\infty z_{11}(x) \alpha \omega(x) dx \right) \psi(x) dx. \] (80)

Using, whenever required, the changes of variable \( \frac{t-x}{g_1} = t_1 \) or \( \frac{m-x}{g_1} = t_1 \), \( \frac{m-x}{g_1} = m_1 \) in the previous terms, we have

\[ J^{11}_1(g_1) = g_1^{-2} J^{11}_{11}(g_1) - 2g_1^{-1} J^{11}_{12}(g_1) + J^{11}_{13}(g_1) \\
+2g_1^{-1} J^{11}_{14}(g_1) - 2J^{11}_{15}(g_1) + J^{11}_{16}(g_1) \] (81)

where

\[ J^{11}_{11}(g_1) = \int_0^\infty \left( \int_0^L K'(t_1) z_{111}(x+g_1 t_1) dt_1 \right)^2 \psi(x) dx, \]

\[ J^{11}_{12}(g_1) = \int_0^\infty \int_0^L K'(t_1) z_{111}(x+g_1 t_1) dt_1 \]
\[ \times \int_0^L K(m_1) z_{112}(x+g_1 m_1) dm_1 \psi(x) dx, \]
\[ J_{13}^{11}(g_1) = \int_0^\infty \left( \int_{-\frac{L}{g_1}}^{L} K(t_1) z_{112}(x + g_1 t_1) dt_1 \right)^2 \psi(x) dx, \]

\[ J_{14}^{11}(g_1) = \int_0^\infty \int_{-\frac{L}{g_1}}^{L} K(t_1) z_{111}(x + g_1 t_1) dt_1 \]
\[ \times \int_\varepsilon^\infty \int_{-\frac{m-x}{g_1}}^{m-x} K(u_1) z_{11}(x + g_1 u_1) \alpha_w(m) du_1 dm \psi(x) dx, \]

\[ J_{15}^{11}(g_1) = \int_0^\infty \int_{-\frac{L}{g_1}}^{L} K(t_1) z_{112}(x + g_1 t_1) dt_1 \]
\[ \times \int_\varepsilon^\infty \int_{-\frac{m-x}{g_1}}^{m-x} K(u_1) z_{11}(x + g_1 u_1) \alpha_w(m) du_1 dm \psi(x) dx, \]

and

\[ J_{16}^{11}(g_1) = \int_0^\infty \left( \int_\varepsilon^\infty \int_{-\frac{m-x}{g_1}}^{m-x} K(r_1) z_{11}(x + g_1 r_1) \alpha_w(t) dr_1 dt \right)^2 \psi(x) dx. \]

For the term \( J_{11}^{11}(g_1) \) it is straightforward to prove

\[ \lim_{g_1 \to 0} J_{11}^{11}(g_1) = \left( \int_{-L}^{L} K'(t_1) dt_1 \right)^2 \int_\varepsilon^\infty z_{111}(x)^2 \psi(x) dx = 0. \]

The first two derivatives of this term are

\[ \frac{d}{dg_1} J_{11}^{11}(g_1) = 2 \int_0^\infty \int_{-\frac{L}{g_1}}^{L} K'(t_1) z_{111}(x + g_1 t_1) dt_1 \]
\[ \times \left( K' \left( \frac{\varepsilon - x}{g_1} \right) \frac{\varepsilon - x}{g_1} z_{111} (\varepsilon) \right) \]
\[ + \int_{-\frac{L}{g_1}}^{L} K'(m_1) m_1 z_{111}(x + g_1 m_1) dm_1 \right) \psi(x) dx \]
\[ = 2 \int_0^\infty \int_{-\frac{L}{g_1}}^{L} K'(t_1) z_{111}(x + g_1 t_1) dt_1 \]
\[ \times \int_{-\frac{L}{g_1}}^{L} K'(m_1) m_1 z_{111}(x + g_1 m_1) dm_1 \psi(x) dx \]
and

\[
\frac{d^2}{dg_1^2} J_{111}^{11}(g_1) = 2 \int_0^\infty \left( K' \left( \frac{\varepsilon - x}{g_1} \right) \frac{\varepsilon - x}{g_1^2} z_{111}(\varepsilon) + \int_{-\varepsilon}^L K'(t_1) t_1 z'_{111}(x + g_1 t_1) dt_1 \right) \\
\times \int_{-\varepsilon}^L K'(m_1) m_1 z''_{111}(x + g_1 m_1) dm_1 \psi(x) dx \\
+ 2 \int_0^\infty \int_{-\varepsilon}^L K'(t_1) z_{111}(x + g_1 t_1) dt_1 \\
\times \left( K' \left( \frac{\varepsilon - x}{g_1} \right) \frac{(\varepsilon - x)^2}{g_1^2} z_{111}'(\varepsilon) \\
+ \int_{-\varepsilon}^L K'(m_1) m_1^2 z''_{111}(x + g_1 m_1) dm_1 \psi(x) dx \right) \\
\times \int_{-\varepsilon}^L K'(m_1) m_1^2 z''_{111}(x + g_1 m_1) dm_1 \psi(x) dx
\]

where the equation \( z_{111}(\varepsilon) = z'_{111}(\varepsilon) = 0 \) have been used.

Therefore, letting \( g_1 \) tending to zero,

\[
\lim_{g_1 \to 0} \frac{d}{dg_1} J_{111}^{11}(g_1) = 2 \int_{-\infty}^L K'(t_1) dt_1 \int_{-\infty}^L K'(m_1) m_1 dm_1 \\
\times \int_{-\varepsilon}^\infty z_{111}(x) z'_{111}(x) \psi(x) dx
\]

and

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\[
\lim_{g_1 \to 0} \frac{d^2}{dg_1^2} J_{11}^{11}(g_1) = 2 \left( \int_{-L}^L K'(t_1) t_1 dt_1 \right)^2 \int_\varepsilon^\infty z_{1111}(x)^2 \psi(x) dx + 2 \int_{-L}^L K'(t_1) dt_1 \int_{-L}^L K'(m_1) m_1^2 dm_1 \\
\times \int_\varepsilon^\infty z_{111}(x) z_{111}(x) \psi(x) dx = 2 \int_\varepsilon^\infty z_{1111}(x)^2 \psi(x) dx,
\]

since

\[
\int_{-L}^L K'(t_1) dt_1 = 0 \quad \text{and} \quad \int_{-L}^L K'(t_1) t_2 dt_1 = -1.
\]

As a consequence, the following Taylor expansion of \( J_{11}^{11}(g_1) \) around 0 can be obtained

\[
J_{11}^{11}(g_1) = g_1^2 \int_\varepsilon^\infty z_{1111}(x)^2 \psi(x) dx + o(g_1^2). \tag{82}
\]

The derivative of \( J_{12}^{11}(g_1) \) is
\[
\begin{align*}
\frac{d}{dg_1} f^{11}_{12}(g_1) & = \int_0^\infty \left( K' \left( \frac{\varepsilon - x}{g_1} \right) \frac{\varepsilon - x}{g_1^2} z_{111}(\varepsilon) + \int_{\frac{\varepsilon}{g_1}}^L K'(t_1) t_1 z_{111}(x + g_1 t_1) dt_1 \right) \\
& \times \int_{-\frac{\varepsilon}{g_1}}^L K(m_1) z_{112}(x + g_1 m_1) dm_1 \psi(x) dx \\
& + \int_0^\infty \int_{\frac{\varepsilon}{g_1}}^L K'(t_1) z_{111}(x + g_1 t_1) dt_1 \\
& \times \left( K \left( \frac{\varepsilon - x}{g_1} \right) \frac{\varepsilon - x}{g_1^2} z_{112}(\varepsilon) \right) \\
& + \int_{\frac{\varepsilon}{g_1}}^L K(m_1) m_1 z'_{112}(x + g_1 m_1) dm_1 \psi(x) dx \\
& = \int_0^\infty \int_{-\frac{\varepsilon}{g_1}}^L K'(t_1) t_1 z_{111}(x + g_1 t_1) dt_1 \\
& \times \int_{-\frac{\varepsilon}{g_1}}^L K(m_1) z_{112}(x + g_1 m_1) dm_1 \psi(x) dx \\
& + \int_0^\infty \int_{\frac{\varepsilon}{g_1}}^L K'(t_1) z_{111}(x + g_1 t_1) dt_1 \\
& \times \int_{-\frac{\varepsilon}{g_1}}^L K(m_1) m_1 z'_{112}(x + g_1 m_1) dm_1 \psi(x) dx,
\end{align*}
\]

where the equation \( z_{111}(\varepsilon) = z_{112}(\varepsilon) = 0 \) has been used.

Using
\[
\lim_{g_1 \to 0} f^{11}_{12}(g_1) = \int_{-L}^L K'(t_1) dt_1 \int_{-L}^L K(m_1) dm_1 \int_0^\infty z_{111}(x) z_{112}(x) \psi(x) dx = 0
\]

and
\[
\lim_{g_1 \to 0} \frac{d}{dg_1} f^{11}_{12}(g_1) = \int_{-L}^L K'(t_1) dt_1 \int_{-L}^L K(m_1) dm_1 \\
\times \int_0^\infty z_{111}(x) z_{112}(x) \psi(x) dx \\
+ \int_{-L}^L K'(t_1) dt_1 \int_{-L}^L K(m_1) m_1 dm_1 \\
\times \int_0^\infty z_{111}(x) z'_{112}(x) \psi(x) dx \\
= \int_0^\infty z_{111}(x) z_{112}(x) \psi(x) dx,
\]

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we obtain the following Taylor expansion of $J_{12}^{11}(g_1)$ around 0

$$J_{12}^{11}(g_1) = -g_1 \int_{\varepsilon}^{\infty} z_{111}'(x)z_{112}(x)\psi(x)dx + o(g_1). \quad (83)$$

The term $J_{13}^{11}(g_1)$ satisfies

$$\lim_{g_1 \to 0} J_{13}^{11}(g_1) = \left( \int_{-L}^{L} K(t_1)dt_1 \right)^2 \int_{\varepsilon}^{\infty} z_{112}(x)^2\psi(x)dx$$

and hence

$$J_{13}^{11}(g_1) = \int_{\varepsilon}^{\infty} z_{112}(x)^2\psi(x)dx + o(1). \quad (84)$$

With respect to $J_{14}^{11}(g_1)$,

$$\lim_{g_1 \to 0} J_{14}^{11}(g_1) = \int_{-L}^{L} K'(t_1)dt_1 \int_{-L}^{L} K(u_1)du_1 \int_{\varepsilon}^{\infty} A_w(x)z_{111}(x)z_{11}'(x)\psi(x)dx = 0,$$

whose derivative is

$$\frac{d}{dg_1} J_{14}^{11}(g_1) = \int_{0}^{\infty} \left( \frac{e - x}{g_1} \right) \left( \frac{e - x}{g_1^2} \right) z_{111}(\varepsilon) + \int_{-L}^{L} K'(t_1)z_{11}'(x + g_1t_1)dt_1$$

$$\times \int_{\varepsilon}^{\infty} \int_{\frac{m-x}{g_1}}^{\frac{m-x}{g_1}} K(u_1)z_{11}'(x + g_1u_1)\alpha_w(m) du_1 dm \psi(x)dx$$

$$+ \int_{0}^{\infty} \int_{-L}^{L} K'(t_1)z_{111}(x + g_1t_1)dt_1$$

$$\times \int_{\varepsilon}^{\infty} \left( -K \left( \frac{m-x}{g_1} \right) \frac{m-x}{g_1^2} z_{11}'(m) + K \left( \frac{e-x}{g_1} \right) \frac{e-x}{g_1^2} z_{11}'(\varepsilon) \right)$$

$$+ \int_{\varepsilon}^{\infty} \int_{\frac{m-x}{g_1}}^{\frac{m-x}{g_1}} K(u_1)z_{11}'(x + g_1u_1)du_1 \alpha_w(m) dm \psi(x)dx$$
\[
\begin{align*}
&= \int_0^\infty \int_{z_{11}(x+g_1t_1)}^L K'(t_1) t_1 z_{111}(x+g_1t_1) dt_1 \\
&\quad \times \int_\varepsilon^\infty \int_{z_{11}(x+g_1u_1)}^{m-x} K(u_1) z_{11}''(x+g_1u_1) \alpha_w(m) du_1 dm \psi(x) dx \\
&\quad - \int_0^\infty \int_{z_{11}(x+g_1t_1)}^L K'(t_1) z_{111}(x+g_1t_1) dt_1 \\
&\quad \times \int_\varepsilon^\infty \left( K \left( \frac{m-x}{g_1} \right) \frac{m-x}{g_1} z_{11}''(m) \right) \\
&\quad \quad - \int_{z_{11}(x+g_1u_1)}^{\infty} K(u_1) u_1 z_{11}^{(3)}(x+g_1u_1) du_1 \right) \alpha_w(m) dm \psi(x) dx \\
&= \int_0^\infty \int_{z_{11}(x+g_1t_1)}^L K'(t_1) t_1 z_{111}(x+g_1t_1) dt_1 \\
&\quad \times \int_\varepsilon^\infty \int_{z_{11}(x+g_1u_1)}^{m-x} K(u_1) z_{11}''(x+g_1u_1) \alpha_w(m) du_1 dm \psi(x) dx \\
&\quad - \int_0^\infty \int_{z_{11}(x+g_1t_1)}^L K'(t_1) z_{111}(x+g_1t_1) dt_1 \\
&\quad \times \int_\varepsilon^\infty \frac{K(m_1) m_1 z_{11}''(x+g_1m_1) \alpha_w(x+g_1m_1) dm_1 \psi(x) dx}{m_1} \\
&\quad + \int_0^\infty \int_{z_{11}(x+g_1t_1)}^L K'(t_1) z_{111}(x+g_1t_1) dt_1 \\
&\quad \times \int_\varepsilon^\infty \int_{z_{11}(x+g_1u_1)}^{\infty} K(u_1) u_1 z_{11}^{(3)}(x+g_1u_1) du_1 \alpha_w(m) dm \psi(x) dx,
\end{align*}
\]

which required the change of variable \( \frac{m-x}{g_1} = m_1 \) and the equation \( z_{111}(\varepsilon) = z_{11}''(\varepsilon) = 0 \).
Therefore,

\[
\lim_{g_1 \to 0} \frac{d}{dg_1} J_{14}^{11}(g_1) = \int_{-L}^{L} K'(t_1) dt_1 \int_{-L}^{L} K(u_1) du_1 \\
\times \int_{\varepsilon}^{\infty} A_w(x) z_{111}(x) z_{11}''(x) \psi(x) dx \\
- \int_{-L}^{L} K'(t_1) dt_1 \int_{-L}^{L} K(m_1) m_1 dm_1 \\
\times \int_{\varepsilon}^{\infty} \alpha_w(x) z_{111}(x) z_{11}''(x) \psi(x) dx \\
+ \int_{-L}^{L} K'(t_1) dt_1 \int_{-L}^{L} K(u_1) u_1 du_1 \\
\times \int_{\varepsilon}^{\infty} A_w(x) z_{111}(x) z_{11}^{(3)}(x) \psi(x) dx \\
= - \int_{\varepsilon}^{\infty} A_w(x) z_{111}(x) z_{11}''(x) \psi(x) dx \\
\]

which leads to a Taylor expansion of \( J_{14}^{11}(g_1) \) around 0:

\[
J_{14}^{11}(g_1) = -g_1 \int_{\varepsilon}^{\infty} A_w(x) z_{111}(x) z_{11}''(x) \psi(x) dx + o(g_1). \tag{85}
\]

For the last two terms in (81), \( J_{15}^{11}(g_1) \) and \( J_{16}^{11}(g_1) \), it is easy to check that

\[
\lim_{g_1 \to 0} J_{15}^{11}(g_1) = \left( \int_{-L}^{L} K(t_1) dt_1 \right)^2 \int_{\varepsilon}^{\infty} A_w(x) z_{112}(x) z_{11}''(x) \psi(x) dx \\
\]

and

\[
\lim_{g_1 \to 0} J_{16}^{11}(g_1) = \left( \int_{-L}^{L} K(r_1) dr_1 \right)^2 \int_{\varepsilon}^{\infty} A_w(x)^2 z_{11}''(x)^2 \psi(x) dx,
\]

which implies

\[
J_{15}^{11}(g_1) = \int_{\varepsilon}^{\infty} A_w(x) z_{112}(x) z_{11}''(x) \psi(x) dx + o(1) \tag{86}
\]

and

\[
J_{16}^{11}(g_1) = \int_{\varepsilon}^{\infty} A_w(x)^2 z_{11}''(x)^2 \psi(x) dx + o(1). \tag{87}
\]
Collecting expressions (82)-(87) we reach to the following result

\[ J_{12}^{11}(g_1) = \int_0^\infty \left( z_{111}'(x)^2 + 2z_{111}'(x)z_{112}(x) + z_{112}(x)^2 - 2A_w(x)z_{111}'(x)z''_{111}(x) \right. \\
\left. - 2A_w(x)z_{112}(x)z''_{111}(x) + A_w(x)2z_{111}'(x)^2 \right) \psi(x)dx + o(1) \]

\[ = \int_0^\infty \left( 2z_{111}'(x)\alpha_w(x) + z_{112}(x)\alpha'_w(x) - A_w(x)z''_{111}(x) \right)^2 \psi(x)dx + o(1). \]

The second term in (78), \( J_{12}^{11}(g_1) \), can be studied in a parallel way to the previous one. With the change of variable \( \frac{t-x}{g_1} = r_1 \),

\[ J_{12}^{11}(g_1) = g_1^{-4} \int_0^\infty \left( \int_0^\infty \int_{r_1}^{t-x} z_{12}(x + g_1r_1)K''(r_1)\alpha_w(t)dr_1dt \right)^2 \phi(x)dx. \]

Now for \( g_1 \) small enough we have \( g_1 < \frac{t-x}{L} \) which implies \( \frac{t-x}{g_1} < -L \) and thus

\[ J_{12}^{11}(g_1) = g_1^{-4} \int_0^\infty \left( \int_{-L}^{t-x} z_{12}(x + g_1r_1)K''(r_1)\alpha_w(t)dr_1dt \right)^2 \phi(x)dx. \]

Integration by parts gives

\[ \int_{-L}^{t-x} z_{12}(x + g_1r_1)K''(r_1)dr_1 = K'(\frac{t-x}{g_1})z_{12}(t) - g_1K(\frac{t-x}{g_1})z_{12}'(t) \]

\[ + g_1^2 \int_{-L}^{t-x} K(r_1)z_{12}'(x + g_1r_1)dr_1, \]

\[(89)\]
and finally

\[ J^{11}_2(g_1) = g_1^{-4} \int_\varepsilon^\infty \left( \int_{-\varepsilon}^{\infty} K' \left( \frac{t-x}{g_1} \right) z_{12}(t) \alpha_w(t) dt \right)^2 \phi(x) dx - 2g_1^{-3} \int_\varepsilon^\infty \left( \int_{-\varepsilon}^{\infty} K' \left( \frac{t-x}{g_1} \right) z_{12}(t) \alpha_w(t) dt \right) \phi(x) dx \times \int_{-\varepsilon}^{\infty} \left( \int_{-\varepsilon}^{\infty} K \left( \frac{t-x}{g_1} \right) z_{12}(m) \alpha_w(m) dm \phi(x) dx \right) \]

\[ + g_1^{-2} \int_\varepsilon^\infty \left( \int_{-\varepsilon}^{\infty} K' \left( \frac{t-x}{g_1} \right) z_{12}(t) \alpha_w(t) dt \right)^2 \phi(x) dx \]

\[ + 2g_1^{-2} \int_\varepsilon^\infty \left( \int_{-\varepsilon}^{\infty} K' \left( \frac{t-x}{g_1} \right) z_{12}(t) \alpha_w(t) dt \right) \phi(x) dx \times \int_{-\varepsilon}^{\infty} \left( \int_{-\varepsilon}^{\infty} K(u_1) z_{12}(x + g_1 u_1) \alpha_w(m) du_1 dm \phi(x) dx \right) \]

\[ - 2g_1^{-1} \int_\varepsilon^\infty \left( \int_{-\varepsilon}^{\infty} K' \left( \frac{t-x}{g_1} \right) z_{12}(t) \alpha_w(t) dt \right) \phi(x) dx \times \int_{-\varepsilon}^{\infty} \left( \int_{-\varepsilon}^{\infty} K(u_1) z_{12}(x + g_1 u_1) \alpha_w(m) du_1 dm \phi(x) dx \right) \]

\[ + \int_\varepsilon^\infty \left( \int_{-\varepsilon}^{\infty} \int_{-\varepsilon}^{\infty} K(r_1) z_{12}(x + g_1 r_1) \alpha_w(t) dr_1 dt \right)^2 \phi(x) dx. \]

(90)

Using the changes of variable \( \frac{\varepsilon-x}{g_1} = t_1 \) or \( \frac{\varepsilon-x}{g_1} = t_1 \), \( \frac{m-x}{g_1} = m_1 \) in the previous expressions, we have

\[ J^{11}_2(g_1) = g_1^{-2} J^{11}_1(g_1) - 2g_1^{-1} J^{11}_2(g_1) + J^{11}_3(g_1) + 2g_1^{-1} J^{11}_4(g_1) - 2J^{11}_5(g_1) + J^{11}_6(g_1) \]

(91)

where the following notation has been introduced

\[ J^{11}_1(g_1) = \int_\varepsilon^\infty \left( \int_{-\varepsilon}^{L} K'(t_1) z_{121}(x + g_1 t_1) dt_1 \right)^2 \phi(x) dx, \]

\[ J^{11}_2(g_1) = \int_\varepsilon^\infty \int_{-\varepsilon}^{L} K'(t_1) z_{121}(x + g_1 t_1) dt_1 \]

\[ \times \int_{-\varepsilon}^{L} K(m_1) z_{122}(x + g_1 m_1) dm_1 \phi(x) dx, \]

\[ J^{11}_3(g_1) = \int_\varepsilon^\infty \left( \int_{-\varepsilon}^{L} K(t_1) z_{122}(x + g_1 t_1) dt_1 \right)^2 \phi(x) dx, \]
\[ J_{24}^{1,1}(g_1) = \int_{\varepsilon}^{\infty} \left( \int_{-L}^{L} K'(t_1) z_{121}(x + g_1 t_1) dt_1 \right. \]
\[ \times \int_{\varepsilon}^{\infty} \left( \int_{-L}^{L} K(t_1) z'_{122}(x + g_1 t_1) dt_1 \right) \]
\[ \left. \times \int_{-L}^{L} K(u_1) \frac{z''_{12}(x + g_1 u_1)}{u_1} \alpha_w(m) du_1 dm \phi(x) dx \right), \]

and

\[ J_{25}^{1,1}(g_1) = \int_{\varepsilon}^{\infty} \left( \int_{-L}^{L} K(t_1) z_{122}(x + g_1 t_1) dt_1 \right. \]
\[ \times \int_{\varepsilon}^{\infty} \left( \int_{-L}^{L} K(u_1) \frac{z''_{12}(x + g_1 u_1)}{u_1} \alpha_w(m) du_1 dm \phi(x) dx \right), \]

\[ J_{26}^{1,1}(g_1) = \int_{\varepsilon}^{\infty} \left( \int_{-L}^{L} K(t_1) z_{122}(x + g_1 t_1) dt_1 \right. \]
\[ \times \int_{\varepsilon}^{\infty} \left( \int_{-L}^{L} K(u_1) \frac{z''_{12}(x + g_1 u_1)}{u_1} \alpha_w(m) du_1 dm \phi(x) dx \right). \]

To handle \( J_{21}^{1,1}(g_1) \) a first order Taylor expansion of \( z_{121}(x + g_1 t_1) \) around \( x \)
gives

\[ J_{21}^{1,1}(g_1) = g_1^2 \int_{\varepsilon}^{\infty} \phi'(x) dx \]

\[ + o\left(g_1^2\right). \]  

The term \( J_{21}^{1,1}(g_1) \) can also be studied by Taylor expansions of \( z_{121}(x + g_1 t_1) \) and \( z_{122}(x + g_1 t_1) \) around \( x \). This leads to

\[ J_{22}^{1,1}(g_1) = -g_1 \int_{\varepsilon}^{\infty} z_{121}(x)z_{122}(x) \phi(x) dx + o(g_1). \]

It is straightforward to check that

\[ J_{23}^{1,1}(g_1) = \int_{\varepsilon}^{\infty} z_{122}(x)^2 \phi(x) dx + o(1). \]

For the term \( J_{24}^{1,1}(g_1) \) let’s first observe that its limit when \( g_1 \) tends to 0 is, obviously, 0. Its derivative is
\[ \frac{d}{dg_1} u_1^{(g_1)} = \int_{-\infty}^{\infty} \int_{-L}^{L} K'(t_1) t_1 z_{121}^H(x + g_1 t_1) dt_1 \]

\[ \times \int_{-\infty}^{\infty} \int_{-L}^{\frac{m-x}{g_1}} K(u_1) z_{12}^H(x + g_1 u_1) \alpha_w(m) du_1 dm \phi(x) dx \]

\[ - \int_{-\infty}^{\infty} \int_{-L}^{L} K'(t_1) z_{121}^H(x + g_1 t_1) dt_1 \]

\[ \times \int_{-\infty}^{\infty} K \left( \frac{m-x}{g_1} \right) \frac{m-x}{g_1^2} z_{12}^H(m) \alpha_w(m) dm \phi(x) dx \]

\[ + \int_{-\infty}^{\infty} \int_{-L}^{L} K'(t_1) z_{121}^H(x + g_1 t_1) dt_1 \]

\[ \times \int_{-\infty}^{\infty} \int_{-L}^{\frac{m-x}{m_1}} K(u_1) u_1 z_{12}^H(x + g_1 u_1) \alpha_w(m) du_1 dm \phi(x) dx \]

\[ = \int_{-\infty}^{\infty} \int_{-L}^{L} K'(t_1) t_1 z_{121}^H(x + g_1 t_1) dt_1 \]

\[ \times \int_{-\infty}^{\infty} \int_{-L}^{\frac{m-x}{m_1}} K(u_1) z_{12}^H(x + g_1 u_1) \alpha_w(m) du_1 dm \phi(x) dx \]

\[ - \int_{-\infty}^{\infty} \int_{-L}^{L} K'(t_1) z_{121}^H(x + g_1 t_1) dt_1 \]

\[ \times \int_{-\infty}^{\infty} K(m_1) m_1 z_{12}^H(x + g_1 m_1) \alpha_w(x + g_1 m_1) dm_1 \phi(x) dx \]

\[ + \int_{-\infty}^{\infty} \int_{-L}^{L} K'(t_1) z_{121}^H(x + g_1 t_1) dt_1 \]

\[ \times \int_{-\infty}^{\infty} \int_{-L}^{\frac{m-x}{m_1}} K(u_1) u_1 z_{12}^H(x + g_1 u_1) \alpha_w(m) du_1 dm \phi(x) dx \]

where the change of variable \( \frac{m-x}{m} = m_1 \) has been used and the assumption \( g_1 < \frac{\epsilon}{t} \) (i.e. \( g_1 \) is small enough) has been made. Since
the following Taylor expansion holds

$$J_{24}^{11}(g_1) = -g_1 \int_{x}^{\infty} z_{121}'(x) z_{12}'(x) A_w(x) \phi(x) \, dx + o(g_1). \quad (95)$$

It is straightforward to prove a similar expression for $J_{25}^{11}(g_1)$:

$$J_{25}^{11}(g_1) = \int_{x}^{\infty} A_w(x) z_{122}(x) z_{12}'(x) \phi(x) \, dx + o(1). \quad (96)$$

as well as for $J_{26}^{11}(g_1)$:

$$J_{26}^{11}(g_1) = \int_{x}^{\infty} A_w(x)^2 z_{12}'(x)^2 \phi(x) \, dx + o(1). \quad (97)$$

Collecting (92)-(97) gives

$$J_{2}^{11}(g_1) = \int_{x}^{\infty} z_{121}'(x)^2 \phi(x) \, dx + 2 \int_{x}^{\infty} z_{121}'(x) z_{122}(x) \phi(x) \, dx$$

$$+ \int_{x}^{\infty} z_{122}(x)^2 \phi(x) \, dx - \int_{x}^{\infty} 2 A_w(x) z_{121}'(x) z_{12}'(x) \phi(x) \, dx$$

$$- 2 \int_{x}^{\infty} A_w(x) z_{122}(x) z_{12}'(x) \phi(x) \, dx$$

$$+ \int_{x}^{\infty} A_w(x)^2 z_{12}'(x)^2 \phi(x) \, dx + o(1)$$

$$= \int_{x}^{\infty} (2 z_{12}'(x) \alpha_w(x) + z_{12}(x) \alpha_w'(x) - A_w(x) z_{12}'(x))^2 \phi(x) \, dx$$

$$+ o(1). \quad (98)$$
(88) and (98) and use (75) to obtain (77).

**Lemma 21** Under the conditions of Lemma 17,

\[
\text{Var}(\tilde{A}_{112}) = n^{-1} \int_{\epsilon}^{\infty} A_w(x)^2 \left( z_{21}(x)^2 \psi(x) + z_{22}(x)^2 \phi(x) \right) dx \\
+ o(n^{-1}) \\
= n^{-1} \int_{\epsilon}^{\infty} z_2(x)^2 A_w(x)^2 (1 - p(x)) p(x) h(x) dx \\
+ o(n^{-1}).
\]  

(99)

**Proof.** We proceed in a parallel way to the proof of the previous lemma. Denote \( I_{112}(g_1) \) the expectation of \( \text{Var}(\tilde{A}_{112}) \).

\[
\text{Var}(\tilde{A}_{112}) = \text{Var} \left( \int_{\epsilon}^{\infty} \int_{\epsilon}^{t} z_2(r) \frac{1}{n} \sum_{i=1}^{n} K_{g_1}(r - T_i) (\delta_i - p(r)) \alpha_w(t) dr dt \right) \\
= \frac{1}{n} \text{Var} \left( \int_{\epsilon}^{\infty} \int_{\epsilon}^{t} z_2(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \alpha_w(t) dr dt \right) \\
= \frac{1}{n} E \left( \left( \int_{\epsilon}^{\infty} \int_{\epsilon}^{t} z_2(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \alpha_w(t) dr dt \right)^2 \right) \\
- \frac{1}{n} I_{112}(g_1)^2.
\]

The term

\[
J_{2,2}^2(g_1) = E \left( \left( \int_{\epsilon}^{\infty} \int_{\epsilon}^{t} z_2(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \alpha_w(t) dr dt \right)^2 \right)
\]

can be decomposed as

\[
J_{2,2}^2(g_1) = J_{1,2}^2(g_1) + J_{2,2}^2(g_1),
\]

where

\[
J_{1,2}^2(g_1) = \int_{\epsilon}^{\infty} \left( \int_{\epsilon}^{\infty} \int_{\epsilon}^{t} z_{21}(r) K_{g_1}(r - x) \alpha_w(t) dr dt \right)^2 \psi(x) dx,
\]

\[
J_{2,2}^2(g_1) = \int_{\epsilon}^{\infty} \left( \int_{\epsilon}^{\infty} \int_{\epsilon}^{t} z_{22}(r) K_{g_1}(r - x) \alpha_w(t) dr dt \right)^2 \phi(x) dx.
\]
The integration limits in the previous expressions have been changed using the fact that the functions $z_{21}$ and $\phi$ vanish within the interval $[0, \epsilon]$.

The change of variable $\frac{r-x}{g_1} = r_1$ gives

$$J_{1}^{2,2}(g_1) = \int_{0}^{\infty} \left( \int_{\epsilon}^{\infty} \left( \int_{\frac{r-x}{g_1}}^{\frac{r-x}{g_1}} z_{21}(x + g_1 r_1) K(r_1) \alpha_w(t) \, dr_1 \, dt \right)^2 \psi(x) \, dx \right)$$

which leads to

$$J_{1}^{2,2}(g_1) = \int_{\epsilon}^{\infty} z_{21}(x) \alpha_w(x)^2 \psi(x) \, dx + o(1). \tag{100}$$

The same change of variable can be used to prove

$$J_{2}^{2,2}(g_1) = \int_{\epsilon}^{\infty} \left( \int_{\epsilon}^{\infty} \left( \int_{\frac{r-x}{g_1}}^{\frac{r-x}{g_1}} z_{22}(x + g_1 r_1) K(r_1) \alpha_w(t) \, dr_1 \, dt \right)^2 \phi(x) \, dx \right)$$

and

$$J_{2}^{2,2}(g_1) = \int_{\epsilon}^{\infty} z_{22}(x) \alpha_w(x)^2 \phi(x) \, dx + o(1). \tag{101}$$

Therefore, recalling

$$\text{Var} \left( \hat{A}_{112} \right) = \frac{1}{n} \left( J_{1}^{2,2}(g_1) + J_{2}^{2,2}(g_1) - I_{112}(g_1)^2 \right),$$

and using (100), (101) and (76) we conclude (99). \(\blacksquare\)

**Lemma 22** Under the assumptions of Lemma 17,

$$\text{Cov} \left( \hat{A}_{111}, \hat{A}_{112} \right) = n^{-1} \int_{\epsilon}^{\infty} (A_w(x) (z_{111}''(x) + z_{112}''(x))$$

$$- z_{111}''(x) - z_{112}''(x) - z_{111}(x) - z_{112}(x)\right)$$

$$\times A_w(x) z_2(x) (1 - p(x)) p(x) h(x) \, dx + o(n^{-1}). \tag{102}$$

**Proof.** First of all, let’s write,
\[
\text{Cov} \left( \tilde{A}_{111}, \tilde{A}_{112} \right) = \text{Cov} \left( \int_{t'}^t \int_{t'}^t z_1(r) \left( \bar{\psi}''(r) - p(r) \tilde{h}''(r) - \zeta(r) \right) \alpha_w(t) dr dt, \right.
\]
\[
\left. \int_{t'}^t \int_{t'}^t z_2(u) \left( \bar{\psi}(u) - p(u) \tilde{h}(u) \right) \alpha_w(m) du dm \right) \]
\[
= \text{Cov} \left( \int_{t'}^t \int_{t'}^t z_1(r) \left( \bar{\psi}''(r) - p(r) \tilde{h}''(r) \right) \alpha_w(t) dr dt, \right.
\]
\[
\left. \int_{t'}^t \int_{t'}^t z_2(u) \left( \bar{\psi}(u) - p(u) \tilde{h}(u) \right) \alpha_w(m) du dm \right) \]
\[
= \text{Cov} \left( \int_{t'}^t \int_{t'}^t z_1(r) \frac{1}{n} \sum_{i=1}^n K_{1i}''(r - T_i) (\delta_i - p(r)) \alpha_w(t) dr dt, \right.
\]
\[
\left. \int_{t'}^t \int_{t'}^t z_2(u) \frac{1}{n} \sum_{j=1}^n K_{1j} (u - T_j) (\delta_j - p(r)) \alpha_w(m) du dm \right) \]
\[
= \frac{1}{n} \text{Cov} \left( \int_{t'}^t \int_{t'}^t z_1(r) K_{1i}''(r - T_i) (\delta_i - p(r)) \alpha_w(t) dr dt, \right.
\]
\[
\left. \int_{t'}^t \int_{t'}^t z_2(u) K_{1i} (u - T_i) (\delta_i - p(r)) \alpha_w(m) du dm \right) \]
\[
= \frac{1}{n} E \left( \int_{t'}^t \int_{t'}^t z_1(r) K_{1i}''(r - T_i) (\delta_1 - p(r)) \alpha_w(t) dr dt \right.
\]
\[
\left. \times \int_{t'}^t \int_{t'}^t z_2(u) K_{1i} (u - T_i) (\delta_1 - p(r)) \alpha_w(m) du dm \right) \]
\[
- \frac{1}{n} f_{111}(g_1) f_{112}(g_1). \]

Define
\[
J_{1.2}^{1.2}(g_1) = E \left( \int_{t'}^t \int_{t'}^t z_1(r) K_{1i}''(r - T_i) (\delta_1 - p(r)) \alpha_w(t) dr dt \right.
\]
\[
\left. \times \int_{t'}^t \int_{t'}^t z_2(u) K_{1i} (u - T_i) (\delta_1 - p(r)) \alpha_w(m) du dm \right) . \]

The term \( J_{1.2}^{1.2}(g_1) \) will be studied following the same lines of the previous lemmas.

We consider the decomposition
\[
J_{1.2}^{1.2}(g_1) = J_{1.2}^{1.2}(g_1) + J_{1.2}^{1.2}(g_1), \]

where
\[
J_{1.2}^{1.2}(g_1) = \int_0^\infty \int_0^\infty \int_{t'}^t z_{11}(r) K_{1i}''(r - x) \alpha_w(t) dr dt \]
\[
\times \int_{t'}^t \int_{t'}^t z_{21}(u) K_{1i} (u - x) \alpha_w(m) du dm \psi(x) dx \]

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and

\[ J_{12}^{1,2}(g_1) = \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_{12}(r) K''_{g_1}(r - x) \alpha_w(t) dr dt \times \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{m} z_{22}(u) K_{g_1}(u - x) \alpha_w(m) du dm \phi(x) dx. \]

The change of variable \( \frac{r}{g_1} = r_1 \quad \frac{u}{m} = u_1 \)

\[ J_{12}^{1,2}(g_1) = g_1^{-2} \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_{11}(x + g_1 r_1) K''(r_1) \alpha_w(t) dr_1 dt \times \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{z_{11}(x + g_1 u_1) K(u_1) \alpha_w(m) du_1 dm \psi(x) dx \]

and partial integration of \( \int_{\varepsilon}^{t} z_{11}(x + g_1 r_1) K''(r_1) dr_1 \), as done for (79), lead to

\[ J_{12}^{1,2}(g_1) = J_{11}^{1,2}(g_1) + J_{12}^{1,2}(g_1) + J_{13}^{1,2}(g_1) \]

where

\[ J_{11}^{1,2}(g_1) = g_1^{-2} \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} K' \left( \frac{t - x}{g_1} \right) z_{11}(t) \alpha_w(t) dt \times \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{z_{11}(x + g_1 u_1) K(u_1) \alpha_w(m) du_1 dm \psi(x) dx, \]

\[ J_{12}^{1,2}(g_1) = -g_1^{-1} \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} K \left( \frac{t - x}{g_1} \right) z_{11}(t) \alpha_w(t) dt \times \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{z_{11}(x + g_1 u_1) K(u_1) \alpha_w(m) du_1 dm \psi(x) dx \]

and

\[ J_{13}^{1,2}(g_1) = \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_{11}(x + g_1 r_1) K(r_1) \alpha_w(t) dr_1 dt \times \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{z_{11}(x + g_1 u_1) K(u_1) \alpha_w(m) du_1 dm \psi(x) dx. \]

These terms are very similar to the last three terms in (80). They can be obtained by replacing the function \( z_{22} \) in those terms by \( z_{11}'' \). Therefore we can directly obtain the following expressions:

\[ J_{11}^{1,2}(g_1) = -\int_{\varepsilon}^{\infty} A_w(x) z_{11}(x) z_{21}(x) \psi(x) dx + o(1), \]
\[
J_{12}^{1,2}(g_1) = - \int_{e}^{\infty} A_w(x)z_{112}(x)z_{21}(x)\psi(x)dx + o(1),
\]
and, consequently,
\[
J_{13}^{1,2}(g_1) = \int_{e}^{\infty} A_w(x)^2 z_{11}''(x)z_{21}(x)\psi(x)dx + o(1)
\]

The term \(J_{12}^{1,2}(g_1)\) can be treated using the changes of variable \(\frac{r}{g_1} = r_1, \frac{u}{g_1} = u_1\) to obtain
\[
J_{12}^{1,2}(g_1) = g_1^{-2} \int_{e}^{\infty} \int_{e'}^{\infty} \int_{-L}^{\infty} \frac{r}{g_1} z_{12}(x + g_1r_1)K''(r_1)\alpha_w(t)dr_1dt \\
\times \int_{e'}^{\infty} \int_{-L}^{\infty} \frac{u}{g_1} z_{22}(x + g_1u_1)K(u_1)\alpha_w(m)du_1dm\phi(x)dx.
\]

Now, for \(g_1 < \frac{c}{L}\),
\[
J_{12}^{1,2}(g_1) = g_1^{-2} \int_{e}^{\infty} \int_{e'}^{\infty} \int_{-L}^{\infty} \frac{r}{g_1} z_{12}(x + g_1r_1)K''(r_1)\alpha_w(t)dr_1dt \\
\times \int_{e'}^{\infty} \int_{-L}^{\infty} \frac{u}{g_1} z_{22}(x + g_1u_1)K(u_1)\alpha_w(m)du_1dm\phi(x)dx.
\]

Partial integration, as done for (89), gives
\[
J_{12}^{1,2}(g_1) = g_1^{-2} \int_{e}^{\infty} \int_{e'}^{\infty} K'\left(\frac{t - x}{g_1}\right) z_{12}(t)\alpha_w(t)dt \\
\times \int_{e'}^{\infty} \int_{-L}^{\infty} K(u_1)z_{22}(x + g_1u_1)\alpha_w(m)du_1dm\phi(x)dx \\
- g_1^{-1} \int_{e}^{\infty} \int_{e'}^{\infty} K\left(\frac{t - x}{g_1}\right) z_{12}'(t)\alpha_w(t)dt \\
\times \int_{e'}^{\infty} \int_{-L}^{\infty} K(u_1)z_{22}(x + g_1u_1)\alpha_w(m)du_1dm\phi(x)dx \\
+ \int_{e}^{\infty} \int_{e'}^{\infty} K(r_1)z_{12}''(x + g_1r_1)\alpha_w(t)dr_1dt \\
\int_{e'}^{\infty} \int_{-L}^{\infty} K(u_1)z_{22}(x + g_1u_1)\alpha_w(m)du_1dm\phi(x)dx \\
= J_{21}^{1,2}(g_1) + J_{22}^{1,2}(g_1) + J_{23}^{1,2}(g_1).
\]

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Comparing these terms with the last three terms in (90), shows that (up to some constants) we can use the results obtained for those terms by replacing $z_{22}$ by $z_{12}^0$ (only once). This gives,

$$J_{12}^1(g_1) = - \int_{\mathbb{R}} A_w(x) z'_{121}(x) z_{22}(x) \phi(x) dx + o(1)$$
$$J_{22}^1(g_1) = - \int_{\mathbb{R}} A_w(x) z_{122}(x) z_{22}(x) \phi(x) dx + o(1),$$
$$J_{23}^1(g_1) = \int_{\mathbb{R}} A_w(x)^2 z'_{12}(x) z_{22}(x) \phi(x) dx + o(1),$$

and, consequently,

$$J_{22}^1(g_1) = \int_{\mathbb{R}} (A_w(x) z''_{12}(x) - z'_{121}(x) - z_{122}(x)) A_w(x) z_{22}(x) \phi(x) dx + o(1).$$

Expression (102) is a straightforward consequence of

$$\text{Cov} \left( \tilde{A}_{111}, \tilde{A}_{112} \right) = \frac{1}{n} \left( J_{11}^{12}(g_1) + J_{12}^{12}(g_1) - I_{111}(g_1) I_{112}(g_1) \right),$$

and expressions (103), (104), (75) and (76).

Now, we start the study of the term $\tilde{A}_{12}$.

**Lemma 23** Assume conditions (K.1), (P.1), (P.2), (H.1), (W.1) and (V.1). Then,

$$E \left( \tilde{A}_{12} \right) = n^{-1} g_1^{-3} \frac{1}{4} K \int_{\mathbb{R}} (1 - H(x))^{-\frac{1}{2}} (1 - p(x)) p(x) h(x) w(x) dx + o \left( n^{-1} g_1^{-3} \right) + O \left( g_1^4 \right)$$

and

$$\text{Var} \left( \tilde{A}_{12} \right) = o \left( n^{-2} g_1^{-6} \right).$$

**Proof.** The term $\tilde{A}_{12}$ is decomposed as follows

$$\tilde{A}_{12} = \tilde{A}_{121} + \tilde{A}_{122} + \tilde{A}_{123},$$

where

$$\tilde{A}_{121} = \frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} z_2(r) \left( \hat{\psi}(r) - p(r) \hat{h}(r) \right) z_2(s) \left( \hat{\psi}(s) - p(s) \hat{h}(s) \right) w(t) dr ds dt,$$
\[
\tilde{A}_{122} = \frac{1}{4} \int_{c'}^\infty \int_{c'}^t \int_{c'}^t z_1(r) \left( \tilde{\psi}''(r) - p(r) \tilde{h}''(r) - \zeta(r) \right) \\
\quad \times z_1(s) \left( \tilde{\psi}''(s) - p(s) \tilde{h}''(s) - \zeta(s) \right) w(t) dr ds dt
\]

\[
\tilde{A}_{123} = \frac{1}{2} \int_{c'}^\infty \int_{c'}^t \int_{c'}^t z_2(r) \left( \tilde{\psi}(r) - p(r) \tilde{h}(r) \right) \\
\quad \times z_1(s) \left( \tilde{\psi}''(s) - p(s) \tilde{h}''(s) - \zeta(s) \right) w(t) dr ds dt.
\]

The thesis of the lemma is a consequence of Lemmas 24, 25, 26, 27, 32, 34 and 35, that are stated and proved below. \(\blacksquare\)

**Lemma 24** Assume the conditions of Lemma 23. Then, it holds

\[
E \left( \tilde{A}_{122} \right) = n^{-1} g_1^{-1} \frac{1}{4} c K' \int_{c'}^\infty (1 - H(x))^{-2} (1 - p(x)) p(x) h(x) w(x) dx \\
\quad + o \left( n^{-1} g_1^{-3} \right) + O \left( g_1^4 \right). \quad (106)
\]

**Proof.** First, observe that

\[
E \left( \tilde{A}_{122} \right) = E \left( \frac{1}{4} \int_{c'}^\infty \int_{c'}^t \int_{c'}^t z_1(r) \left( \tilde{\psi}''(r) - p(r) \tilde{h}''(r) - \zeta(r) \right) \\
\quad \times z_1(s) \left( \tilde{\psi}''(s) - p(s) \tilde{h}''(s) - \zeta(s) \right) w(t) dr ds dt \right)
\]

\[
= \frac{n(n - 1)}{4n^2} I_{1221}(g_1) + \frac{1}{4n} I_{1222}(g_1)
\]

where

\[
I_{1221}(g_1) = E \left( \int_{c'}^\infty \int_{c'}^t \int_{c'}^t z_1(r) \left( K_{g_1}''(r - T_1) (\delta_1 - p(r)) - \zeta(r) \right) \\
\quad z_1(s) \left( K_{g_1}''(s - T_2) (\delta_2 - p(s)) - \zeta(s) \right) dr ds dw(t) dt \right)
\]

and

\[
I_{1222}(g_1) = E \left( \int_{c'}^\infty \int_{c'}^t \int_{c'}^t z_1(r) \left( K_{g_1}''(r - T_1) (\delta_1 - p(r)) - \zeta(r) \right) \\
\quad z_1(s) \left( K_{g_1}''(s - T_1) (\delta_1 - p(s)) - \zeta(s) \right) dr ds dw(t) dt \right).
\]

The term \(I_{1221}(g_1)\) can be written as

\[
I_{1221}(g_1) = \int_{c'}^\infty \left( \int_{c'}^t z_1(r) \left( \int_0^\infty K_{g_1}''(r - x) (p(x) - p(r)) h(x) dx - \zeta(r) \right) dr \right)^2 \\
\quad \times w(t) dt
\]

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which, using the change of variable \( \frac{1221}{g_1} = x_1 \), gives

\[
I_{1221}(g_1) = g_1^4 \int_{\varepsilon}^{\infty} \left( \int_{\varepsilon}^{t} z_1(r) \left( \int_{-\infty}^{r} K''(x_1) (p(r - g_1 x_1) - p(r)) \right. \right.
\]
\[ \times h(r - g_1 x_1) dx_1 - g_1^2 \zeta(r) \left. \right) dr \right)^2 w(t) dt.
\]

This expression can be simplified for \( g_1 < \varepsilon \),

\[
I_{1221}(g_1) = g_1^4 \int_{\varepsilon}^{\infty} \left( \int_{\varepsilon}^{t} z_1(r) \left( \int_{-L}^{L} K''(x_1) (p(r - g_1 x_1) - p(r)) \right. \right.
\]
\[ \times h(r - g_1 x_1) dx_1 - g_1^2 \zeta(r) \left. \right) dr \right)^2 w(t) dt.
\]

Using Taylor expansions of \( p(r - g_1 x_1) - p(r) \) and \( h(r - g_1 x_1) \) around \( r \) gives

\[
I_{1221}(g_1) = g_1^4 \int_{\varepsilon}^{\infty} \left( \int_{\varepsilon}^{t} z_1(r) \left( \int_{-L}^{L} K''(x_1) \right. \right.
\]
\[ \times \left( \sum_{i=1}^{4} \frac{(-1)^i}{i!} g_1^i x_1^i p^{(i)} (r) - \frac{1}{6} g_1^5 x_1^5 p^{(5)} (\theta_1) \right)
\]
\[ \times \left( \sum_{i=0}^{3} \frac{(-1)^i}{i!} g_1^i x_1^i h^{(i)} (r) + \frac{1}{4} g_1^4 x_1^4 h^{(4)} (\theta_2) \right) dx_1 - g_1^2 \zeta(r) \right) dr \right)^2
\]
\[ \times w(t) dt.
\]

where \( \theta_1 \) and \( \theta_2 \) are intermediate points between \( r - g_1 x_1 \) and \( r \). Elementary algebra involving equations (74) leads to

\[
I_{1221}(g_1) = g_1^4 \mu^2 \int_{\varepsilon}^{\infty} \left( \int_{\varepsilon}^{t} z_1(r) \left( \frac{1}{2} h(r) p^{(4)} (r) + 2 h'(r) p'^{(3)} (r) \right. \right.
\]
\[ \left. + 3 h''(r) p''(r) + 2 h^{(3)}(r) p'(r) \right) dr \right)^2 w(t) dt
\]
\[ + O \left( g_1^6 \right). \]

The term \( I_{1222}(g_1) \) can be handled using standard calculations and recalling that the functions \( z_{11} \) and \( \phi \) vanish in \( [0, \varepsilon] \),

\[
I_{1222}(g_1) = \int_{0}^{\infty} \int_{0}^{\infty} \left( \int_{\varepsilon}^{t} z_{11}(r) K''_{g_1} (r - x) dr \right) \left. \right)^2 w(t) \psi(x) dt dx
\]
\[ + \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \left( \int_{\varepsilon}^{t} z_{12}(r) K''_{g_1} (r - x) dr \right) \left. \right)^2 w(t) \phi(x) dt dx
\]
\[ - 4 \int_{0}^{\infty} \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_1(r) K''_{g_1} (r - x) (p(x) - p(r)) \alpha_w(t) h(x) dr dx dt dx
\]
\[ + 4 \int_{\varepsilon}^{\infty} \alpha(t)^2 w(t) dt.
\]
Now, the change of variable \( \frac{t-x}{g_1} = r_1 \) gives

\[
I_{1222}(g_1) = g_1^{-4} \int_0^\infty \int_{\epsilon}^{\infty} \left( \int_{\frac{t-x}{g_1}}^{\infty} z_{11}(x + g_1 r_1) K''(r_1) dr_1 \right)^2 w(t) \psi(x) dtdx
\]

\[
+ g_1^{-4} \int_0^\infty \int_{\epsilon}^{\infty} \left( \int_{\frac{t-x}{g_1}}^{\infty} z_{12}(x + g_1 r_1) K''(r_1) dr_1 \right)^2 w(t) \phi(x) dtdx
\]

\[
-4g_1^{-2} \int_0^\infty \int_{\epsilon}^{\infty} \left( \int_{\frac{t-x}{g_1}}^{\infty} z_1(x + g_1 r_1) K''(r_1) (p(x) - p(x + g_1 r_1)) \cdot \alpha_w(t) \psi(x) dt \right) dtdx
\]

\[
+ 4 \int_\epsilon^\infty \alpha(t)^2 w(t) dt
\]

\[
= I_{12221}(g_1) + I_{12222}(g_1) + I_{12223}(g_1) + I_{12224}.
\]

Using expression (79),

\[
I_{12221}(g_1) = g_1^{-4} \int_0^\infty \int_{\epsilon}^{\infty} K' \left( \frac{t-x}{g_1} \right) z_{11}(t)^2 w(t) \psi(x) dtdx
\]

\[
-2g_1^{-3} \int_0^\infty \int_{\epsilon}^{\infty} K' \left( \frac{t-x}{g_1} \right) z_{11}(t) K \left( \frac{t-x}{g_1} \right) z_{11}(t) w(t) \psi(x) dtdx
\]

\[
+ g_1^{-2} \int_0^\infty \int_{\epsilon}^{\infty} K \left( \frac{t-x}{g_1} \right) z_{11}(t)^2 w(t) \psi(x) dtdx
\]

\[
+ 2g_1^{-2} \int_0^\infty \int_{\epsilon}^{\infty} K' \left( \frac{t-x}{g_1} \right) z_{11}(t) \int_{\frac{t-x}{g_1}}^{\infty} K(r_1) z_{11}'(x + g_1 r_1) dr_1 \times w(t) \psi(x) dtdx
\]

\[
-2g_1^{-1} \int_0^\infty \int_{\epsilon}^{\infty} K \left( \frac{t-x}{g_1} \right) z_{11}'(t) \int_{\frac{t-x}{g_1}}^{\infty} K(r_1) z_{11}'(x + g_1 r_1) dr_1 \times w(t) \psi(x) dtdx
\]

\[
+ \int_0^\infty \int_{\epsilon}^{\infty} \left( \int_{\frac{t-x}{g_1}}^{\infty} K(r_1) z_{11}'(x + g_1 r_1) dr_1 \right)^2 w(t) \psi(x) dtdx.
\]

The change of variable \( \frac{t-x}{g_1} = t_1 \) can be used in the first five terms of the
previous expression to obtain

\[ I_{12221}(g_1) = g_1^{-3} \int_0^\infty \int_{-\frac{L}{2}}^L K'(t_1)^2 z_{11}(x + g_1 t_1)^2 w(x + g_1 t_1) \psi(x) dt_1 dx \]

\[ -2g_1^{-2} \int_0^\infty \int_{-\frac{L}{2}}^L K'(t_1) z_{11}(x + g_1 t_1) K(t_1) z'_{11}(x + g_1 t_1) \]

\[ \times w(x + g_1 t_1) \psi(x) dt_1 dx \]

\[ + g_1^{-1} \int_0^\infty \int_{-\frac{L}{2}}^L K(t_1)^2 z'_{11}(x + g_1 t_1)^2 w(x + g_1 t_1) \psi(x) dt_1 dx \]

\[ + 2g_1^{-1} \int_0^\infty \int_{-\frac{L}{2}}^L K'(t_1) z_{11}(x + g_1 t_1) \int_{-\frac{L}{2}}^{t_1} K(r_1) z''_{11}(x + g_1 r_1) dr_1 \]

\[ \times w(x + g_1 t_1) \psi(x) dt_1 dx \]

\[ - 2 \int_0^\infty \int_{-\frac{L}{2}}^L K(t_1) z'_{11}(x + g_1 t_1) \int_{-\frac{L}{2}}^{t_1} K(r_1) z''_{11}(x + g_1 r_1) dr_1 \]

\[ \times w(x + g_1 t_1) \psi(x) dt_1 dx \]

\[ + \int_0^\infty \int_{\varepsilon}^\infty \left( \int_{-\frac{L}{2}}^{t_1} K(r_1) z''_{11}(x + g_1 r_1) dr_1 \right)^2 w(t) \psi(x) dt dx \]

\[ = \sum_{i=1}^6 I_{12221i}(g_1). \]  

(110)

Since

\[ \lim_{g_1 \to 0} g_1^{-3} I_{122211}(g_1) = c_{K'} \int_{\varepsilon}^\infty z_{11}(x)^2 w(x) \psi(x) dx, \]

we conclude

\[ I_{122211}(g_1) = g_1^{-3} c_{K'} \int_{\varepsilon}^\infty z_{11}(x)^2 w(x) \psi(x) dx + o(g_1^{-3}). \]

Using direct bounds is straightforward to check that the remainder terms in (110) are of order \( O(g_1^{-2}) \). Therefore,

\[ I_{12221}(g_1) = g_1^{-3} c_{K'} \int_{\varepsilon}^\infty z_{11}(x)^2 w(x) \psi(x) dx + o(g_1^{-3}). \]

To deal with \( I_{12222}(g_1) \) we consider \( g_1 < \frac{\varepsilon}{L} \) to obtain

\[ I_{12222}(g_1) = g_1^{-4} \int_{\varepsilon}^\infty \int_{\varepsilon}^\infty \left( \int_{-\frac{L}{2}}^L z_{12}(x + g_1 r_1) K''(r_1) dr_1 \right)^2 w(t) \phi(x) dt dx \]
which, using (89), gives

\[
I_{12222}(g_1) = g_1^{-4} \int_{\varepsilon}^{\infty} \int_{-L}^{L} \left( K' \left( \frac{t-x}{g_1} \right) z_{12}(t) \right)^2 w(t) \phi(x) dt dx
\]

\[
- 2g_1^{-3} \int_{\varepsilon}^{\infty} \int_{-L}^{L} \left( K' \left( \frac{t-x}{g_1} \right) z_{12}(t) \right) K \left( \frac{t-x}{g_1} \right) z'_{12}(t) w(t) \phi(x) dt dx
\]

\[
+ g_1^{-2} \int_{\varepsilon}^{\infty} \int_{-L}^{L} \left( K \left( \frac{t-x}{g_1} \right) z'_{12}(t) \right)^2 w(t) \phi(x) dt dx
\]

\[
+ 2g_1^{-2} \int_{\varepsilon}^{\infty} \int_{-L}^{L} K' \left( \frac{t-x}{g_1} \right) z_{12}(t) \int_{-L}^{L} K(r_1) z''_{12}(x + g_1 t) dt \phi(x) dt dx
\]

\[
- 2g_1^{-2} \int_{\varepsilon}^{\infty} \int_{-L}^{L} K \left( \frac{t-x}{g_1} \right) z'_{12}(t) \int_{-L}^{L} K(r_1) z''_{12}(x + g_1 t) dt \phi(x) dt dx
\]

\[
+ \int_{\varepsilon}^{\infty} \int_{-L}^{L} \left( \int_{-L}^{L} K(r_1) z''_{12}(x + g_1 t) dt \right)^2 w(t) \phi(x) dt dx.
\]

Now, the change of variable \( \frac{t-x}{g_1} = t_1 \) can be carried out in all the terms except the last one. As a consequence,

\[
I_{12222}(g_1) = g_1^{-3} \int_{\varepsilon}^{\infty} \int_{-L}^{L} K' \left( t_1 \right)^2 z_{12}(x + g_1 t_1) w(x + g_1 t_1) \phi(x) dt_1 dx
\]

\[
- 2g_1^{-2} \int_{\varepsilon}^{\infty} \int_{-L}^{L} K' \left( t_1 \right) z_{12}(x + g_1 t_1) K \left( t_1 \right) z'_{12}(x + g_1 t_1)
\times w(x + g_1 t_1) \phi(x) dt_1 dx
\]

\[
+ g_1^{-1} \int_{\varepsilon}^{\infty} \int_{-L}^{L} K \left( t_1 \right)^2 z'_{12}(x + g_1 t_1) w(x + g_1 t_1) \phi(x) dt_1 dx
\]

\[
+ 2g_1^{-1} \int_{\varepsilon}^{\infty} \int_{-L}^{L} K' \left( t_1 \right) z_{12}(x + g_1 t_1) \int_{-L}^{L} K(r_1) z''_{12}(x + g_1 r_1) dt_1 \phi(x) dt_1 dx
\]

\[
- 2g_1^{-1} \int_{\varepsilon}^{\infty} \int_{-L}^{L} K \left( t_1 \right) z'_{12}(x + g_1 t_1) \int_{-L}^{L} K(r_1) z''_{12}(x + g_1 r_1) dt_1 \phi(x) dt_1 dx
\]

\[
+ \int_{\varepsilon}^{\infty} \int_{-L}^{L} \left( \int_{-L}^{L} K(r_1) z''_{12}(x + g_1 r_1) dt_1 \right)^2 w(t) \phi(x) dt_1 dx.
\]

\[
= \sum_{i=1}^{6} I_{12222_i}(g_1).
\]
The limit
\[
\lim_{g_1 \to 0} g_1^3 I_{12221}(g_1) = c_{K'} \int_{\varepsilon}^{\infty} z_{12}(x)^2 w(x) \phi(x) dx,
\]
leads to the following expansion
\[
I_{12221}(g_1) = g_1^{-3} c_{K'} \int_{\varepsilon}^{\infty} z_{12}(x)^2 w(x) \phi(x) dx + o(g_1^{-3}).
\]

The remainder terms in (111) can be directly bounded obtaining an order \(O(g_1^{-2})\) for all of them.

In summary,
\[
I_{12222}(g_1) = g_1^{-3} c_{K'} \int_{\varepsilon}^{\infty} z_{12}(x)^2 w(x) \phi(x) dx + o(g_1^{-3}).
\]

On the other hand, it is straightforward to conclude, as well, that the term (109) satisfies
\[
I_{12223}(g_1) = O(g_1^{-2}),
\]
which implies
\[
I_{1222}(g_1) = g_1^{-3} c_{K'} \int_{\varepsilon}^{\infty} z_{11}(x)^2 w(x) \phi(x) dx + g_1^{-3} c_{K'} \int_{\varepsilon}^{\infty} z_{12}(x)^2 w(x) \phi(x) dx
+ o(g_1^{-3})
= g_1^{-3} c_{K'} \int_{\varepsilon}^{\infty} z_{1}(x)^2 (1 - p(x)) p(x) h(x) w(x) dx + o(g_1^{-3}).
\]  
(112)

Collecting expressions (108) and (112) and recalling
\[
E(\hat{A}_{122}) = \frac{n(n-1)}{4n^2} I_{1221}(g_1) + \frac{1}{4n} I_{1222}(g_1)
\]
it results in (106). \(\Box\)

**Lemma 25** Under the same conditions of Lemma 23,
\[
E(\hat{A}_{121}) = O(g_1^4) + O(n^{-1}).
\]  
(113)
Proof. Let’s write

\[ E \left( \hat{A}_{121} \right) = E \left( \frac{1}{4} \int_{c_t}^{t} \int_{c_t}^{t} \int_{c_t}^{t} z_2(r) \left( \hat{\psi}(r) - p(r) \hat{h}(r) \right) z_2(s) \times \left( \hat{\psi}(s) - p(s) \hat{h}(s) \right) w(t) dr ds dt \right) \]

\[ = \frac{n(n-1)}{4T^2} E \left( \int_{c_t}^{t} \int_{c_t}^{t} \int_{c_t}^{t} z_2(r) K_{g_1} (r - T_1) (\delta_1 - p(r)) z_2(s) K_{g_1} (s - T_2) (\delta_2 - p(s)) dr ds dt \right) \]

\[ + \frac{1}{4T} E \left( \int_{c_t}^{t} \int_{c_t}^{t} \int_{c_t}^{t} z_2(r) K_{g_1} (r - T_1) (\delta_1 - p(r)) z_2(s) K_{g_1} (s - T_1) (\delta_1 - p(s)) dr ds dt \right) . \]

Now, define

\[ I_{1211}(g_1) = E \left( \int_{c_t}^{t} \int_{c_t}^{t} \int_{c_t}^{t} z_2(r) K_{g_1} (r - T_1) (\delta_1 - p(r)) z_2(s) \times K_{g_1} (s - T_2) (\delta_2 - p(s)) dr ds dt \right) \]

and

\[ I_{1212}(g_1) = E \left( \int_{c_t}^{t} \int_{c_t}^{t} \int_{c_t}^{t} z_2(r) K_{g_1} (r - T_1) (\delta_1 - p(r)) z_2(s) \times K_{g_1} (s - T_1) (\delta_1 - p(s)) dr ds dt \right) . \]

The term \( I_{1211}(g_1) \) can be represented as

\[ I_{1211}(g_1) = \int_{c_t}^{t} \left( \int_{c_t}^{t} z_2(r) \int_{0}^{\infty} K_{g_1} (r - x) (p(x) - p(r)) h(x) dx dr \right)^2 w(t) dt. \]

Now, the change of variable \( \frac{x}{g_1} = x_1 \) leads to

\[ I_{1211}(g_1) = \int_{c_t}^{t} \left( \int_{c_t}^{t} z_2(r) \int_{-\infty}^{\frac{r}{g_1}} K(x_1) (p(r - g_1 x_1) - p(r)) h(r - g_1 x_1) dx_1 dr \right)^2 w(t) dt, \]

which, for \( g_1 < \frac{T}{t} \), is

\[ I_{1211}(g_1) = \int_{c_t}^{t} \left( \int_{c_t}^{t} z_2(r) \int_{-L}^{L} K(x_1) (p(r - g_1 x_1) - p(r)) h(r - g_1 x_1) dx_1 dr \right)^2 w(t) dt. \]
Using Taylor expansions of \(p(r - g_1 x_1) - p(r)\) and \(h(r - g_1 x_1)\) around \(r\) we have

\[
I_{1211}(g_1) = \int_{c'}^L \left( \int_{c'}^t z_2(r) \int_{-L}^L K(x_1) \left( \sum_{i=1}^{2} \frac{(-1)^i}{i!} g_1^i x_1^i p^{(i)}(r) - \frac{1}{3!} g_1^3 x_1^3 p^{(3)}(\theta_1) \right) \right) \left( \sum_{i=0}^{1} \frac{(-1)^i}{i!} g_1^i x_1^i h^{(i)}(r) + \frac{1}{2!} g_1^2 x_1^2 h^{(2)}(\theta_2) \right) dx_1 dr \right)^2 w(t) dt
\]

\[
= g_1^4 \mu^2 K \int_{c'}^L \left( \int_{c'}^t z_2(r) \left( p'(r) h'(r) + \frac{1}{2} p''(r) h(r) \right) dx \right)^2 w(t) dt
\]

\[+ o(g_1^4), \tag{114}\]

where \(\theta_1\) and \(\theta_2\) are some intermediate points between \(r - g_1 x_1\) and \(r\) and we have used \(\int_{-L}^L K(x_1) x_1 dx_1 = 0\).

The term \(I_{1212}(g_1)\) can be written in the following way

\[
I_{1212}(g_1) = \int_{0}^{\infty} \int_{c'}^L \left( \int_{c'}^t z_{21}(r) K_{g_1}(r - x) dx \right)^2 w(t) \phi(x) dt dx
\]

\[+ \int_{0}^{\infty} \int_{c'}^L \left( \int_{c'}^t z_{22}(r) K_{g_1}(r - x) dx \right)^2 w(t) \phi(x) dt dx\]

\[= \int_{0}^{\infty} \int_{c'}^L \left( \int_{c'}^{t - x} z_{21}(r + g_1 r_1) K(r_1) dr_1 \right)^2 w(t) \psi(x) dt dx
\]

\[+ \int_{0}^{\infty} \int_{c'}^L \left( \int_{c'}^{t - x} z_{22}(r + g_1 r_1) K(r_1) dr_1 \right)^2 w(t) \phi(x) dt dx,\]

which only requires the change of variable \(\frac{r - x}{g_1} = r_1\).

Therefore, it is evident that

\[I_{1212}(g_1) = O(1). \tag{115}\]

The final result in the lemma can be obtained using (114) and (115) and recalling

\[E \left( \hat{A}_{121} \right) = \frac{n(n-1)}{4n^2} I_{1211}(g_1) + \frac{1}{4n} I_{1212}(g_1).\]

Lemma 26 Under the conditions of Lemma 23,

\[E \left( \hat{A}_{123} \right) = O(g_1^4) + O\left( n^{-1} g_1^2 \right). \tag{116}\]
Proof. The expectation of $\hat{A}_{123}$ is

$$
E\left(\hat{A}_{123}\right) \ = \ E\left(\frac{1}{2} \int_{\epsilon'}^t \int_{\epsilon'}^t z_2(r) \left(\tilde{\psi}(r) - p(r)\tilde{h}(r)\right) \times z_1(s) \left(\tilde{\psi}''(s) - p(s)\tilde{h}''(s) - \zeta(s)\right) w(t) dr ds dt\right)
$$

$$
= \frac{n(n-1)}{2n^2} E\left(\int_{\epsilon'}^t \int_{\epsilon'}^t z_2(r) K_{g_1} (r - T_1) (\delta_1 - p(r))
\times z_1(s) \left(K''_{g_1} (s - T_2) (\delta_2 - p(s)) - \zeta(s)\right) dr ds dt\right)
$$

$$
\quad + \frac{1}{2n} E\left(\int_{\epsilon'}^t \int_{\epsilon'}^t z_2(r) K_{g_1} (r - T_1) (\delta_1 - p(r))
\times z_1(s) \left(K''_{g_1} (s - T_1) (\delta_1 - p(s)) - \zeta(s)\right) dr ds dt\right).
$$

Define

$$
I_{1231}(g) \ = \ E\left(\int_{\epsilon'}^t \int_{\epsilon'}^t z_2(r) K_{g_1} (r - T_1) (\delta_1 - p(r))
\times z_1(s) \left(K''_{g_1} (s - T_2) (\delta_2 - p(s)) - \zeta(s)\right) dr ds dt\right)
$$

and

$$
I_{1232}(g_1) \ = \ E\left(\int_{\epsilon'}^t \int_{\epsilon'}^t z_2(r) K_{g_1} (r - T_1) (\delta_1 - p(r))
\times z_1(s) \left(K''_{g_1} (s - T_1) (\delta_1 - p(s)) - \zeta(s)\right) dr ds dt\right).
$$

These terms are studied next. The first one is

$$
I_{1231}(g_1) \ = \ \int_{\epsilon'}^t \int_{\epsilon'}^t z_2(r) \int_0^\infty K_{g_1} (r - x) (p(x) - p(r)) h(x) dx dr
\times \int_{\epsilon'}^t z_1(s) \left(\int_0^\infty K''_{g_1} (s - y) (p(y) - p(s)) h(y) dy - \zeta(s)\right) ds dt
$$

$$
= \ g_1^{-2} \int_{\epsilon'}^t \int_{\epsilon'}^t z_2(r) \int_{-\infty}^{\infty} K'(x_1) (p(r - g_1 x_1) - p(r)) h(r - g_1 x_1) dx_1 dr
\times \int_{\epsilon'}^t z_1(s) \left(\int_{-\infty}^{\infty} K''(y_1) (p(s - g_1 y_1) - p(s)) h(s - g_1 y_1) dy_1
- g_1^2 \zeta(s)\right) ds dt
$$

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where we have used the changes of variable $\frac{x}{y_1} = x_1$, $\frac{x}{y_1} = y_1$. For $g_1 < \frac{t}{r}$

$I_{1231}(g_1) = g_1^{-2} \int_{c'}^{\infty} \int_{c'}^{t} z_2(r) \int_{-L}^{L} K(x_1) (p(r - g_1 x_1) - p(r)) h(r - g_1 x_1) dx_1 dr \times \int_{c'}^{t} z_1(s) \left( \int_{-L}^{L} K''(y_1) (p(s - g_1 y_1) - p(s)) h(s - g_1 y_1) dy_1 \right)
\times g_1 \zeta(s) ds w(t) dt."

Taylor expansions of $(p(r - g_1 x_1) - p(r))$ and $h(r - g_1 x_1)$ around $r$ and of $(p(s - g_1 y_1) - p(s))$ and $h(s - g_1 y_1)$ around $s$ can be used to obtain

$I_{1231}(g_1) = g_1^{-2} \int_{c'}^{\infty} \int_{c'}^{t} z_2(r) \int_{-L}^{L} K(x_1) \left( \frac{2}{\gamma} \sum_{i=1}^{2} \frac{(-1)^i}{i!} g_1 x_1^i \psi(i)(r) - \frac{1}{\gamma} g_1^2 x_1^2 \psi^{(3)}(\theta_1) \right) \times \left( \frac{1}{\gamma} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} g_1 x_1^i \psi(i)(r) + \frac{1}{\gamma} g_1^2 x_1^2 \psi^{(2)}(\theta_2) \right) dx_1 dr
\times \int_{c'}^{t} z_1(s) \left( \int_{-L}^{L} K''(y_1) \left( \frac{4}{\gamma} \sum_{i=1}^{4} \frac{(-1)^i}{i!} g_1 y_1^i \psi(i)(s) - \frac{1}{\gamma} g_1^2 y_1^2 \psi^{(4)}(\theta_3) \right) \right.
\times \left( \frac{3}{\gamma} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} g_1 y_1^i \psi(i)(s) + \frac{1}{\gamma} g_1^2 y_1^2 \psi^{(4)}(\theta_4) \right) dy_1 - g_1^2 \zeta(s) \right) ds w(t) dt
= g_1^4 \mu^2 K \int_{c'}^{\infty} \int_{c'}^{t} z_2(r) \left( p'(r) h'(r) + \frac{1}{2} h''(r) h(r) \right) dr
\times \int_{c'}^{t} z_1(s) \left( \frac{1}{2} h(s) p^{(4)}(s) + 2h'(s) p^{(3)}(s)
+ 3h''(s) p''(s) + 2h^{(3)}(s) p'(s) \right) ds w(t) dt
+ o(g_1^4) \quad (117)

where $\theta_1$ and $\theta_2$ are intermediate points between $r - g_1 x_1$; $\theta_3$ and $\theta_4$ are intermediate points between $s - g_1 y_1$ and $s$ and the equation $\int_{-L}^{L} K(x_1)x_1dx_1 = 0$, as well as expressions (74) have been used.

Let’s consider the term $I_{1232}(g_1)$, that may be represented as

$I_{1232}(g_1) = \int_0^{\infty} \int_{c'}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_2(r) K_{g_1}(r - x) z_1(s) K''_{g_1}(s - x) w(t)
\times \psi(x) dr ds dt dx
+ \int_0^{\infty} \int_{c'}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_2(r) K_{g_1}(r - T_1) z_2(s) K''_{g_1}(s - T_1) w(t)
\times \phi(x) dr ds dt dx
- \int_0^{\infty} \int_{c'}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_2(r) K_{g_1}(r - x) (p(x) - p(r)) \alpha w(t) h(x) dr dt dx.$
By using the changes of variable $\frac{t}{g_1} = r_1$, $\frac{\epsilon}{g_1} = s_1$ and $\frac{\epsilon^2}{g_1} = r_1$ and assuming $g_1 < \frac{\epsilon^3}{r_1}$,

\[ I_{1232}(g_1) = g_1^{-2} \int_0^\infty \int_{\epsilon'}^\infty \int_0^{\frac{\epsilon^2}{g_1}} \int_{\frac{\epsilon}{g_1}}^{\frac{\epsilon^2}{g_1}} z_{21}(x + g_1r_1)K(r_1)z_{11}(x + g_1s_1) \times K''(s_1)w(t)\psi(x)dr_1ds_1dt \]
\[ + g_1^{-2} \int_{\epsilon'}^\infty \int_{\epsilon'}^\infty \int_{\frac{\epsilon}{g_1}}^{\frac{\epsilon^2}{g_1}} \int_{\frac{\epsilon}{g_1}}^{\frac{\epsilon^2}{g_1}} z_{22}(x + g_1r_1)K(r_1)z_{12}(x + g_1s_1) \times K''(s_1)w(t)\phi(x)dr_1ds_1dt \]
\[ - \int_0^\infty \int_{\epsilon'}^\infty \int_0^{\frac{\epsilon^2}{g_1}} \int_{\frac{\epsilon}{g_1}}^{\frac{\epsilon^2}{g_1}} z_2(x + g_1r_1)K(r_1)(p(x) - p(x + g_1r_1)) \times \alpha_w(t)h(x)dr_1ds_1dt. \]

These three summands may be directly bounded. Thus

\[ I_{1232}(g_1) = O\left(g_1^{-2}\right) + O\left(g_1^{-2}\right) + O(1) = O\left(g_1^{-2}\right). \quad (118) \]

Finally, from expressions (117), (118) and

\[ E\left(\tilde{A}_{123}\right) = \frac{n(n - 1)}{2n^2} I_{1231}(g_1) + \frac{1}{2n} I_{1232}(g_1). \]

it is easy to derive (116). \( \square \)

**Lemma 27** Assume the same conditions as in Lemma 23. Then

\[ \text{Var}\left(\tilde{A}_{122}\right) = o\left(n^{-2}g_1^{-6}\right). \quad (119) \]

**Proof.** We have

\[ \text{Var}\left(\tilde{A}_{122}\right) = \frac{1}{4n^4} \text{Var}\left(\sum_{i=1}^{n} \sum_{j=1}^{n} \int_{c}^{t} \int_{c}^{t} z_1(r) (K''_i(r - T_i)(s - p(s) - \zeta(s)) ds dr \right) \]
\[ = \frac{1}{4n^4} \left( nM^{1,1,1}(g_1) + 2n(n - 1)M^{1,2,2,2}(g_1) + 2n(n - 1)M^{1,2,1,1}(g_1) + n(n - 1)M^{1,2,1,2}(g_1) + n(n - 1)(n - 2)M^{1,2,1,3}(g_1) + n(n - 1)(n - 2)M^{1,2,3,1}(g_1) + n(n - 1)(n - 2)M^{1,2,2,3}(g_1) + n(n - 1)(n - 2)M^{1,2,3,2}(g_1) \right) \]

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where

\[ M^{i,j,k,l}(g_1) = \text{Cov} \left( \int_0^\infty \int_{v'} z_1(r) \left( K''_{g_1}(r - T_1) (\delta_1 - p(r)) - \zeta(r) \right) dr \right. \]
\[ \times \left. \int_{v'}^t z_1(s) \left( K''_{g_1}(s - T_2) (\delta_2 - p(s)) - \zeta(s) \right) dsw(t) dt, \right. \]
\[ \int_{v'}^\infty \int_{v'}^m z_1(u) \left( K''_{g_1}(u - T_3) (\delta_3 - p(u)) - \zeta(u) \right) du \]
\[ \times \left. \int_{v'}^m z_1(v) \left( K''_{g_1}(v - T_1) (\delta_1 - p(v)) - \zeta(v) \right) dsw(m) dm \right) . \]

The proof of the following equations is straightforward:

\[ M^{1,2,1,2}(g_1) = M^{1,2,2,1}(g_1), \quad M^{1,2,2,2}(g_1) = M^{1,2,1,1}(g_1) \]
and

\[ M^{1,2,1,3}(g_1) = M^{1,2,3,1}(g_1) = M^{1,2,2,3}(g_1) = M^{1,2,3,2}(g_1), \]
and then it suffices to study the order of \( M^{1,1,1,1}(g_1), M^{1,2,2,2}(g_1), M^{1,2,1,2}(g_1) \)
and \( M^{1,2,1,3}(g_1) \).

The thesis of the lemma is a consequence of the following results:

\[ M^{1,1,1,1}(g_1) = O \left( g_1^{-8} \right), \quad M^{1,2,1,3}(g_1) = o(1), \]
\[ M^{1,2,2,2}(g_1) = o \left( g_1^{-6} \right), \quad M^{1,2,1,2}(g_1) = o \left( g_1^{-6} \right), \]

that are collected in Lemmas 28-31, and condition (V.1), which implies \( ng_1^2 \to \infty \)
and \( ng_1^6 \to 0 \).

**Lemma 28** Under the conditions of Lemma 27,

\[ M^{1,2,1,3}(g_1) = o(1). \]  \hspace{1cm} (120)

**Proof.** Let’s start studying \( M^{1,2,1,3}(g_1) \). First, we compute the expectation

\[ E \left( \int_0^\infty \int_{v'} z_1(r) \left( K''_{g_1}(r - T_1) (\delta_1 - p(r)) - \zeta(r) \right) dr \right. \]
\[ \times \left. \int_{v'}^t z_1(s) \left( K''_{g_1}(s - T_2) (\delta_2 - p(s)) - \zeta(s) \right) dsw(t) dt, \right. \]
\[ \int_{v'}^\infty \int_{v'}^m z_1(u) \left( K''_{g_1}(u - T_3) (\delta_3 - p(u)) - \zeta(u) \right) du \]
\[ \times \left. \int_{v'}^m z_1(v) \left( K''_{g_1}(v - T_1) (\delta_1 - p(v)) - \zeta(v) \right) dsw(m) dm \right) . \]

\[ = M^{1,2,1,3}(g_1) + M^{1,2,1,3}(g_1) + M^{1,2,1,3}(g_1) + M^{1,2,1,3}(g_1) \]  \hspace{1cm} (121)
where

\[
M_{1,2,1,3}^1(g_1) = \int_0^\infty \left( \int_{-\infty}^\infty \int_{c^*}^1 z_{11}(r) K''_g(r - x) dr \right. \\
\times \left. \int_{c^*}^1 z_1(s) \left( \int_0^\infty K''_g(s - y) (p(y) - p(s)) h(y) dy - \zeta(s) \right) dsw(t) dt \right)^2 \psi(x) dx,
\]

\[
M_{1,2,1,3}^2(g_1) = \int_0^\infty \left( \int_{c^*}^\infty \int_{c^*}^1 z_{12}(r) K''_g(r - x) dr \right. \\
\times \left. \int_{c^*}^1 z_1(s) \left( \int_0^\infty K''_g(s - y) (p(y) - p(s)) h(y) dy - \zeta(s) \right) dsw(t) dt \right)^2 \phi(x) dx,
\]

\[
M_{1,2,1,3}^3(g_1) = -2 \int_{c^*}^t \int_{c^*}^1 z_1(s) \left( \int_{-\infty}^\infty K''_g(s - y) (p(y) - p(s)) h(y) dy - \zeta(s) \right) dsw(t) dt \\
\times \int_{c^*}^1 z_1(v) \left( \int_0^\infty K''_g(v - \eta) (p(\eta) - p(v)) h(\eta) d\eta - \zeta(v) \right) d\omega(m) dm
\]

and

\[
M_{1,2,1,3}^4(g_1) = 4 \left( \int_{c^*}^t \int_{c^*}^1 z_1(r) \left( \int_{-\infty}^\infty K''_g(r - x) (p(x) - p(r)) h(x) dx - \zeta(r) \right) d\omega(t) dt \right)^2.
\]

Using the changes of variable \( \frac{y_1}{g_1} = y_1, \frac{y_1}{r_1} = r_1 \),

\[
M_{1,2,1,3}^1(g_1) = g_1^8 \int_0^1 \left( \int_0^\infty \int_{c^*}^1 \frac{z_1(x + g_1 r_1) K''(r_1) dr_1}{g_1} \right. \\
\times \left. \int_{c^*}^1 z_1(s) \left( \int_{-\infty}^\infty K''(y_1) (p(s - g_1 y_1) - p(s)) \right. \right. \\
\times \left. \left. h(s - g_1 y_1) dy_1 - g_1^2 \zeta(s) \right) dsw(t) dt \right)^2 \psi(x) dx
\]

or, equivalently, for \( g_1 < \frac{c^*}{r_1} \),

\[
M_{1,2,1,3}^1(g_1) = g_1^8 \int_0^1 \left( \int_0^\infty \int_{c^*}^1 \frac{z_1(x + g_1 r_1) K''(r_1) dr_1}{g_1} \right. \\
\times \left. \int_{c^*}^1 z_1(s) \left( \int_{-\infty}^L K''(y_1) (p(s - g_1 y_1) - p(s)) \right. \right. \\
\times \left. \left. h(s - g_1 y_1) dy_1 - g_1^2 \zeta(s) \right) dsw(t) dt \right)^2 \psi(x) dx.
\]
Taylor expansions of \( p(s - g_1 y_1) - p(s) \) and \( h(s - g_1 y_1) \), as used in (107), give

\[
M_{1}^{1,2,1.3}(g_1) = \mu_2^2 \int_0^\infty \left( \int_{z_1}^t \int_{z_1}^t z_{11}(x + g_1 r_1) K''(r_1) dr_1 \right. \\
\left. \times \int_{z_1}^t z_1(r) \left( \frac{1}{2} h(r)p^{(4)}(r) + 2 h'(r)p^{(3)}(r) + 3 h''(r)p''(r) + 2 h'''(r)p''(r) \right) dr w(t) dt \right)^2 \psi(x) dx \\
+ o(1).
\]

Now using \( \int_{-L}^L K''(r_1) dr_1 = 0 \),

\[
\lim_{g_1 \to 0} M_{1}^{1,2,1.3}(g_1) = \left( \int_{-L}^L K''(r_1) dr_1 \right)^2 \mu_2^2 \\
\times \int_{z_1}^t \int_{z_1}^t z_{11}(x) \left( 2 h'(r)p^{(3)}(r) + 3 h''(r)p''(r) \right) dr w(t) dt \\
\times z_{11}(x)^2 \psi(x) dx \\
= 0,
\]

which implies

\[
M_{1}^{1,2,1.3}(g_1) = o(1).
\]  \hspace{1cm} (122)

A simple inspection of the term \( M_2^{1,2,1.3}(g_1) \) and a direct comparison with the previous one suffices to conclude

\[
M_2^{1,2,1.3}(g_1) = o(1).
\]  \hspace{1cm} (123)

To deal with \( M_3^{1,2,1.3}(g_1) \) we used the changes of variable \( \frac{r-x}{g_1} = x_1, \frac{s-y}{g_1} = y_1 \) and \( \frac{v-m}{g_1} = \eta_1 \),

\[
M_3^{1,2,1.3}(g_1) = -2g_1^{-6} \int_{z_1}^t z_1(r) \int_{z_1}^t K''(x_1) \left( p(r - g_1 x_1) - p(r) \right) h(r - g_1 x_1) dx_1 dr \\
\times \int_{z_1}^t z_1(s) \left( \int_{-\infty}^{z_1} K'''_{g_1}(y_1) (p(s - g_1 y_1) - p(s)) h(s - g_1 y_1) dy_1 - g_1^2 \zeta(s) \right) ds w(t) dt \\
\times \int_{z_1}^t z_1(v) \left( \int_{-\infty}^{z_1} K''_{g_1}(\eta_1) (p(v - g_1 \eta_1) - p(v)) h(v - g_1 \eta_1) d\eta_1 - g_1^2 \zeta(v) \right) dv \\
\times a_w(m) dm
\]
and, assuming \( g_1 < \frac{1}{L} \).

\[
M_{3,1,2,1,3}^{1,2,1,3} (g_1) = -2g_1^{-6} \int_{c'} \int_{c'} z_1(r) \int_{-L}^{L} K''(x_1) (p(r - g_1 x_1) - p(r)) \\
\times h(r - g_1 x_1) dx_1 dr \\
\times \int_{c'} z_1(s) \left( \int_{-L}^{L} K''_m (r_1) (p(s - g_1 y_1) - p(s)) h(s - g_1 y_1) dy_1 \\
- g_1^2 \zeta(s) d \omega_0 (t) dt \\
\times \int_{c} \int_{c'} z_1(v) \left( \int_{-L}^{L} K''_n (\eta_1) (p(v - g_1 \eta_1) - p(v)) h(v - g_1 \eta_1) d \eta_1 \\
- g_1^2 \zeta(v) d \omega_0 (m) dm \right).
\]

Taylor expansions of \( p(r - g_1 x_1) - p(r) \) and \( h(r - g_1 x_1) \) around \( r \); of \( p(s - g_1 y_1) - p(s) \) and \( h(s - g_1 y_1) \) around \( s \) and of \( p(v - g_1 \eta_1) - p(v) \) and \( h(v - g_1 \eta_1) \) around \( v \), as done for previous results, lead to

\[
M_{3,1,2,1,3}^{1,2,1,3} (g_1) = -4g_1^2 \mu K \int_{c} \int_{c'} z_1(s) \left( \frac{1}{2} h(s) p^{(4)} (s) + 2 h'(s) p^{(3)} (s) \\
+ 3 h''(s) p''(s) + 2 h'''(s) p'(s) \right) d \omega_0 (t) dt \\
\times \int_{c} \int_{c'} z_1(v) \left( \frac{1}{2} h(v) p^{(4)} (v) + 2 h'(v) p^{(3)} (v) \\
+ 3 h''(v) p''(v) + 2 h'''(v) p'(v) \right) d \omega_0 (m) dm \\
+ o (g_1^2). \tag{124}
\]

To analyze the term \( M_{4}^{1,2,1,3} (g_1) \) it is sufficient to study the order of

\[
\int_{c} \int_{c'} z_1(r) \left( \int_{0}^{\infty} K''_m (r - x) (p(x) - p(r)) h(x) dx - \zeta (r) \right) d \omega_0 (t) dt,
\]

which becomes

\[
\int_{c} \int_{c'} z_1(r) \left( \int_{0}^{\infty} K''_m (r - x) (p(x) - p(r)) h(x) dx - \zeta (r) \right) d \omega_0 (t) dt \\
= g_1^{-2} \int_{c} \int_{c'} z_1(r) \left( \int_{-L}^{L} K''(x_1) (p(r - g_1 x_1) - p(r)) h(r - g_1 x_1) dx_1 \\
- g_1^2 \zeta(r) d \omega_0 (t) dt \\
= g_1^2 \mu K \int_{c} \int_{c'} z_1(r) \left( \frac{1}{2} h(r) p^{(4)} (r) + 2 h'(r) p^{(3)} (r) \\
+ 3 h''(r) p''(r) + 2 h'''(r) p'(r) \right) d \omega_0 (m) dm + o (g_1^2)
\]

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after using the change of variable \( \frac{r - x}{g_1} = x_1 \), assuming \( g_1 < \frac{\varepsilon}{2} \) and performing a Taylor expansion of \( p(r - g_1 x_1) - p(r) \) and \( h(r - g_1 x_1) \) around \( r \). Therefore,

\[
M_i^{1,2,1,3}(g_1) = O(g_1^2). \tag{125}
\]

From (122)-(125) the order of the expectation in (121) can be proved to be \( o(1) \). On the other hand, the order of the product of expectations can be proved, via (108), to be

\[
E \left( \int_{\varepsilon}^{\infty} z_1(r) \left( K_{g_1}(r - T_1) (\delta_1 - p(r)) - \zeta(r) \right) dr \right.
\]

\[
\times \int_{\varepsilon}^{t} z_1(s) \left( K_{g_1}(s - T_2) (\delta_2 - p(s)) - \zeta(s) \right) ds \left( w(t) dt \right)
\]

\[
\times E \left( \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{m} z_1(u) \left( K_{g_1}(u - T_1) (\delta_1 - p(u)) - \zeta(u) \right) du \right.
\]

\[
\times \int_{\varepsilon}^{m} z_1(v) \left( K_{g_1}(v - T_3) (\delta_3 - p(v)) - \zeta(v) \right) dw \left( m dm \right)
\]

\[
= I_{1221}(g_1)^2,
\]

which is \( O(g_1^2) \). As a consequence expression (120) holds. \( \Box \)

**Lemma 29** Under the conditions of Lemma 27,

\[
M_i^{1,2,2,2}(g_1) = o(g_1^{-6}). \tag{126}
\]

**Proof.** To study \( M_i^{1,2,2,2}(g_1) \), let’s start with the expectation

\[
E \left( \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{t} z_1(r) \left( K_{g_1}(r - T_1) (\delta_1 - p(r)) - \zeta(r) \right) dr \right.
\]

\[
\times \int_{\varepsilon}^{t} z_1(s) \left( K_{g_1}(s - T_2) (\delta_2 - p(s)) - \zeta(s) \right) ds \left( w(t) dt \right)
\]

\[
\times \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{m} z_1(u) \left( K_{g_1}(u - T_1) (\delta_1 - p(u)) - \zeta(u) \right) du \right.
\]

\[
\times \int_{\varepsilon}^{m} z_1(v) \left( K_{g_1}(v - T_2) (\delta_2 - p(v)) - \zeta(v) \right) dw \left( m dm \right)
\]

\[
= \sum_{i=1}^{11} M_i^{1,2,2,2}(g_1), \tag{127}
\]

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where

\[ M_{1}^{1,2,2,2}(g_1) = \int_{0}^{\infty} \int_{e^t}^{\infty} \int_{e^t}^{t} z_1(r) \]
\[ \times \left( \int_{0}^{\infty} K''_{g_1}(r-y)(p(y) - p(r)) h(y)dy - \zeta(r) \right) dr \]
\[ \times \int_{e^t}^{t} z_{11}(s) K''_{g_1}(s-x) ds w(t) dt \int_{e^t}^{\infty} \int_{e^t}^{m} z_{11}(u) K''_{g_1}(u-x) du \]
\[ \times \int_{e^t}^{m} z_{11}(v) K''_{g_1}(v-x) dw(m) dm \psi(x) dx, \]

\[ M_{2}^{1,2,2,2}(g_1) = -\int_{0}^{\infty} \int_{e^t}^{\infty} \int_{e^t}^{t} z_1(r) \]
\[ \times \left( \int_{0}^{\infty} K''_{g_1}(r-y)(p(y) - p(r)) h(y)dy - \zeta(r) \right) dr \]
\[ \times \int_{e^t}^{t} z_{12}(s) K''_{g_1}(s-x) ds w(t) dt \int_{e^t}^{\infty} \int_{e^t}^{m} z_{12}(u) K''_{g_1}(u-x) du \]
\[ \times \int_{e^t}^{m} z_{12}(v) K''_{g_1}(v-x) dw(m) dm \phi(x) dx, \]

\[ M_{3}^{1,2,2,2}(g_1) = -4 \int_{0}^{\infty} \int_{e^t}^{\infty} \int_{e^t}^{t} z_1(r) \]
\[ \times \left( \int_{0}^{\infty} K''_{g_1}(r-y)(p(y) - p(r)) h(y)dy - \zeta(r) \right) dr \]
\[ \times \int_{e^t}^{t} z_{11}(s) K''_{g_1}(s-x) ds w(t) dt \int_{e^t}^{\infty} \int_{e^t}^{m} z_{11}(u) K''_{g_1}(u-x) du \]
\[ \times \int_{e^t}^{m} z_{11}(v) K''_{g_1}(v-x) dw(m) dm \psi(x) dx, \]

\[ M_{4}^{1,2,2,2}(g_1) = -2 \int_{0}^{\infty} \int_{e^t}^{\infty} \int_{e^t}^{t} z_1(r) \]
\[ \times \left( \int_{0}^{\infty} K''_{g_1}(r-y)(p(y) - p(r)) h(y)dy - \zeta(r) \right) dr \alpha(t) dt \]
\[ \times \int_{e^t}^{\infty} \int_{e^t}^{m} z_{11}(u) K''_{g_1}(u-x) du \]
\[ \times \int_{e^t}^{m} z_{11}(v) K''_{g_1}(v-x) dw(m) dm \psi(x) dx, \]
\begin{align*}
M_{5}^{1,2,2,2}(g_1) &= -4 \int_{0}^{\infty} \int_{\varepsilon}^{t} \int_{\varepsilon}^{t} z_1(r) \times \left( \int_{0}^{\infty} K_{g_1}''(r-y)(p(y) - p(r)) h(y) dy - \zeta(r) \right) dr \\
& \quad \times \int_{\varepsilon}^{t} z_{12}(s) K_{g_1}''(s-x) dsw(t) dt \\
& \quad \times \int_{\varepsilon}^{t} \int_{\varepsilon}^{m} z_{12}(u) K_{g_1}''(u-x) d\alpha_w(m) d\phi(x) dx,
\end{align*}

\begin{align*}
M_{6}^{1,2,2,2}(g_1) &= -2 \int_{0}^{\infty} \int_{\varepsilon}^{t} \int_{\varepsilon}^{t} z_1(r) \times \left( \int_{0}^{\infty} K_{g_1}''(r-y)(p(y) - p(r)) h(y) dy - \zeta(r) \right) d\alpha_w(t) dt \\
& \quad \times \int_{\varepsilon}^{t} \int_{\varepsilon}^{m} z_{12}(u) K_{g_1}''(u-x) du \\
& \quad \times \int_{\varepsilon}^{t} z_{12}(v) K_{g_1}''(v-x) d\nu w(m) d\phi(x) dx,
\end{align*}

\begin{align*}
M_{7}^{1,2,2,2}(g_1) &= 4 \int_{0}^{\infty} \int_{\varepsilon}^{t} \int_{\varepsilon}^{t} z_1(r) \times \left( \int_{0}^{\infty} K_{g_1}''(r-y)(p(y) - p(r)) h(y) dy - \zeta(r) \right) dr \\
& \quad \times \int_{\varepsilon}^{t} z_{11}(s) K_{g_1}''(s-x) dsw(t) dt \\
& \quad \times \int_{\varepsilon}^{t} \alpha(m)^2 w(m) d\nu \psi(x) dx,
\end{align*}

\begin{align*}
M_{8}^{1,2,2,2}(g_1) &= 8 \int_{0}^{\infty} \int_{\varepsilon}^{t} \int_{\varepsilon}^{t} z_1(r) \times \left( \int_{0}^{\infty} K_{g_1}''(r-y)(p(y) - p(r)) h(y) dy - \zeta(r) \right) d\alpha_w(t) dt \\
& \quad \times \int_{\varepsilon}^{t} \int_{\varepsilon}^{m} z_{11}(u) K_{g_1}''(u-x) d\alpha_w(m) d\nu \psi(x) dx,
\end{align*}

\begin{align*}
M_{9}^{1,2,2,2}(g_1) &= 4 \int_{0}^{\infty} \int_{\varepsilon}^{t} \int_{\varepsilon}^{t} z_1(r) \times \left( \int_{0}^{\infty} K_{g_1}''(r-y)(p(y) - p(r)) h(y) dy - \zeta(r) \right) dr \\
& \quad \times \int_{\varepsilon}^{t} z_{12}(s) K_{g_1}''(s-x) dsw(t) dt \int_{\varepsilon}^{t} \alpha(m)^2 w(m) d\phi(x) dx,
\end{align*}
\[ M_{10}^{1,2,2}(g_1) = 8 \int_0^\infty \int_0^\infty \int_{c'}^t z_1(r) \times \left( \int_0^\infty K''_g(r - y) (p(y) - p(r)) h(y) dy - \zeta(r) \right) d\alpha_w(t) dt \] 
\[ \times \int_{c'}^m \int_{c'}^{m_1} z_{12}(u) K''_g(u - x) d\alpha_w(m) d\psi(x) dx \]

and

\[ M_{11}^{1,2,2}(g_1) = -8 \int_0^\infty \int_0^\infty \int_{c'}^t z_1(r) \times \left( \int_0^\infty K''_g(r - y) (p(y) - p(r)) h(y) dy - \zeta(r) \right) d\alpha_w(t) dt \] 
\[ \times \int_{c'}^\infty \alpha(m)^2 w(m) d\mu(x) dx. \]

The first term will be handled in a similar way to \( M_{1}^{1,2,3,1}(g_1) \), i.e. using the change of variable \( \frac{r - y}{g_1} = y_1, \frac{s - x}{g_1} = s_1, \frac{u - x}{g_1} = u_1, \frac{v - x}{g_1} = v_1 \) and assuming \( g_1 < \frac{1}{2} \).

\[ M_{11}^{1,2,2}(g_1) = g_1^{-8} \int_0^\infty \int_{c'}^t \int_{c'}^{t_1} z_1(r) \times \left( \int_{-L}^L K''(y_1) (p(r - g_1 y_1) - p(r)) h(r - g_1 y_1) dy_1 - g_1^2 \zeta(r) \right) dr \] 
\[ \times \int_{s_1}^{s_1 + g_1} z_{11}(x + g_1 s_1) K''(s_1) ds_1 w(t) dt \] 
\[ \times \int_{c'}^\infty \int_{c'}^{m_1} z_{11}(x + g_1 u_1) K''(u_1) du_1 \] 
\[ \times \int_{c'}^\infty \int_{c'}^{m_1} z_{11}(x + g_1 v_1) K''(v_1) dv_1 w(m) d\psi(x) dx. \]
Taylor expansions of \( p(r - g_1y_1) - p(r) \) and \( h(r - g_1y_1) \) lead to

\[
M_1^{1,2,2,2}(g_1) = g_1^4 \mu_K \int_0^\infty \int_{c'}^1 \int_{c'}^1 z_1(r) \times \left( \frac{1}{2} h(r)p^{(4)}(r) + 2h'(r)p^{(3)}(r) + 3h''(r)p''(r) + 2h^{(3)}(r)p'(r) \right) dr \\
\times \int_{c'-z_1} \int_{c'-z_1} z_{11}(x + g_1 s_1)K''(s_1)ds_1 w(t) dt \\
\times \int_{c'-z_1} \int_{c'-z_1} z_{11}(x + g_1 u_1)K''(u_1)du_1 \\
\times \int_{c'-z_1} \int_{c'-z_1} z_{11}(x + g_1 v_1)K''(v_1)dv_1 w(m) dm \psi(x) dx + o(g_1^{-4}) .
\]

Hence,

\[
M_1^{1,2,2,2}(g_1) = o(g_1^{-4}) ,
\]

since

\[
\lim_{g_1 \to 0} g_1^4 M_1^{1,2,2,2}(g_1) = \mu_K \left( \int_{-L}^{L} K''(s_1)ds_1 \right)^3 \int_{c'}^1 \int_{c'}^1 \int_{c'}^1 z_1(r) \times \left( \frac{1}{2} h(r)p^{(4)}(r) + 2h'(r)p^{(3)}(r) \\
+ 3h''(r)p''(r) + 2h^{(3)}(r)p'(r) \right) drw(t) dt \\
\times \int_{x} w(m) dm z_{11}(x)^3 \psi(x) dx = 0 .
\]

On the other hand, a simple inspection of the term \( M_2^{1,2,2,2}(g_1) \) and its comparison with \( M_1^{1,2,2,2}(g_1) \) suffices to conclude that

\[
M_2^{1,2,2,2}(g_1) = o(g_1^{-4}) .
\]

To determine the order of the next term in (127) is enough to use the change

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of variable $\frac{r - y}{y_1} = y_1$, $\frac{s - x}{s_1} = s_1$, $\frac{u - x}{u_1} = x_1$, to obtain

$$M_{3}^{1,2,2,2}(g_1) = -4g_1^{-6} \int_{0}^{\infty} \int_{e'}^{t} z_1(r) \times \left( \int_{-\infty}^{\frac{r}{y_1}} K''(y_1)(p(r - g_1y_1) - p(r))h(r - g_1y_1)dy_1 - g_1^2\zeta(r) \right) dr$$

$$\times \int_{e'}^{t} z_{11}(x + g_1s_1)K''(s_1)ds_1w(t) dt$$

$$\times \int_{e'}^{t} \int_{e''}^{s_1} z_{11}(x + g_1u_1)K''(u_1)du_1\alpha_w(m)dm\psi(x) dx$$

$$= o(g_1^{-6}) \quad (130)$$

since, obviously,

$$\lim_{g_1 \to 0} g_1^6 M_{3}^{1,2,2,2}(g_1) = 0.$$

The next three terms can be handled as the previous one. We omit the details and present their order:

$$M_{4}^{1,2,2,2}(g_1) = o(g_1^{-6}) \quad (131)$$

$$M_{5}^{1,2,2,2}(g_1) = o(g_1^{-6}) \quad (132)$$

and

$$M_{6}^{1,2,2,2}(g_1) = o(g_1^{-6}) \quad (133)$$

The next four terms in (127) can be studied very similarly. Only details about the first one will be given. After using the change of variable $\frac{r - y}{y_1} = y_1$, $\frac{z - x}{s_1} = s_1$, this term becomes

$$M_{7}^{1,2,2,2}(g_1) = 4g_1^{-4} \int_{0}^{\infty} \int_{e'}^{t} z_1(r) \times \left( \int_{-\infty}^{\frac{r}{y_1}} K''(y_1)(p(r - g_1y_1) - p(r))h(r - g_1y_1)dy_1 - g_1^2\zeta(r) \right) dr$$

$$\times \int_{e'}^{t} z_{11}(x + g_1s_1)K''(s_1)ds_1w(t) dt \int_{e'}^{t} \alpha(m)^2\omega(m)dm\psi(x) dx$$

$$= o(g_1^{-4}) \quad (134)$$
since, obviously,

$$\lim_{g_1 \to 0} g_1^4 M_{11}^{1,2,2,2}(g_1) = 0.$$  

Similar results may be proved for the three other terms:

$$M_8^{1,2,2,2}(g_1) = o \left( g_1^{-4} \right),$$  \hfill (135) 

$$M_9^{1,2,2,2}(g_1) = o \left( g_1^{-4} \right),$$  \hfill (136) 

and

$$M_{10}^{1,2,2,2}(g_1) = o \left( g_1^{-4} \right).$$  \hfill (137) 

Finally, the change of variable $\frac{x}{g_1} = y_1$ leads to

$$M_{11}^{1,2,2,2}(g_1) = -8g_1^{-2} \int_0^\infty \int_{\epsilon_1}^t z_1(r) \left( \int_{-\infty}^{\frac{x_1}{g_1}} K_m(y_1) (p(r - g_1y_1) - p(r)) \right)$$

$$\times h(r - g_1y_1)dy_1 - g_1^2 \zeta(r) \ dx_s \omega(t) dt$$

$$\times (\int_{\epsilon_1}^t \alpha(m) w(m) dm) dx.$$

Using this expression is not difficult to prove that

$$\lim_{g_1 \to 0} g_1^2 M_{11}^{1,2,2,2}(g_1) = 0$$

and hence

$$M_{11}^{1,2,2,2}(g_1) = o \left( g_1^{-2} \right).$$  \hfill (138) 

Using (128)-(138) the order of the expectation (127) is $o \left( g_1^{-6} \right)$. On the other hand the product of the expectations is

$$E \left( \int_{\epsilon_1}^t \int_{\epsilon_1}^t z_1(r) \left( K_m^\mu (r - T_1) (\delta_1 - p(r)) + \zeta(r) \right) dr \right)$$

$$\times \int_{\epsilon_1}^t z_1(s) \left( K_m^\mu (s - T_2) (\delta_2 - p(s)) + \zeta(s) \right) ds \omega(t) dt$$

$$\times E \left( \int_{\epsilon_1}^t \int_{\epsilon_1}^t z_1(u) \left( K_m^\mu (u - T_2) (\delta_2 - p(u)) + \zeta(u) \right) du \right)$$

$$\times \int_{\epsilon_1}^t z_1(v) \left( K_m^\mu (v - T_2) (\delta_2 - p(v)) + \zeta(v) \right) dw(m) dm$$

$$= I_{1Z221}(g_1) I_{1Z222}(g_1)$$

whose order is $O(g_1)$, according to (108) and (112). All these orders give for the order of $M_{11}^{1,2,2,2}(g_1)$ the result in (126). \hfill \blacksquare
Lemma 30 Assume the same conditions of Lemma 27. It holds

\[ M^{1,2,1,2}_i(g_1) = o(\delta_1^{-6}) \]  

(139)

Proof. Let’s start by computing the expectation

\[
E \left( \int_{-\infty}^{t} \int_{-\infty}^{t} z_1(r) \left( K''_{g_1}(r - T_1) (\delta_1 - p(r)) - \zeta(r) \right) dr \times \int_{r'}^{t} z_1(s) \left( K''_{g_1}(s - T_2) (\delta_2 - p(s)) - \zeta(s) \right) ds w(t) dt \right.
\]

\[
\times \int_{-\infty}^{m} \int_{-\infty}^{m} z_1(u) \left( K''_{g_1}(u - T_1) (\delta_1 - p(u)) - \zeta(u) \right) du \times \int_{m'}^{m} z_1(v) \left( K''_{g_1}(v - T_2) (\delta_2 - p(v)) - \zeta(v) \right) dv w(m) dm \right)
\]

\[ = \sum_{i=1}^{16} M^{1,2,1,2}_i(g_1) + 16 \left( \int_{-\infty}^{\infty} \alpha(t)^2 w(t) dt \right), \]

(140)

where

\[ M^{1,2,1,2}_1(g_1) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{m} z_{11}(r) K''_{g_1}(r - x) z_{11}(u) K''_{g_1}(u - x) \times \psi(x) dr du dx \]

\[ \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_{12}(s) K''_{g_1}(s - x) z_{12}(v) K''_{g_1}(v - x) \times \phi(y) ds dw dv w(t) w(m) dm dt, \]

\[ M^{1,2,1,2}_2(g_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{t} z_{11}(r) K''_{g_1}(r - x) z_{11}(u) K''_{g_1}(u - x) \psi(x) dr du dx \right) \times \phi(y) ds dw dv w(t) w(m) dm dt \]

\[ \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_{12}(s) K''_{g_1}(s - x) z_{12}(v) K''_{g_1}(v - x) \times \phi(y) ds dw dv w(t) w(m) dm dt, \]

\[ M^{1,2,1,2}_3(g_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{t} z_{11}(r) K''_{g_1}(r - x) z_{11}(u) K''_{g_1}(u - x) \phi(x) dr du dx \right) \times \psi(x) dr du dx \]

\[ \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_{12}(s) K''_{g_1}(s - x) z_{12}(v) K''_{g_1}(v - x) \times \phi(y) ds dw dv w(t) w(m) dm dt, \]

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\[ M_{5}^{1.2.1.2}(g_1) = -8 \int_{c}^{\infty} \int_{c}^{t} \int_{c}^{s} \int_{c}^{m} z_{12}(r)z_{12}(u) \int_{0}^{\infty} K''_{g_1}(r - x)K''_{g_1}(u - x) \times \phi(x)dxdu \alpha_{w}(t) \times \int_{c}^{m} z_{1}(v) \int_{0}^{\infty} K''_{g_1}(v - y) \left( p(y) - p(v) \right) h(y)dydv \alpha_{w}(m)dm dt, \]

\[ M_{6}^{1.2.1.2}(g_1) = 8 \int_{c}^{\infty} \alpha(t)^{2} w(t)dt \int_{c}^{\infty} \int_{c}^{t} \left( \int_{c}^{m} \int_{c}^{\infty} z_{1}(u)K''_{g_1}(u - x) \left( p(x) - p(u) \right) h(x)dxdu \right)^{2} \times \alpha_{w}(m)dm, \]

\[ M_{7}^{1.2.1.2}(g_1) = 8 \left( \int_{c}^{\infty} \int_{c}^{t} \int_{0}^{\infty} z_{1}(r)K''_{g_1}(r - x) \left( p(x) - p(r) \right) h(x)\alpha_{w}(t)dxdr \right)^{2}, \]

\[ M_{8}^{1.2.1.2}(g_1) = 8 \int_{c}^{\infty} \int_{c}^{t} \int_{c}^{s} \int_{c}^{m} z_{11}(r)z_{11}(u) \times \int_{0}^{\infty} K''_{g_1}(r - x)K''_{g_1}(u - x)\psi(x)dx \alpha_{w}(t)\alpha_{w}(m)drdu \times \alpha_{w}(m)drdudm, \]

\[ M_{9}^{1.2.1.2}(g_1) = 8 \int_{c}^{\infty} \int_{c}^{t} \int_{c}^{s} \int_{c}^{m} z_{12}(r)z_{12}(u) \times \int_{0}^{\infty} K''_{g_1}(r - x)K''_{g_1}(u - x)\phi(x)dx \alpha_{w}(t)\alpha_{w}(m)drdu \times \alpha_{w}(m)drdudm, \]

and

\[ M_{10}^{1.2.1.2}(g_1) = -32 \int_{c}^{\infty} \int_{c}^{t} \int_{c}^{s} \int_{c}^{m} z_{1}(r)K''_{g_1}(r - x) \left( p(x) - p(r) \right) \times \alpha_{w}(t)\alpha(m)^{2} w(m)drdtdm. \]

The term \( M_{10}^{1.2.1.2}(g_1) \) may be rewritten with different integration limits taking into account that the functions \( z_{11} \) and \( \phi \) vanish in \([0, \varepsilon]\).

\[ M_{1}^{1.2.1.2}(g_1) = 2 \int_{c}^{\infty} \int_{c}^{t} \int_{c}^{s} \int_{c}^{m} z_{11}(r)K''_{g_1}(r - x)z_{11}(u)K''_{g_1}(u - x) \times \psi(x)drdudx \times \int_{c}^{m} \int_{c}^{t} \int_{c}^{s} z_{12}(s)K''_{g_1}(s - x)z_{12}(v)K''_{g_1}(v - x)\phi(y)dsdvdy \times w(t)w(m)drdtdm. \]

The change of variable \( \frac{r-x}{g_1} = r_1, \frac{u-x}{g_1} = u_1, \frac{s-x}{g_1} = s_1, \frac{v-x}{g_1} = v \), leads to
\[ M_{1,1,2}^{1,2,1,2}(g_1) = 2g_1^{-8} \int_{\varepsilon}^{\infty} \int_{-L}^{L} \int_{0}^{\infty} \int_{0}^{\infty} z_{11}(x + g_1 v_1)K''(v_1) \times z_{11}(x + g_1 u_1)K''(u_1) \phi(x) dx \] 
\[ \times \int_{\varepsilon}^{\infty} \int_{-L}^{L} \int_{0}^{\infty} \int_{0}^{\infty} z_{12}(y + g_1 s_1)K''(s_1) z_{12}(y + g_1 v_1) \times K''(v_1) \phi(y) ds_1 dv_1 dy \] 

or, assuming \( g_1 < \frac{\varepsilon}{2} \),

\[ M_{1,1,2}^{1,2,1,2}(g_1) = 2g_1^{-8} \int_{\varepsilon}^{\infty} \int_{-L}^{L} \int_{0}^{\infty} \int_{0}^{\infty} z_{11}(x + g_1 v_1)K''(v_1) \times \int_{\varepsilon}^{\infty} \int_{-L}^{L} \int_{0}^{\infty} \int_{0}^{\infty} z_{12}(y + g_1 s_1)K''(s_1) z_{12}(y + g_1 v_1) \times K''(v_1) \phi(y) ds_1 dv_1 dy \] 

where, for every \( t, m > \varepsilon \) we have defined

\[ M_{1,1,1}^{1,2,1,2}(g_1, t, m) = M_{1,1,2}^{1,2,1,2}(g_1, t, m) M_{1,1,2}^{1,2,1,2}(g_1, t, m) w(t) w(m) dm dt \]

and

\[ M_{1,2,1,2}^{1,2,1,2}(g_1, t, m) = \frac{\varepsilon}{8} \int_{-L}^{L} \int_{-L}^{L} z_{12}(y + g_1 s_1)K''(s_1) z_{12}(y + g_1 v_1) K''(v_1) \times \phi(y) ds_1 dv_1 dy. \]

Obviously,

\[ \lim_{g_1 \to 0} M_{1,1,1}^{1,2,1,2}(g_1, t, m) = \left( \int_{-L}^{L} K''(r_1) dr_1 \right)^2 \int_{\varepsilon}^{t \wedge m} z_{11}(x)^2 \phi(x) dx = 0 \]

and

\[ \lim_{g_1 \to 0} M_{1,2,1,2}^{1,2,1,2}(g_1, t, m) = \left( \int_{-L}^{L} K''(s_1) ds_1 \right)^2 \int_{\varepsilon}^{t \wedge m} z_{12}(y)^2 \phi(y) dy = 0. \]

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On the other hand the partial derivative of $M_{1,1}^{1,2,1,2}(g_1, t, m)$ with respect to $g_1$ is

$$
\frac{\partial}{\partial g_1} M_{1,1}^{1,2,1,2}(g_1, t, m) = \int_{0}^{\infty} \psi(x) \left( -z_{11}(t)K'' \left( \frac{t-x}{g_1^2} \right) \frac{t-x}{g_1^2} \right)
+ z_{11}(\varepsilon) K'' \left( \frac{\varepsilon-x}{g_1^2} \right) \frac{\varepsilon-x}{g_1^2}
+ \int_{\varepsilon-\delta}^{\varepsilon+\delta} z_{11}(x+g_1r_1)K''(r_1) dr_1
\times \int_{\varepsilon-\delta}^{\varepsilon+\delta} z_{11}(x+g_1u_1)K''(u_1) du_1 dx
+ \int_{0}^{\infty} \psi(x) \left( -z_{11}(m)K'' \left( \frac{m-x}{g_1^2} \right) \frac{m-x}{g_1^2} \right)
+ z_{11}(\varepsilon) K'' \left( \frac{\varepsilon-x}{g_1^2} \right) \frac{\varepsilon-x}{g_1^2}
+ \int_{\varepsilon-\delta}^{\varepsilon+\delta} z_{11}(x+g_1r_1)K''(r_1) dr_1
\times \int_{\varepsilon-\delta}^{\varepsilon+\delta} z_{11}(x+g_1u_1)K''(u_1) du_1 dx
= - \int_{-L}^{\varepsilon} \psi(t-g_1x_1)z_{11}(t)K''(x_1) x_1
\times \int_{\varepsilon-x_1}^{\varepsilon+x_1} z_{11}(t+g_1(u_1-x_1))K''(u_1) du_1 dx x_1
- \int_{-L}^{\varepsilon} \psi(m-g_1x_1)z_{11}(m)K''(x_1) x_1
\times \int_{\varepsilon-x_1}^{\varepsilon+x_1} z_{11}(m+g_1(u_1-x_1))K''(u_1) du_1 dx x_1
+ \int_{0}^{\infty} \psi(x) \int_{\varepsilon-\delta}^{\varepsilon+\delta} z_{11}'(x+g_1r_1)r_1K''(r_1) dr_1
\times \int_{\varepsilon-\delta}^{\varepsilon+\delta} z_{11}(x+g_1u_1)K''(u_1) du_1 dx
+ \int_{0}^{\infty} \psi(x) \int_{\varepsilon-\delta}^{\varepsilon+\delta} z_{11}'(x+g_1r_1)r_1K''(r_1) dr_1
\times \int_{\varepsilon-\delta}^{\varepsilon+\delta} z_{11}(x+g_1u_1)K''(u_1) du_1 dx
$$
where the whole expression has been decomposed in several summands and the changes of variable \( \frac{t-x}{g_1} = x_1 \) and \( \frac{m-x}{g_1} = x_1 \) have been used for the first two.

Similarly, the partial derivative of \( M_{1,2}^{1,2,1,2}(g_1,t,m) \) with respect to \( g_1 \) is

\[
\frac{\partial}{\partial g_1} M_{1,2}^{1,2,1,2}(g_1,t,m) = \int_{-\infty}^{\infty} \phi(y) \left( -z_{12}(t)K'' \left( \frac{t-y}{g_1} \right) \frac{t-y}{g_1^2} \right) dy
\]

\[
+ \int_{-L}^{\infty} \int_{-L}^{m-y} z_{12}'(y + g_1 r_1) K''(r_1) dr_1 dy
\]

\[
\times \int_{-L}^{m-y} z_{12}(y + g_1 u_1) K''(u_1) du_1 dy
\]

\[
+ \int_{\epsilon}^{\infty} \phi(y) \left( -z_{12}(m)K'' \left( \frac{m-y}{g_1} \right) \frac{m-y}{g_1^2} \right) dy
\]

\[
+ \int_{-L}^{m-y} z_{12}'(y + g_1 r_1) K''(r_1) dr_1
\]

\[
\times \int_{-L}^{m-y} z_{12}(y + g_1 u_1) K''(u_1) du_1 dy
\]

\[
= - \int_{-L}^{m-y} \phi(t - g_1 y_1) z_{12}(t)K''(y_1) y_1 dy_1
\]

\[
\times \int_{-L}^{m-y} z_{12}(t + g_1 (v_1 - y_1)) K''(v_1) dv_1 dy_1
\]

\[
- \int_{-L}^{m-y} \phi(m - g_1 y_1) z_{12}(m)K''(y_1) y_1 dy_1
\]

\[
\times \int_{-L}^{m-y} z_{12}(m + g_1 (v_1 - y_1)) K''(v_1) dv_1 dy_1
\]

\[
+ \int_{\epsilon}^{\infty} \phi(y) \int_{-L}^{m-y} z_{12}'(y + g_1 s_1) K''(s_1) ds_1 dy
\]

\[
\times \int_{-L}^{m-y} z_{12}(y + g_1 v_1) K''(v_1) dv_1 dy
\]

\[
+ \int_{\epsilon}^{\infty} \phi(y) \int_{-L}^{m-y} z_{12}'(y + g_1 s_1) K''(s_1) ds_1
\]

\[
\times \int_{-L}^{m-y} z_{12}(y + g_1 v_1) K''(v_1) dv_1 dy.
\]

When letting \( g_1 \) tend to zero, the limits of these derivatives are, respectively,\[89\]
\[
\lim_{g_1 \to 0} \frac{\partial}{\partial g_1} M_{1,1}^{1,2,1,2}(g_1, t, m) = - \int_{-L}^{L} K''(x_1) x_1 dx_1 \int_{-L}^{L} K''(u_1) du_1 \\
\times (1(m > t)\psi(t)z_{11}(t)^2 + 1(t > m)\psi(m)z_{11}(m)^2) \\
+ 2 \int_{-L}^{L} r_1 K''(r_1) dr_1 \int_{-L}^{L} K''(u_1) du_1 \\
\times \int_{t \wedge m}^{t \vee m} \psi(x)z'_{11}(x)z_{11}(x) dx = 0
\]

and

\[
\lim_{g_1 \to 0} \frac{\partial}{\partial g_1} M_{1,2}^{1,2,1,2}(g_1, t, m) = - \int_{-L}^{L} K''(y_1) y_1 dy_1 \int_{-L}^{L} K''(v_1) dv_1 \\
\times (1(m > t)\phi(t)z_{12}(t)^2 + 1(t > m)\phi(m)z_{12}(m)^2) \\
+ 2 \int_{-L}^{L} s_1 K''(s_1) ds_1 \int_{-L}^{L} K''(v_1) dv_1 \\
\times \int_{t \wedge m}^{t \vee m} \phi(y)z'_{12}(y)z_{12}(y) dy = 0.
\]

Therefore, applying Taylor formula for \( M_{1,1}^{1,2,1,2}(g_1, t, m) \) and \( M_{1,2}^{1,2,1,2}(g_1, t, m) \)
as functions of \( g_1 \) gives

\[
M_{1,1}^{1,2,1,2}(g_1, t, m) = o(g_1)
\]

and

\[
M_{1,2}^{1,2,1,2}(g_1, t, m) = o(g_1).
\]

Using the Dominated Convergence Theorem,

\[
\lim_{g_1 \to 0} 6\gamma^6 M_{1}^{1,2,1,2}(g_1) = 0,
\]
or equivalently

\[
M_{1}^{1,2,1,2}(g_1) = o (g_1^{-6}). \tag{141}
\]

The same order remains valid for the terms \( M_{2}^{1,2,1,2}(g_1) \) and \( M_{3}^{1,2,1,2}(g_1) \). This may be checked by a direct inspection of these terms and a comparison with \( M_{1}^{1,2,1,2}(g_1) \). Therefore,

\[
M_{2}^{1,2,1,2}(g_1) = o (g_1^{-6}) \tag{142}
\]
and

\[ M_3^{1.2.1.2}(g_1) = o \left( g_1^{-6} \right) \]  \hspace{1cm} (143)

The next term in representation (140) can be analyzed by using the change of variable \( \frac{u - x}{g_1} = r_1, \frac{u - z}{g_1} = u_1, \frac{u - z}{g_1} = y_1 \).

\[ M_4^{1.2.1.2}(g_1) = -8g_1^{-6} \int_{c'}^{\infty} \int_{c'}^{\infty} \int_0^{\infty} \int_0^{\infty} z_{11}(x + g_1r_1)z_{11}(x + g_1u_1) \]
\[ \times K''(r_1)K''(u_1)\psi(x)du_1dr_1dx_1} \]
\[ \times \int_{c'}^{\infty} z_1(v) \int_{\infty}^{\infty} K''(y_1) (p(v - g_1y_1) - p(v)) h(v - g_1y_1) \]
\[ \times dy_1dw(m)dm \]

and, obviously,

\[ M_4^{1.2.1.2}(g_1) = o \left( g_1^{-6} \right) \]  \hspace{1cm} (144)

In a similar way,

\[ M_5^{1.2.1.2}(g_1) = o \left( g_1^{-6} \right) \]  \hspace{1cm} (145)

since \( M_5^{1.2.1.2}(g_1) \) can be obtained from \( M_4^{1.2.1.2}(g_1) \) by simply replacing \( z_{11} \) and \( \psi \) by \( z_{12} \) and \( \phi \), respectively. These modifications do not affect the arguments used to derive the order in the previous case.

To deal with the term \( M_6^{1.2.1.2}(g_1) \) the change of variable \( \frac{u - x}{g_1} = x_1 \) applies to obtain

\[ M_6^{1.2.1.2}(g_1) = 8g_1^{-4} \int_{c'}^{\infty} \alpha(t)^2w(t)dt \]
\[ \times \int_{c'}^{\infty} \left( \int_{c'}^{\infty} \int_{\infty}^{\infty} z_1(u)K''(x_1) (p(u - g_1x_1) - p(u)) \right. \]
\[ \times h(u - g_1x_1)dx_1du \] \( w(m)dm \]

and, hence,

\[ M_6^{1.2.1.2}(g_1) = o \left( g_1^{-4} \right) \]  \hspace{1cm} (146)

Similarly, the change of variable \( \frac{u - x}{g_1} = x_1 \)

\[ M_7^{1.2.1.2}(g_1) = 8g_1^{-4} \int_{c'}^{\infty} \int_{c'}^{\infty} \int_{\infty}^{\infty} z_1(r)K''(x_1) (p(r - g_1x_1) - p(r)) \]
\[ \times h(r - g_1x_1)\alpha_u(t)dx_1drdt \]

\[ 91 \]
which implies

$$M^{1,2,1,2}_7(g_1) = o(g_1^{-4}). \quad (147)$$

The next term is, after using the change of variable \( \frac{t-x}{g_1} = r_1, \frac{u-x}{g_1} = u_1. \)

$$M^{1,2,1,2}_8(g_1) = 8g_1^{-4} \int_{c'}^{\infty} \int_{c'}^{\infty} \int_{l}^{\infty} \int_{\frac{x}{g_1}}^{\frac{u}{g_1}} \int_{\frac{w}{g_1}}^{\frac{v}{g_1}} z_11(x + g_1r_1)z_11(x + g_1u_1)$$
$$\times K''(r_1)K''(u_1)\psi(x)d\alpha_w(t)d\alpha_w(m)dr_1du_1dm dt$$

which implies

$$M^{1,2,1,2}_8(g_1) = o(g_1^{-4}). \quad (148)$$

A completely similar result is valid for \( M^{1,2,1,2}_9(g_1): \)

$$M^{1,2,1,2}_9(g_1) = o(g_1^{-4}). \quad (149)$$

Finally, the change of variable \( \frac{t-x}{g_1} = r_1 \) gives

$$M^{1,2,1,2}_{10}(g_1) = -32g_1^{-2} \int_{c'}^{\infty} \int_{c'}^{\infty} \int_{\frac{x}{g_1}}^{\frac{u}{g_1}} \int_{\frac{w}{g_1}}^{\frac{v}{g_1}} z_1(x + g_1r_1)K''(r_1)(p(x + g_1r_1) - p(r))$$
$$\times d\alpha_w(t)d\alpha_w(m)dr_1du_1dm$$

and then it is immediate to check that

$$M^{1,2,1,2}_{10}(g_1) = o(g_1^{-2}). \quad (150)$$

Using expressions (141)-(150) in the expectation (140) gives the order \( o(g^{-6}). \)

To end with the proof, which states that this is also the order of \( M^{1,2,1,2}(g_1), \) it only remains to deal with the product of the expectations:

$$E \left( \int_{c'}^{\infty} \int_{c'}^{t} z_1(r) \left( K''_{g_1}(r - T_1)(\delta_1 - p(r)) - \zeta(r) \right) dr \right)$$
$$\times \left( \int_{c'}^{\infty} z_1(s) \left( K''_{g_1}(s - T_2)(\delta_2 - p(s)) - \zeta(s) \right) dsw(t)dt \right)$$
$$\times E \left( \int_{c'}^{\infty} \int_{c'}^{m} z_1(u) \left( K''_{g_1}(u - T_1)(\delta_1 - p(u)) - \zeta(u) \right) du \right)$$
$$\times \left( \int_{c'}^{m} z_1(v) \left( K''_{g_1}(v - T_2)(\delta_2 - p(v)) - \zeta(v) \right) dw(m)dm \right)$$
$$= I_{1221}(g_1)^2$$

which, using (108), may be proved to be \( O(g_1^8). \)
Lemma 31 Under the assumptions of Lemma 27,

\[ M^{1,1,1}(g_1) = O \left( g_1^{-8} \right). \]  

(151)

Proof. We start by computing the expectation

\[
E \left( \int_{e'}^\infty z_1(r) \left( K''_{g_1} (r-T_1)(\delta_1 - p(r)) - \zeta(r) \right) dr \right. \\
\times \int_{e'}^t z_1(s) \left( K''_{g_1} (s-T_1)(\delta_1 - p(s)) - \zeta(s) \right) ds \, dt \\
\times \int_{e'}^\infty \int_{e'}^m z_1(u) \left( K''_{g_1} (u-T_1)(\delta_1 - p(u)) - \zeta(u) \right) du \\
\times \left. \int_{e'}^m z_1(v) \left( K''_{g_1} (v-T_1)(\delta_1 - p(v)) - \zeta(v) \right) dv \, dm \right). 
\]

which, after some standard algebra, leads to

\[
\int_0^\infty \left( \int_{e'}^\infty \left( \int_{e'}^t z_{11}(r) K''_{g_1} (r-x) \, dr \right) \right)^2 w(t) \, dt \right)^2 \psi(x) \, dx \\
+ \int_0^\infty \left( \int_{e'}^\infty \left( \int_{e'}^t z_{12}(r) K''_{g_1} (r-x) \, dr \right) \right)^2 w(t) \, dt \right)^2 \phi(x) \, dx \\
-4 \int_0^\infty \left( \int_{e'}^\infty \int_{e'}^m z_{11}(u) K''_{g_1} (u-x) \, du \, dm \right) \psi(x) \, dx \\
+4 \int_0^\infty \left( \int_{e'}^\infty \int_{e'}^m z_{12}(u) K''_{g_1} (u-x) \, du \, dm \right) \phi(x) \, dx
\]
+8 \int_{\varepsilon_1}^{\infty} \alpha(m)^2w(m)dm \int_{0}^{\infty} \int_{\varepsilon_1'}^{\infty} \left( \int_{\varepsilon_1'}^{t} z_{11}(r)K''_{11}(r-x)dr \right)^2 w(t)dt \psi(x)dx

+8 \int_{\varepsilon_1}^{\infty} \alpha(m)^2w(m)dm \int_{0}^{\infty} \int_{\varepsilon_1'}^{\infty} \left( \int_{\varepsilon_1'}^{t} z_{12}(r)K''_{11}(r-x)dr \right)^2 w(t)dt \phi(x)dx

+4 \int_{0}^{\infty} \left( \int_{\varepsilon_1'}^{t} z_{11}(r)K''_{11}(r-x)dr \alpha_w(t) \right) \psi(x)dx

+4 \int_{0}^{\infty} \left( \int_{\varepsilon_1'}^{t} z_{12}(r)K''_{11}(r-x)dr \alpha_w(t) \right) \phi(x)dx

-16 \int_{\varepsilon_1}^{\infty} \alpha(m)^2w(m)dm \int_{0}^{\infty} \int_{\varepsilon_1'}^{t} z_{1}(r)K''_{11}(r-x) (p(x)-p(r)) dr \times \alpha_w(t) dt h(x)dx

+16 \left( \int_{\varepsilon_1}^{\infty} \alpha(t)^2w(t)dt \right)^2 .

Using the changes of variable \( \frac{r-\varepsilon}{\varepsilon_1} = r_1 \) or \( \frac{r-\varepsilon}{\varepsilon_1} = r_1 \), \( \frac{r-\varepsilon}{\varepsilon_1} = u_1 \), the previous expression becomes

\[ g_1^{-8} \int_{0}^{\infty} \left( \int_{\varepsilon_1'}^{t} \left( \int_{\frac{r-\varepsilon}{\varepsilon_1_1}}^{\frac{r-\varepsilon}{\varepsilon_1_1}} z_{11}(x + g_1r_1)K''_{11}(r_1)dr_1 \right)^2 w(t)dt \right)^2 \psi(x)dx \]

\[ + g_1^{-8} \int_{0}^{\infty} \left( \int_{\varepsilon_1'}^{t} \left( \int_{\frac{r-\varepsilon}{\varepsilon_1_1}}^{\frac{r-\varepsilon}{\varepsilon_1_1}} z_{12}(x + g_1r_1)K''_{11}(r_1)dr_1 \right)^2 w(t)dt \right)^2 \phi(x)dx \]

\[ -4g_1^{-6} \int_{0}^{\infty} \int_{\varepsilon_1'}^{t} \left( \int_{\frac{r-\varepsilon}{\varepsilon_1_1}}^{\frac{r-\varepsilon}{\varepsilon_1_1}} z_{11}(x + g_1u_1)K''_{11}(u_1)du_1 \alpha_w(m) dm \psi(x)dx \]

\[ +4g_1^{-6} \int_{0}^{\infty} \int_{\varepsilon_1'}^{t} \left( \int_{\frac{r-\varepsilon}{\varepsilon_1_1}}^{\frac{r-\varepsilon}{\varepsilon_1_1}} z_{12}(x + g_1u_1)K''_{11}(u_1)du_1 \alpha_w(m) dm \phi(x)dx \]
Assume the conditions of Lemma 23. Then, it holds

\[ \text{Lemma 32} \]

Now (151) becomes a direct consequence of (152) and (153).

\[ +8g_1^{-4} \int_{\epsilon'}^{\infty} \alpha(m)^2 w(m)dm \int_0^{\infty} \int_{\epsilon'}^{\infty} \left( \int_{g_1 r_1}^{x} z_{11}(x + g_1 r_1)K''(r_1)dr_1 \right)^2 \times w(t)dt \psi(x)dx \]

\[ +8g_1^{-4} \int_{\epsilon'}^{\infty} \alpha(m)^2 w(m)dm \int_0^{\infty} \int_{\epsilon'}^{\infty} \left( \int_{g_1 r_1}^{x} z_{12}(x + g_1 r_1)K''(r_1)dr_1 \right)^2 \times w(t)dt \phi(x)dx \]

\[ +4g_1^{-4} \int_0^{\infty} \left( \int_{\epsilon'}^{\infty} \left[ \int_{g_1 r_1}^{x} z_{11}(x + g_1 r_1)K''(r_1)dr_1 \right] \alpha_w(t)dt \right)^2 \psi(x)dx \]

\[ +4g_1^{-4} \int_0^{\infty} \left( \int_{\epsilon'}^{\infty} \left[ \int_{g_1 r_1}^{x} z_{11}(x + g_1 r_1)K''(r_1)dr_1 \right] \alpha_w(t)dt \right)^2 \phi(x)dx \]

\[ -16g_1^{-2} \int_{\epsilon'}^{\infty} \alpha(m)^2 w(m)dm \]

\[ \times \int_0^{\infty} \int_{\epsilon'}^{\infty} \int_{\epsilon'}^{t} z_1(r)K''(r - x)(p(x) - p(r))dr \alpha_w(t)dt \phi(t)dx \]

\[ +16 \left( \int_{\epsilon'}^{\infty} \alpha(t)^2 w(t)dt \right)^2 \]

\[ = O(g_1^{-8}). \tag{152} \]

On the other hand, according to (112), the product of the expectations is

\[ E \left( \int_{\epsilon'}^{\infty} \int_{\epsilon'}^{t} z_1(r) \left( K''(r - T_1) (\delta_1 - p(r)) - \zeta(r) \right) dr \times \int_{\epsilon'}^{\infty} z_1(s) \left( K''(s - T_1) (\delta_1 - p(s)) - \zeta(s) \right) ds \right) \]

\[ \times E \left( \int_{\epsilon'}^{\infty} \int_{\epsilon'}^{m} z_1(u) \left( K''(u - T_1) (\delta_1 - p(u)) - \zeta(u) \right) du \times \int_{\epsilon'}^{m} z_1(v) \left( K''(v - T_1) (\delta_1 - p(v)) - \zeta(v) \right) dv \right) \]

\[ = I_{1222}(g_1)^2 \]

\[ = O(g_1^{-6}). \tag{153} \]

Now (151) becomes a direct consequence of (152) and (153). \[ \square \]

**Lemma 32** Assume the conditions of Lemma 23. Then, it holds

\[ \text{Var} \left( \hat{A}_{121} \right) = o(n^{-1}). \tag{154} \]
Proof. The variance in the lemma is

$$Var \left( \hat{A}_{121} \right) = \frac{1}{4n^4} Var \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{t'}^t z_2(r) K_{g_1} (r - T_i) \left( \delta_i - p(r) \right) dr \right)$$

$$= \frac{1}{4n^4} \left( n N^{1,1,1,1}_1 (g_1) + 2n(n-1)N^{1,2,2,2}_1 (g_1) + 2n(n-1)N^{1,2,1,1}_1 (g_1) + n(n-1)N^{1,1,2,1}_1 (g_1) + n(n-1)(n-2)N^{1,2,1,3}_1 (g_1) + n(n-1)(n-2)N^{1,2,2,3}_1 (g_1) + n(n-1)(n-2)N^{1,2,3,2}_1 (g_1) \right),$$

where

$$N^{i,j,k,l}_1 (g_1) = Cov \left( \int_{t'}^t z_2(r) K_{g_1} (r - T_i) \left( \delta_i - p(r) \right) dr \right) \times \int_{t'}^t \int_{t'}^m \int_{t'}^n \int_{t'}^s z_2(s) K_{g_1} (s - T_j) \left( \delta_j - p(s) \right) ds \times \int_{t'}^m \int_{t'}^n \int_{t'}^s z_2(s) K_{g_1} (s - T_k) \left( \delta_k - p(s) \right) ds \times \int_{t'}^n \int_{t'}^m \int_{t'}^s z_2(s) K_{g_1} (s - T_l) \left( \delta_l - p(s) \right) ds \times \int_{t'}^m \int_{t'}^n \int_{t'}^s z_2(s) K_{g_1} (s - T_m) \left( \delta_m - p(s) \right) ds \times \int_{t'}^n \int_{t'}^m \int_{t'}^s z_2(s) K_{g_1} (s - T_n) \left( \delta_n - p(s) \right) ds \times \int_{t'}^m \int_{t'}^n \int_{t'}^s z_2(s) K_{g_1} (s - T_p) \left( \delta_p - p(s) \right) ds.$$ 

As discussed when dealing with $Var \left( \hat{A}_{122} \right)$, it suffices to study the four following terms: $N^{1,2,1,3}_1 (g_1)$, $N^{1,2,2,2}_1 (g_1)$, $N^{1,2,1,2}_1 (g_1)$ and $N^{1,1,1,1}_1 (g_1)$. The thesis of the lemma is a consequence of the following results

$$N^{1,2,1,3}_1 (g_1) = o(1), \quad N^{1,2,2,2}_1 (g_1) = O (g_1^{-4}), \quad N^{1,2,1,2}_1 (g_1) = O (1), \quad N^{1,1,1,1}_1 (g_1) = O (g_1^{-4})$$

included below as Lemma 33, and condition (V.1), that ensures $ng_1^2 \to \infty$. \[ \]  

**Lemma 33** Under the assumptions of Lemma 32,

$$N^{1,2,1,3}_1 (g_1) = o(1), \quad (155)$$

$$N^{1,2,2,2}_1 (g_1) = O (g_1^{-4}), \quad (156)$$

and

$$N^{1,2,1,2}_1 (g_1) = O (1), \quad (157)$$

$$N^{1,1,1,1}_1 (g_1) = O (g_1^{-4}). \quad (158)$$
Proof. Let’s start the study of $N^{1,2,1,3}(g_1)$ by analyzing its expectation

$$
E \left( \int_{e}^{\infty} \int_{e}^{t} z_2(r)K_{g_1}(r-T_1)(\delta_1 - p(r)) \, dr \int_{e}^{t} z_2(s)K_{g_1}(s-T_2)(\delta_2 - p(s)) \, ds \right) \, dt
\times \int_{e}^{\infty} \int_{e}^{m} z_2(u)K_{g_1}(u-T_1)(\delta_1 - p(u)) \, du \int_{e}^{m} z_2(v)K_{g_1}(v-T_3)(\delta_3 - p(v)) \, dv \, dm \, dm
$$

$$
= N_1^{1,2,1,3}(g_1) + N_2^{1,2,1,3}(g_1)
$$

where

$$
N_1^{1,2,1,3}(g_1) = \int_{0}^{\infty} \int_{e}^{\infty} \int_{e}^{t} z_2(r)K_{g_1}(r-x) \, dr
\times \int_{e}^{t} z_2(s) \int_{0}^{\infty} K_{g_1}(s-y)(p(y) - p(s)) \, dy \, ds \, dt
\times \int_{e}^{\infty} \int_{e}^{m} z_2(u)K_{g_1}(u-x) \, du
\times \int_{e}^{m} z_2(v) \int_{0}^{\infty} K_{g_1}(v-\eta)(p(\eta) - p(v)) \, h(\eta) \, dy \, dv \, dm \, \phi(x) \, dx
$$

and

$$
N_2^{1,2,1,3}(g_1) = \int_{0}^{\infty} \int_{e}^{\infty} \int_{e}^{t} z_2(r)K_{g_1}(r-x) \, dr
\times \int_{e}^{t} z_2(s) \int_{0}^{\infty} K_{g_1}(s-y)(p(y) - p(s)) \, dy \, ds \, dt
\times \int_{e}^{\infty} \int_{e}^{m} z_2(u)K_{g_1}(u-x) \, du
\times \int_{e}^{m} z_2(v) \int_{0}^{\infty} K_{g_1}(v-\eta)(p(\eta) - p(v)) \, h(\eta) \, dy \, dv \, dm \, \psi(x) \, dx
$$

The change of variable $r_1 = \frac{r-x}{g_1}$, $y_1 = \frac{s-x}{g_1}$, $u_1 = \frac{u-x}{g_1}$, $\eta_1 = \frac{v-x}{g_1}$, implies now:

$$
N_1^{1,2,1,3}(g_1) = \int_{0}^{\infty} \int_{e}^{\infty} \int_{e}^{t} \int_{-\infty}^{\frac{r-x}{g_1}} z_2(x + g_1y_1)K'(r_1) \, dr_1
\times \int_{e}^{t} z_2(s) \int_{-\infty}^{\frac{s-x}{g_1}} K(y_1)(p(s-g_1y_1) - p(s)) \, h(s-g_1y_1) \, dy_1 \, ds \, dt
\times \int_{0}^{\infty} \int_{-\infty}^{\frac{u-x}{g_1}} z_2(u + g_1u_1)K'(u_1) \, du_1
\times \int_{e}^{m} z_2(v) \int_{-\infty}^{\frac{v-x}{g_1}} K(\eta_1)(p(v-g_1\eta_1) - p(v)) \, h(v-g_1\eta_1) \, dy_1 \, dv \, dm \, \phi(x) \, dx
$$

$$
= o(1).
$$

Obviously, $N_2^{1,2,1,3}(g_1)$ has the same asymptotic order.
Now, since the order of the product of expectations

\[
E \left( \int_{\varepsilon'}^\infty \int_{\varepsilon'}^t z_2(r)K_{g_1}(r - T_1)(\delta_1 - p(r)) \, dr \int_{\varepsilon'}^t z_2(s)K_{g_1}(s - T_2)(\delta_2 - p(s)) \, dw(t)dt \right) \\
\times E \left( \int_{\varepsilon'}^\infty \int_{\varepsilon'}^m z_2(u)K_{g_1}(u - T_1)(\delta_1 - p(u)) \, du \int_{\varepsilon'}^m z_2(v)K_{g_1}(v - T_3)(\delta_3 - p(v)) \, dw(m)dm \right)
\]

\[
= I_{1211}(g_1)^2
\]

is \(O(g_1^0)\), according to (114), we conclude (155).

For \(N_{1222}^1(g_1)\), the expectation of the product becomes

\[
E \left( \int_{\varepsilon'}^\infty \int_{\varepsilon'}^t z_2(r)K_{g_1}(r - T_1)(\delta_1 - p(r)) \, dr \right) \\
\times \int_{\varepsilon'}^\infty \int_{\varepsilon'}^m z_2(s)K_{g_1}(s - T_2)(\delta_2 - p(s)) \, dw(t)dt \\
\times \int_{\varepsilon'}^\infty \int_{\varepsilon'}^m z_2(u)K_{g_1}(u - T_1)(\delta_1 - p(u)) \, du \\
\times \int_{\varepsilon'}^m z_2(v)K_{g_1}(v - T_3)(\delta_3 - p(v)) \, dw(m)dm
\]

\[
= N_{1222}^1(g_1) + N_{1222}^2(g_1),
\]

where

\[
N_{1222}^1(g_1) = \int_0^\infty \int_{\varepsilon'}^\infty \int_{\varepsilon'}^t z_2(r) \int_0^\infty K_{g_1}(r - x)(p(x) - p(r)) \, h(x) \, dx \, dr \\
\times \int_{\varepsilon'}^\infty \int_{\varepsilon'}^m z_2(s)K_{g_1}(s - y) \, ds(\theta) \, d\theta dt \\
\times \int_{\varepsilon'}^\infty \int_{\varepsilon'}^m z_2(u)K_{g_1}(u - y) \, du \\
\times \int_{\varepsilon'}^m z_2(v)K_{g_1}(v - y) \, dw(m) \, dm \psi(y)dy
\]

and

\[
N_{1222}^2(g_1) = -\int_0^\infty \int_{\varepsilon'}^\infty \int_{\varepsilon'}^t z_2(r) \int_0^\infty K_{g_1}(r - x)(p(x) - p(r)) \, h(x) \, dxdr \\
\times \int_{\varepsilon'}^\infty \int_{\varepsilon'}^m z_2(s)K_{g_1}(s - y) \, ds(\theta) \, d\theta dt \\
\times \int_{\varepsilon'}^\infty \int_{\varepsilon'}^m z_2(u)K_{g_1}(u - y) \, du \\
\times \int_{\varepsilon'}^m z_2(v)K_{g_1}(v - y) \, dw(m) \, dm \phi(y)dy.
\]

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Now, the change of variable \( \frac{\alpha}{g_1} = x_1, \frac{\alpha}{g_1} = s_1, \frac{\alpha}{g_1} = u_1, \frac{\alpha}{g_1} = v_1 \) and the conditions \( g_1 < \frac{\alpha}{\theta} \) imply

\[
N_{1}^{1,2,2}(g_1) = \int_{0}^{\infty} \int_{c}^{\infty} \int_{c}^{t} z_2(r) \int_{-L}^{L} K(x_1) (p(r - g_1 x_1) - p(r)) h(r - g_1 x_1) dx_1 dr
\]
\[
\times \int_{c}^{\infty} \int_{\frac{m-x}{g_1}}^{\infty} z_2(y + g_1 s_1) K(s_1) ds_1 w(t) dt
\]
\[
\times \int_{c}^{\infty} \int_{\frac{m-x}{g_1}} \int_{\frac{m-x}{g_1}} \int_{\frac{m-x}{g_1}} z_2(y + g_1 u_1) K(u_1) du_1
\]
\[
\times \int_{c}^{\infty} \int_{\frac{m-x}{g_1}} z_2(y + g_1 u_1) K(v_1) dv_1 w(m) dm \psi(y) dy
\]

which, after Taylor expansions of \( p(r - g_1 x_1) - p(r) \) and \( h(r - g_1 x_1) \) around \( r \), becomes

\[
N_{1}^{1,2,2}(g_1) = g_1^2 \mu K \int_{0}^{\infty} \int_{c}^{\infty} \int_{c}^{t} z_1(r)
\]
\[
\times \left( \frac{1}{2} h(r)p''(r) + h'(r)p'(r) \right) dr
\]
\[
\times \int_{c}^{\infty} \int_{\frac{m-x}{g_1}} z_2(y + g_1 s_1) K(s_1) ds_1 w(t) dt
\]
\[
\times \int_{c}^{\infty} \int_{\frac{m-x}{g_1}} \int_{\frac{m-x}{g_1}} \int_{\frac{m-x}{g_1}} z_2(y + g_1 u_1) K(u_1) du_1
\]
\[
\times \int_{c}^{\infty} \int_{\frac{m-x}{g_1}} z_2(y + g_1 u_1) K(v_1) dv_1 w(m) dm \psi(y) dy + o(g_1^2)
\]
\[
= O(g_1^2)
\]

Using similar arguments we conclude

\[
N_{2}^{1,2,2}(g_1) = O(g_1^2)
\]

and to end up with (156), it only remains to deal with the product of expectations

\[
E \left( \int_{c}^{\infty} \int_{c}^{t} z_2(r) K_{g_1} (r - T_1) (\delta_1 - p(r)) dr \int_{c}^{t} z_2(s) K_{g_1} (s - T_2) (\delta_2 - p(s)) dsw(t) dt \right)
\]
\[
\times E \left( \int_{c}^{\infty} \int_{c}^{m} z_2(u) K_{g_1} (u - T_1) (\delta_1 - p(u)) du \int_{c}^{m} z_2(v) K_{g_1} (v - T_2) (\delta_1 - p(v)) dsw(m) dm \right)
\]
\[
= f_{1211}(g_1) f_{1222}(g_1)
\]

which may be proved to be \( O(g_1^4) \), using (114) and (115).
To deal with \( N_{1,2,1,2}(g_1) \) the expectation of the product is
\[
E \left( \left( \int_{c'}^t \int_{c'}^t z_2(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \, dr \times \int_{c'}^t z_2(s) K_{g_1}(s - T_2) (\delta_2 - p(s)) \, ds \right) dt \right)^2
\]
\[
= N_{1,2,1,2}^{1,2,1,2}(g_1) + N_{2,2,1,2}^{1,2,1,2}(g_1) + N_{3,2,1,2}^{1,2,1,2}(g_1),
\]
where
\[
N_{1,2,1,2}^{1,2,1,2}(g_1) = \int_0^\infty \int_0^\infty \left( \int_{c'}^t z_{21}(r) K_{g_1}(r - x) \, dr \times \int_{c'}^t z_{21}(s) K_{g_1}(s - y) \, ds \right)^2 \psi(x) \psi(y) \, dx \, dy.
\]
\[
N_{3,2,1,2}^{1,2,1,2}(g_1) = 2 \int_0^\infty \int_0^\infty \left( \int_{c'}^t z_{22}(r) K_{g_1}(r - x) \, dr \times \int_{c'}^t z_{22}(s) K_{g_1}(s - y) \, ds \right)^2 \phi(x) \phi(y) \, dx \, dy.
\]

After the change of variable \( r/g_1 = r_1, \quad s/g_1 = s_1 \) the first term becomes
\[
N_{1,2,1,2}^{1,2,1,2}(g_1) = \int_0^\infty \int_0^\infty \left( \int_{c'}^t \frac{1}{g_1} z_{21}(x + g_1 r_1) K(r_1) \, dr_1 \times \int_{c'}^t \frac{1}{g_1} z_{21}(y + g_1 s_1) K(s_1) \, ds_1 \right)^2 \psi(x) \psi(y) \, dx \, dy
\]
\[
= O(1).
\]

In a completely similar way the last two terms are
\[
N_{2,2,1,2}^{1,2,1,2}(g_1) = O(1)
\]
and
\[
N_{3,2,1,2}^{1,2,1,2}(g_1) = O(1).
\]
Since the product of the expectation is
\[
E \left( \int_{c'}^{t} \int_{c'}^{t} z_2(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \, dr \right.
\]
\[
\times \left. \int_{c'}^{t} z_2(s) K_{g_1}(s - T_2) (\delta_2 - p(s)) \, ds \right) \int_{c'}^{t} w(t) \, dt \right)^2
\]
\[
= I_{1211}(g_1)^2 = O \left( g_1^{\frac{1}{4}} \right),
\]
equation (157) becomes obvious.

For \( N_1^{1,1,1,1} \) we have the following bound
\[
\left| N_1^{1,2,2,2} \right| \leq g_1^{-4} \| K \|_{\infty}^4 \left( \int_{c'}^{t} \left( \int_{c'}^{t} |z_{2}(r)| \, dr \right)^2 w(t) \, dt \right)^2.
\]
Since conditions (H.1) and (W.1) imply integrability of \( \int_{c'}^{t} \left( \int_{c'}^{t} |z_{2}(r)| \, dr \right)^2 w(t) \, dt \) the result (158) follows.

**Lemma 34** Under the same conditions of Lemma 23, \( Var \left( \hat{A}_{123} \right) = O \left( n^{-\frac{1}{4}} g_1^{\frac{1}{4}} \right). \) \( (159) \)

**Proof.** The variance under study is
\[
Var \left( \hat{A}_{123} \right) = \frac{1}{2^{n^4}} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{c'}^{t} \int_{c'}^{t} z_2(r) K_{g_1}(r - T_1) (\delta_i - p(r)) \, dr \right.
\]
\[
\left. \int_{c'}^{t} z_1(s) \left( K''_{g_1}(s - T_j) (\delta_j - p(s)) - \zeta(s) \right) \, ds \right) \int_{c'}^{t} w(t) \, dt \right)^2
\]
\[
= \frac{1}{2^{n^4}} \left( n Q_{1,1,1,1}^{1,1,1,1} + 2 n(n - 1) Q_{1,2,2,2}^{1,2,2,2} + 2 n(n - 1) Q_{1,2,1,1}^{1,2,1,1} \right.
\]
\[
+ n(n - 1) Q_{1,2,2,2}^{1,2,2,1} + n(n - 1) Q_{1,2,2,2}^{1,2,2,1} \left( g_1 \right)
\]
\[
+ n(n - 1)(n - 2) Q_{1,2,1,3}^{1,2,1,3} + n(n - 1)(n - 2) Q_{1,2,3,1}^{1,2,3,1} \right)
\]
\[
+ n(n - 1)(n - 2) Q_{1,2,2,3}^{1,2,2,3} + n(n - 1)(n - 2) Q_{1,2,3,2}^{1,2,3,2} \left( g_1 \right) \right),
\]
where
\[
Q_{i,j,k,l}^{i,j,k,l}(g_1) = Cov \left( \int_{c'}^{t} \int_{c'}^{t} z_2(r) K_{g_1}(r - T_1) (\delta_i - p(r)) \, dr \right.
\]
\[
\times \left. \int_{c'}^{t} z_1(s) \left( K''_{g_1}(s - T_j) (\delta_j - p(s)) - \zeta(s) \right) \, ds \right) \int_{c'}^{t} w(t) \, dt \right)
\]
\[
\int_{c'}^{t} \int_{c'}^{t} z_2(u) K_{g_1}(u - T_k) (\delta_k - p(u)) \, du \int_{c'}^{t} \int_{c'}^{t} z_1(v) \left( K''_{g_1}(v - T_l) (\delta_l - p(v)) - \zeta(v) \right) \, dv \, dw(m) \, dm \right).
\]

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Similar to what was discussed for \( \text{Var}(\tilde{A}_{122}) \), it suffices to study the order of the terms \( Q^{1,2,1,3}(g_1) \), \( Q^{1,2,2,2}(g_1) \), \( Q^{1,2,1,2}(g_1) \) and \( Q^{1,1,1,1}(g_1) \). Let’s start by studying \( Q^{1,2,1,3}(g_1) \). The expectation may be decomposed as

\[
E \left( \int_{t'}^t \int_{t'}^t z_2(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \, dr \times \int_{t'}^t z_1(s) \left( K''_{g_1}(s - T_2) (\delta_2 - p(s)) - \zeta(s) \right) \, ds \right) \, dt \\
\times \int_{t'}^t \int_{t'}^t z_2(u) K_{g_1}(u - T_1) (\delta_1 - p(u)) \, du \\
\times \int_{t'}^t z_1(v) \left( K''_{g_1}(v - T_3) (\delta_3 - p(v)) - \zeta(v) \right) \, dw(m) dm
= Q^{1,2,1,3}_1(g_1) + Q^{1,2,1,3}_2(g_1)
\]

where

\[
Q^{1,2,1,3}_1(g_1) = \int_0^\infty \left( \int_{t'}^t \int_{t'}^t z_2(r) K_{g_1}(r - x) \, dr \right) \times \int_{t'}^t z_1(s) \left( \int_0^\infty K''_{g_1}(s - y) (p(y) - p(s)) h(y) dy - \zeta(s) \right) \, ds \, dt \right)^2 \psi(x) dx
\]

and

\[
Q^{1,2,1,3}_2(g_1) = \int_0^\infty \left( \int_{t'}^t \int_{t'}^t z_2(r) K_{g_1}(r - x) \, dr \right) \times \int_{t'}^t z_1(s) \left( \int_0^\infty K''_{g_1}(s - y) (p(y) - p(s)) h(y) dy - \zeta(s) \right) \, ds \, dt \right)^2 \phi(x) dx.
\]

The steps already used to analyze the term \( M^{1,2,1,3}_1(g_1) \) give, after using the change of variable \( \frac{x-y_1}{g_1} = y_2, \frac{r-y_1}{g_1} = r_1 \) and condition \( g_1 < \frac{r}{t'}, \)

\[
Q^{1,2,1,3}_1(g_1) = g_1^4 \int_0^\infty \left( \int_{t'}^t \int_{t'}^t z_2(x + g_1 r_1) K(r_1) dr_1 \right) \times \int_{t'}^t z_1(s) \left( \int_{-L}^L K''(y_1) (p(s - g_1 y_1) - p(s)) h(s - g_1 y_1) dy_1 - g_1^2 \zeta(s) \right) \, ds \, dt \right)^2 \psi(x) dx.
\]

Further Taylor of \( p(s - g_1 y_1) - p(s) \) and \( h(s - g_1 y_1) \) around \( s \) can be used to
obtain

\[
Q_1^{1,2,1,3}(g_1) = g_1^4 K \int_0^\infty \left( \int_{r_1}^\infty \int_{r_1}^{r_1+T_1} z_2(x + r_1) K(r_1) dr_1 \right.
\]
\[
\times \int_{r'}^t z_1(r) \left( \frac{1}{2} h(r) p(4)(r) + 2h'(r)p(3)(r) + 3h''(r)p''(r) + 2h^{(3)}(r)p'(r) \right) drw(t) dt \bigg)^2 \times \psi(x) dx + o(g_1^4) \bigg] = O(g_1^4). \]

Since the terms \(Q_1^{1,2,1,3}(g_1)\) and \(Q_2^{1,2,1,3}(g_1)\) are very similar, it is straightforward to check that

\[
Q_2^{1,2,1,3}(g_1) = O(g_1^4)
\]

and, consequently,

\[
Q^{1,2,1,3}(g_1) = Q_1^{1,2,1,3}(g_1) + Q_2^{1,2,1,3}(g_1) - I_{1231}(g_1)^2 = O(g_1^4) + O(g_1^8) = O(g_1^4), \tag{160}
\]

which comes from the following expression

\[
E \left( \int_{r'}^t \int_{r'}^t z_2(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) dr \right.
\]
\[
\times \int_{r'}^t z_1(s) \left( K_{g_1}''(s - T_2) (\delta_2 - p(s)) - \zeta(s) \right) ds w(t) dt \bigg)
\]
\[
\times \left( \int_{r'}^t \int_{r'}^t z_2(u) K_{g_1}(u - T_1) (\delta_1 - p(u)) du \right.
\]
\[
\times \int_{r'}^t z_1(v) \left( K_{g_1}''(v - T_3) (\delta_3 - p(v)) - \zeta(v) \right) dv w(t) dt \bigg)
\]
\[
= I_{1231}(g_1)^2 \]

and from expression (117).
To deal with $Q^{1,2,2,2}_{1}(g_1)$ we first consider the expectation

$$E \left( \int_{c'}^{t} \int_{c'}^{m} z_2(r) K_{g_1}(r - T_1) \left( \delta_1 - p(r) \right) dr \right) \times \int_{c'}^{t} z_1(s) \left( K''_{g_1}(s - T_2) \left( \delta_2 - p(s) \right) - \zeta(s) \right) dsw(t) dt$$

$$\times \int_{c'}^{m} \int_{c'}^{m} z_2(u) K_{g_1}(u - T_2) \left( \delta_2 - p(u) \right) du$$

$$\times \int_{c'}^{m} z_1(v) \left( K''_{g_1}(v - T_2) \left( \delta_2 - p(v) \right) - \zeta(v) \right) dsw(m) dm$$

$$= \sum_{i=1}^{7} Q^{1,2,2,2}_{i}(g_1), \quad (161)$$

with

$$Q^{1,2,2,2}_{1}(g_1) = \int_{0}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_2(r) \int_{0}^{\infty} K_{g_1}(r - y) \left( p(y) - p(r) \right) h(y) dy dr$$

$$\times \int_{c'}^{t} z_{11}(s) K''_{g_1}(s - x) dsw(t) dt$$

$$\times \int_{c'}^{m} \int_{c'}^{m} z_{21}(u) K_{g_1}(u - x) du \int_{c'}^{m} z_{11}(v) K''_{g_1}(v - x) dsw(m) dm \psi(x) dx,$$

$$Q^{1,2,2,2}_{2}(g_1) = - \int_{0}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_2(r) \int_{0}^{\infty} K_{g_1}(r - y) \left( p(y) - p(r) \right) h(y) dy dr$$

$$\times \int_{c'}^{t} z_{12}(s) K''_{g_1}(s - x) dsw(t) dt$$

$$\times \int_{c'}^{m} \int_{c'}^{m} z_{22}(u) K_{g_1}(u - x) du \int_{c'}^{m} z_{12}(v) K''_{g_1}(v - x) dsw(m) dm \phi(x) dx,$$

$$Q^{1,2,2,2}_{3}(g_1) = - 2 \int_{0}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_2(r) \int_{0}^{\infty} K_{g_1}(r - y) \left( p(y) - p(r) \right) h(y) dy dr$$

$$\times \int_{c'}^{t} z_{11}(s) K''_{g_1}(s - x) dsw(t) dt$$

$$\times \int_{c'}^{m} \int_{c'}^{m} z_{21}(u) K_{g_1}(u - x) du \int_{c'}^{m} z_{11}(v) dsw(m) dm \psi(x) dx,$$

$$Q^{1,2,2,2}_{4}(g_1) = - 2 \int_{0}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_2(r) \int_{0}^{\infty} K_{g_1}(r - y) \left( p(y) - p(r) \right) h(y) dy dr$$

$$\times \int_{c'}^{t} z_{12}(s) K''_{g_1}(s - x) dsw(t) dt$$

$$\times \int_{c'}^{m} \int_{c'}^{m} z_{22}(u) K_{g_1}(u - x) du \int_{c'}^{m} z_{12}(v) dsw(m) dm \phi(x) dx,$$
\[ Q_5^{1,2,2,2}(g_1) = -2 \int_0^\infty \int_0^t \int_{r'}^\infty z_2(r) \int_0^\infty K_{g_1}(r-y) (p(y) - p(r)) h(y)dy \alpha_w(t)dt \]
\[ \times \int_{r'}^\infty \int_{r'}^m z_{12}(u) K_{g_1}(u-x)du \int_{r'}^m z_{11}(v) K_{g_1}''(v-x)dvw(m)dm\psi(x)dx, \]

\[ Q_6^{1,2,2,2}(g_1) = -2 \int_0^\infty \int_0^\infty \int_{r'}^\infty z_2(r) \int_0^\infty K_{g_1}(r-y) (p(y) - p(r)) h(y)dy \alpha_w(t)dt \]
\[ \times \int_{r'}^\infty \int_{r'}^m z_{12}(u) K_{g_1}(u-x)du \int_{r'}^m z_{11}(v) K_{g_1}''(v-x)dvw(m)dm\phi(x)dx, \]

and

\[ Q_7^{1,2,2,2}(g_1) = 4 \int_0^\infty \int_{r'}^\infty \int_{r'}^\infty z_2(r) \int_0^\infty K_{g_1}(r-y) (p(y) - p(r)) h(y)dy \alpha_w(t)dt \]
\[ \times \int_{r'}^\infty \int_{r'}^m z_{12}(u) K_{g_1}(u-x) (p(x) - p(u))du \alpha_w(m)dh(x)dx. \]

The change of variable \( \frac{r-w}{y_1} = y_1, \frac{r-w}{y_2} = s_1, \frac{r-w}{y_3} = u_1, \frac{r-w}{y_4} = v_1 \), gives

\[ Q_1^{1,2,2,2}(g_1) = g_1^4 \int_0^\infty \int_{r'}^\infty \int_{r'}^\infty z_2(r) \int_{-\infty}^\infty K(r-y) (p(r-y_1 y_1) - p(r)) h(r-y_1 y_1)dy_1 \]
\[ \times \int_{r'}^\infty \int_{r'}^m z_{11}(x + g_1 s_1) K''(s_1)ds_1 w(t)dt \]
\[ \times \int_{r'}^\infty \int_{r'}^m z_{21}(x + g_1 u_1) K(u_1)du_1 \]
\[ \times \int_{r'}^\infty \int_{r'}^m z_{11}(x + g_1 v_1) K''(v_1)dv_1 w(m)dm\psi(x)dx \]
\[ = o(g_1^{-4}). \]

Obviously, the same order is valid for \( Q_2^{1,2,2,2}(g_1) \). The remainder terms in (161) can be directly bounded. Here, we only present some details for \( Q_3^{1,2,2,2}(g_1) \):

\[ \left| Q_3^{1,2,2,2}(g_1) \right| \leq g_1^{-5} \left\| K \right\|_\infty^2 \left\| K'' \right\|_\infty \int_{r'}^\infty \left( \int_{r'}^t \left| z_2(r) \right| dr \right)^2 w(t)dt \]
\[ \times \int_{r'}^\infty \int_{r'}^m \left| z_{21}(u) \alpha_w(m) \right| dm\alpha_w(t), \]

which implies

\[ Q_3^{1,2,2,2}(g_1) = O(g_1^{-5}). \]

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by using conditions (H.1) and (W.1) to obtain integrability of \( \int_{\varepsilon'}^{\infty} \left( \int_{\varepsilon'}^{t} |z_2(r)| \, dr \right)^2 w(t) \, dt \) and \( \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{m} |z_{21}(u) \alpha_w(m)| \, dudm \). The same order is trivially valid for the remainder terms in (161). Therefore, taking, once more, into account

\[
E \left( \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{t} z_2(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \, dr \\
\times \int_{\varepsilon'}^{t} z_1(s) (K_{g_1}''(s - T_2) (\delta_2 - p(s)) - \zeta(s)) \, dw(t) \, dt \\
\times \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{m} z_2(u) K_{g_1}(u - T_2) (\delta_2 - p(u)) \, du \\
\times \int_{\varepsilon'}^{m} z_1(v) (K_{g_1}''(v - T_2) (\delta_2 - p(v)) - \zeta(v)) \, d(w(m) \, dm) \right) \\
= I_{1231}(g_1) I_{1232}(g_1)
\]

as well as (117) and (118) we conclude

\[
Q^{1,2,2,2}(g_1) = O(g_1^{-5}) - O(g_1^4)O(g_1^{-2}) = O(g_1^{-5}).
\] (162)

The study of \( Q^{1,2,1,2}(g_1) \) starts by considering the expectation

\[
E \left( \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{t} z_2(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \, dr \\
\times \int_{\varepsilon'}^{t} z_1(s) (K_{g_1}''(s - T_2) (\delta_2 - p(s)) - \zeta(s)) \, dw(t) \, dt \\
\times \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{m} z_2(u) K_{g_1}(u - T_1) (\delta_1 - p(u)) \, du \\
\times \int_{\varepsilon'}^{m} z_1(v) (K_{g_1}''(v - T_2) (\delta_2 - p(v)) - \zeta(v)) \, d(w(m) \, dm) \right) \\
= \sum_{i=1}^{8} Q_i^{1,2,1,2}(g_1)
\] (163)

where

\[
Q_1^{1,2,1,2}(g_1) = \int_{0}^{\infty} \int_{0}^{\infty} \left( \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{t} z_{21}(r) K_{g_1}(r - x) \, dr \right. \\
\left. \times \int_{\varepsilon'}^{t} z_{11}(s) K_{g_1}''(s - y) \, dw(t) \, dt \right)^2 \psi(y) \psi(x) \, dxdy,
\]

\[
Q_2^{1,2,1,2}(g_1) = \int_{0}^{\infty} \int_{0}^{\infty} \left( \int_{\varepsilon'}^{\infty} \int_{\varepsilon'}^{t} z_{22}(r) K_{g_1}(r - x) \, dr \right. \\
\left. \times \int_{\varepsilon'}^{t} z_{11}(s) K_{g_1}''(s - y) \, dw(t) \, dt \right)^2 \psi(y) \phi(x) \, dxdy,
\]

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This order will obviously be valid for the terms

\[ Q_{3}^{1.2.1.2}(g_1) = \int_{0}^{\infty} \int_{0}^{\infty} \left( \int_{c'}^{t} \int_{r'}^{r} z_{21}(r) K_{g_1}(r-x) dr \right. \int_{c'}^{t} z_{22}(s) K''_{g_1}(s-y) ds \left. w(t) dt \right)^{2} \times \psi(y) \psi(x) dx dy, \]

\[ Q_{4}^{1.2.1.2}(g_1) = \int_{0}^{\infty} \int_{0}^{\infty} \left( \int_{c'}^{t} \int_{r'}^{r} z_{22}(r) K_{g_1}(r-x) dr \right. \int_{c'}^{t} z_{21}(s) K''_{g_1}(s-y) ds \left. w(t) dt \right)^{2} \times \psi(y) \psi(x) dx dy, \]

\[ Q_{5}^{1.2.1.2}(g_1) = -2 \int_{0}^{\infty} \int_{0}^{\infty} \int_{c'}^{t} \int_{r'}^{r} \int_{c'}^{t} z_{21}(r) K_{g_1}(r-x) dr \int_{c'}^{t} z_{21}(s) \left( K''_{g_1}(s-y) (p(y) - p(s)) \right) ds \left. w(t) dt \right)^{2} \times \psi(y) \psi(x) dx dy, \]

\[ \int_{c'}^{t} \int_{r'}^{r} z_{22}(r) K_{g_1}(r-x) dr \int_{c'}^{t} z_{21}(s) K''_{g_1}(s-y) ds \left. w(t) dt \right)^{2} \times \psi(y) \psi(x) dx dy, \]

\[ Q_{7}^{1.2.1.2}(g_1) = \int_{0}^{\infty} \left( \int_{c'}^{t} \int_{r'}^{r} z_{21}(r) K_{g_1}(r-x) dr \alpha_{w}(t) dt \right)^{2} \psi(x) dx \]

and

\[ Q_{8}^{1.2.1.2}(g_1) = \int_{0}^{\infty} \left( \int_{c'}^{t} \int_{r'}^{r} z_{22}(r) K_{g_1}(r-x) dr \alpha_{w}(t) dt \right)^{2} \phi(x) dx. \]

The first four terms can be handled using the change of variable \( \frac{r-x}{g_1} = r_1 \), \( \frac{s-y}{g_1} = s_1 \). This gives

\[ Q_{1}^{1.2.1.2}(g_1) = g_1^{-4} \int_{0}^{\infty} \int_{0}^{\infty} \left( \int_{c'}^{t} \int_{r'}^{r} \frac{r-x}{g_1} z_{21}(x + g_1 r_1) K(r_1) dr_1 \int_{c'}^{t} \frac{s-y}{g_1} z_{21}(y + g_1 s_1) K''(s_1) ds_1 \right) \times \psi(y) \psi(x) dx dy = o(g_1^{-4}). \]

This order will obviously be valid for the terms \( Q_{i}^{1.2.1.2}(g_1) \), \( i = 2, 3, 4 \).

It is not difficult to find direct bound of the terms in (163) . For instance,

\[ \left| Q_{5}^{1.2.1.2}(g_1) \right| = 2 g_1^{-5} \left\| K \right\|_{\infty}^{2} \left\| K'' \right\|_{\infty} \int_{c'}^{t} \left( \int_{c'}^{t} |z_{21}(r)| dr \right)^{2} w(t) dt \times \int_{c'}^{t} \int_{r'}^{r} |z_{21}(u) \alpha_{w}(m)| du dm, \]

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which implies \[ Q_5^{1.2.2.2}(g_1) = O(g_1^{-5}). \] (164)

Once more, conditions (H.1) and (W.1) have been used to prove that the integrals \( t_{z_1} \left( \int_{r}^{t} |z_2(r)| \, dr \right)^2 \) and \( t_{z_1} \left( \int_{r}^{t} |z_2(u)| \alpha_w(m) \, du \right)^2 \) are finite.

The same procedure is useful to obtain the orders \( O(g_1^{-5}) \), for \( Q_6^{1.2.2.2}(g_1) \), and \( O(g_1^{-5}) \), for \( Q_7^{1.2.2.2}(g_1) \) and for \( Q_8^{1.2.2.2}(g_1) \) as well.

Similar arguments as those used in (117) can be used to prove

\[
E \left( \int_{r}^{t} \int_{r}^{t} z_2(r)K_{g_1}(r-T_1)(\delta_1-p(r)) \, dr \right. \\
\left. \times \int_{r}^{t} z_1(s) \left( K_{g_1}''(s-T_2)(\delta_2-p(s)) - \zeta(s) \right) \, ds \right) \\
E \left( \int_{r}^{t} \int_{r}^{t} z_2(u)K_{g_1}(u-T_1)(\delta_1-p(u)) \, du \right. \\
\left. \times \int_{r}^{t} z_1(v) \left( K_{g_1}''(v-T_2)(\delta_2-p(v)) - \zeta(v) \right) \, dw \right)
\]

which implies \[ Q_1^{1.2.1.2}(g_1) = O(g_1^{-5}). \]

Similarly, we start the study of \( Q_1^{1.1.1.1}(g_1) \) by considering the expectation:

\[
E \left( \int_{r}^{t} \int_{r}^{t} z_2(r)K_{g_1}(r-T_1)(\delta_1-p(r)) \, dr \right. \\
\left. \times \int_{r}^{t} z_1(s) \left( K_{g_1}''(s-T_1)(\delta_1-p(s)) - \zeta(s) \right) \, ds \right) \\
E \left( \int_{r}^{t} \int_{r}^{t} z_2(u)K_{g_1}(u-T_1)(\delta_1-p(u)) \, du \right. \\
\left. \times \int_{r}^{t} z_1(v) \left( K_{g_1}''(v-T_1)(\delta_1-p(v)) - \zeta(v) \right) \, dw \right)
\]

where

\[
Q_1^{1.1.1.1}(g_1) = \int_{0}^{\infty} \left( \int_{r}^{t} \int_{r}^{t} z_2(r)K_{g_1}(r-x) \, dr \right) \left( \int_{r}^{t} z_1(s)K_{g_1}''(s-x) \, ds \right) \, dy \\
\psi(x) \, dx,
\]

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\[ Q_2^{1,1,1}(g_1) = \int_0^\infty \left( \int_{t'}^\infty \int_{t'}^t z_22(r)K_{g_1}(r-x)dr \int_{t'}^t z_{12}(s)K''_{g_1}(s-x)dw(s)dt \right)^2 \phi(x)dx, \]

\[ Q_3^{1,1,1}(g_1) = -4 \int_0^\infty \int_{t'}^\infty \int_{t'}^t z_{21}(r)K_{g_1}(r-x)dr\alpha_w(t)dt \times \int_{t'}^\infty \int_{t'}^m z_{21}(u)K_{g_1}(u-x)du \int_{t'}^m z_{11}(v)K''_{g_1}(v-x)dw(m)m\psi(x)dx, \]

\[ Q_4^{1,1,1}(g_1) = 4 \int_0^\infty \int_{t'}^\infty \int_{t'}^t z_22(r)K_{g_1}(r-x)dr\alpha_w(t)dt \times \int_{t'}^\infty \int_{t'}^m z_{22}(u)K_{g_1}(u-x)du \int_{t'}^m z_{12}(v)K''_{g_1}(v-x)dw(m)m\phi(x)dx, \]

\[ Q_5^{1,1,1}(g_1) = 4 \int_0^\infty \left( \int_{t'}^\infty \int_{t'}^t z_{21}(r)K_{g_1}(r-x)dr\alpha_w(t)dt \right)^2 \psi(x)dx, \]

and

\[ Q_6^{1,1,1}(g_1) = 4 \int_0^\infty \left( \int_{t'}^\infty \int_{t'}^t z_{22}(r)K_{g_1}(r-x)dr\alpha_w(t)dt \right)^2 \phi(x)dx. \]

The change of variable \( \frac{r}{g_1} = r_1, \frac{u}{g_1} = s_1 \) gives

\[ Q_1^{1,1,1}(g_1) = g_1^{-4} \int_0^\infty \left( \int_{t'}^\infty \int_{t'}^t z_{21}(r + g_1r_1)K(r_1)dr_1 \int_{t'}^m z_{11}(x + g_1s_1)K''(s_1)ds_1dw(t)dt \right)^2 \psi(x) \]

and the same order is obviously valid for \( Q_2^{1,1,1}(g_1) \). On the other hand the bound

\[ \left| Q_3^{1,1,1}(g_1) \right| \leq 4g_1^{-5} \|K\|_{\infty}^2 \|K''\|_{\infty} \int_{t'}^\infty \int_{t'}^t |z_{21}(r)|dr|\alpha_w(t)|dt \times \int_{t'}^\infty \left( \int_{t'}^m |z_{21}(u)|du \right)^2 w(m)m \]

gives \( Q_3^{1,1,1}(g_1) = O\left( g_1^{-5} \right) \). The same type of arguments can be used to obtain that \( Q_4^{1,1,1}(g_1) \) is of order \( O\left( g_1^{-5} \right) \) and \( Q_5^{1,1,1}(g_1) \) and \( Q_6^{1,1,1}(g_1) \) are of order \( O\left( g_1^{-2} \right) \).
Finally, using (118),

\[
E \left( \int_{c'}^{t} \int_{c'}^{t} z_2(r) K_{g_1} \left( r - T_1 \right) (\delta_1 - p(r)) \, dr \right) \times \int_{c'}^{t} z_1(s) \left( K_{g_1}''(s - T_1) (\delta_1 - p(s)) - \zeta(s) \right) \, dsw(t) \, dt ) \\
\times E \left( \int_{c'}^{t} \int_{c'}^{m} z_2(u) K_{g_1} \left( u - T_1 \right) (\delta_1 - p(u)) \, du \right) \times \int_{c'}^{m} z_1(v) \left( K_{g_1}''(v - T_1) (\delta_1 - p(v)) - \zeta(v) \right) \, dvw(m) \, dm \right) \\
= I_{1232}(g_1)^2 = O \left( g_1^{-4} \right),
\]

which implies

\[
Q^{1,1,1,1}(g_1) = O \left( g_1^{-5} \right). \tag{165}
\]

The final result in the lemma is a direct consequence of (160)-(165) and condition \( ng \to \infty \). \( \Box \)

**Lemma 35** Under the conditions of Lemmas 27, 32 and 33, the covariances \( \text{Cov} \left( \hat{A}_{121}, \hat{A}_{122} \right) \), \( \text{Cov} \left( \hat{A}_{121}, \hat{A}_{123} \right) \) and \( \text{Cov} \left( \hat{A}_{122}, \hat{A}_{123} \right) \) are of order \( o \left( n^{-2} g_1^{-6} \right) \).

**Proof.** Using expressions (119), (154) and (159) it is straightforward to prove that the terms \( \text{Var} \left( \hat{A}_{121} \right) \), \( \text{Var} \left( \hat{A}_{122} \right) \) and \( \text{Var} \left( \hat{A}_{123} \right) \) are of order \( o \left( n^{-2} g_1^{-6} \right) \).

Using Cauchy-Schwarz inequality,

\[
\left| \text{Cov} \left( \hat{A}_{121}, \hat{A}_{122} \right) \right| \leq \text{Var} \left( \hat{A}_{121} \right)^{\frac{1}{2}} \text{Var} \left( \hat{A}_{122} \right)^{\frac{1}{2}} = o \left( n^{-2} g_1^{-6} \right)
\]

and the same order it is obviously valid for \( \text{Cov} \left( \hat{A}_{121}, \hat{A}_{123} \right) \) and \( \text{Cov} \left( \hat{A}_{122}, \hat{A}_{123} \right) \). \( \Box \)

We now consider the term \( \hat{A}_{13} \).

**Lemma 36** Under conditions (K.1), (P.1), (P.2), (H.1), (W.1) and (V.1),

\[
E \left( \hat{A}_{13} \right) = O \left( g_1^4 \right) + o \left( n^{-1} g_1^{-3} \right) \tag{166}
\]

and

\[
\text{Var} \left( \hat{A}_{13} \right) = o \left( n^{-2} g_1^{-5} \right) + o \left( n^{-1} \right). \tag{167}
\]
Proof. After several cancellations in (42), we obtain the following expression for $\hat{A}_{13}$

$$
\hat{A}_{13} = \int_{c}^{\infty} \int_{c'}^{t} z_3(r) \left( \bar{\phi}(r) - p(r)\bar{h}(r) \right) \left( h(r)\bar{h}''(r) - h''(r)\bar{h}(r) \right) \alpha_w(t) drdt.
$$

Its expectation is

$$
E \left( \hat{A}_{13} \right) = E \left( \int_{c}^{\infty} \int_{c'}^{t} \frac{1}{n^2} z_3(r) \left( \sum_{i=1}^{n} K_{g_1}(r - T_i) (\delta_i - p(r)) \right) \right.
$$

$$
\times \left( h(r)\sum_{j=1}^{n} K_{g_2}'(r - T_j) - h''(r)\sum_{j=1}^{n} K_{g_1}(r - T_j) \right) \alpha_w(t) drdt \bigg) = \frac{n(n-1)}{n^2} I_{131}(g_1) + \frac{1}{n} I_{132}(g_1)
$$

where

$$
I_{131}(g_1) = E \left( \int_{c}^{\infty} \int_{c'}^{t} z_3(r) \left( K_{g_1}(r - T_1) (\delta_1 - p(r)) \right) \right.
$$

$$
\times \left( h(r)K_{g_1}''(r - T_2) - h''(r)K_{g_1}(r - T_2) \right) \alpha_w(t) drdt \bigg) \quad (168)
$$

and

$$
I_{132}(g_1) = E \left( \int_{c}^{\infty} \int_{c'}^{t} z_3(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \right.
$$

$$
\times \left( h(r)K_{g_1}''(r - T_1) - h''(r)K_{g_1}(r - T_1) \right) \alpha_w(t) drdt \bigg) \quad . \quad (169)
$$

Now the term

$$
I_{131}(g_1) = \int_{0}^{\infty} \int_{0}^{\infty} \int_{c'}^{t} z_3(r) K_{g_1}(r - x) (p(x) - p(r)) \right.
$$

$$
\times \left( h(r)K_{g_1}''(r - y) - h''(r)K_{g_1}(r - y) \right) \alpha_w(t) h(y) dy dr dt
$$

$$
= \int_{c'}^{t} \int_{c'}^{t} z_3(r) \int_{0}^{\infty} K_{g_1}(r - x) (p(x) - p(r)) h(x) dx
$$

$$
\times \int_{0}^{\infty} \left( h(r) K_{g_1}''(r - y) - h''(r) K_{g_1}(r - y) \right) h(y) dy \alpha_w(t) dr dt
$$

$$
= g_{1}^{-2} \int_{c'}^{t} \int_{c'}^{t} z_3(r) \int_{-\infty}^{\infty} K(x_1) (p(r - g_1 x_1) - p(r)) h(r - g_1 x_1) dx_1
$$

$$
\times \int_{-\infty}^{\infty} \left( h(r) K_{g_1}''(y_1) - g_{1}^2 h''(r) K(y_1) \right) h(r - g_1 y_1) dy_1 \alpha_w(t) dr dt
$$

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where we have applied Fubini’s Theorem and used the change of variable $\frac{r-y}{g_1} = y_1$.

Assuming that $g_1 < \frac{1}{7}$,

$$I_{131}(g_1) = g_1^{-2} \int_{x_1}^{\infty} \int_{x_1}^{\infty} z_3(r) \int_{-L}^{L} K(x_1) (p(r-g_1x_1) - p(r)) h(r-g_1x_1) dx_1$$

$$\times \left( h(r) \int_{-L}^{L} K''(g_1) h(r-g_1y_1) dy_1 - g_1^2 h''(r) \int_{-L}^{L} K(y_2) h(r-g_2y_2) dy_2 \right) \alpha_w(t) dr dt$$

and using Taylor expansions of $p(r-g_1x_1) - p(r)$, $h(r-g_1x_1)$, and $h(r-g_1y_1)$ around $r$ it is easy to prove that

$$I_{131}(g_1) = g_1^4 \frac{\delta}{\delta x_1} \int_{x_1}^{\infty} \int_{x_1}^{\infty} z_3(r) \left( \frac{1}{\gamma} \gamma''(r) h(r) + p'(r) h'(r) \right)$$

$$\times \left( h(r) h''(r) - h''(r)^2 \right) \alpha_w(t) dr dt + o(g_1^4). \quad (170)$$

Similarly, the change of variable $\frac{r-x}{g_1} = x_1$ leads to

$$I_{132}(g_1) = g_1^3 \int_{x_1}^{\infty} \int_{x_1}^{\infty} \int_{-\infty}^{\infty} z_3(r) (p(r-g_1x_1) - p(r)) h(r-g_1x_1)$$

$$\times (h(r) K''(x_1) - g_1^2 h''(r) K(x_1)) K(x_1) \alpha_w(t) dx_1 dr dt$$

$$= o \left( g_1^{-3} \right). \quad (171)$$

Now (166) is a direct consequence of (170) and (171).

Let’s now consider the variance of $\hat{A}_{13}$,

\begin{align*}
Var \left( \hat{A}_{13} \right) &= Var \left( \int_{x_1}^{\infty} \int_{x_1}^{\infty} z_3(r) \left( \frac{1}{n} \sum_{i=1}^{n} K_{g_1}(r-T_i) \delta_i - p(r) \frac{1}{n} \sum_{i=1}^{n} K_{g_1}(r-T_i) \right) \right) \\
&= \frac{1}{n^2} Var \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{x_1}^{\infty} \int_{x_1}^{\infty} z_3(r) K_{g_1}(r-T_i) (\delta_i - p(r)) \right) \\
&= \frac{1}{n^2} \left( nR^{1,1,1} (g_1) + 2n(n-1)R^{1,2,2,2} (g_1) + 2n(n-1)R^{1,2,1,1} (g_1) + n(n-1)R^{1,2,1,2} (g_1) \\
&\quad + n(n-1)R^{1,2,2,1} (g_1) + n(n-1)(n-2)R^{1,2,1,3} (g_1) + n(n-1)(n-2)R^{1,2,3,1} (g_1) \\
&\quad + n(n-1)(n-2)R^{1,2,3,2} (g_1) + n(n-1)(n-2)R^{1,2,3,3} (g_1) \right) \quad (172)
\end{align*}
where

\[
R^{i,j,k,l}(g_1) = Cov \left( \int_{t'}^\infty \int_{t'}^\infty z_2(r)K_{g_1}(r-T_i)(\delta_i - p(r))
\times (h(r)K''_{g_1}(r-T_j) - h''(r)K_{g_1}(r-T_j)) \alpha_w(t)drdt,
\int_{s'}^\infty \int_{s'}^\infty z_3(s)K_{g_1}(s-T_k)(\delta_k - p(s))
\times (h(s)K''_{g_1}(s-T_i) - h''(s)K_{g_1}(s-T_i)) \alpha_w(m)dsdm \right).
\]

Since \(R^{1,2,3,1}(g_1) = R^{1,2,2,3}(g_1)\) we can concentrate on studying only the terms
\(R^{1,2,1,3}(g_1), R^{1,2,3,1}(g_1), R^{1,2,3,2}(g_1), R^{1,2,2,2}(g_1), R^{1,2,1,1}(g_1), R^{1,2,1,2}(g_1), R^{1,2,2,1}(g_1)\)
and \(R^{1,1,1,1}(g_1)\).

Now (167) is a consequence of the following results

\[
\begin{align*}
R^{1,2,1,3}(g_1) &= O(g_1^4), & R^{1,2,3,1}(g_1) &= o(1), & R^{1,2,3,2}(g_1) &= o(1), \\
R^{1,2,2,2}(g_1) &= o(g_1^{-5}), & R^{1,2,1,1}(g_1) &= o(g_1^{-5}), & R^{1,2,1,2}(g_1) &= o(g_1^{-5}), \\
R^{1,2,2,1}(g_1) &= o(g_1^{-5}), & R^{1,1,1,1}(g_1) &= O(g_1^{-8}).
\end{align*}
\]

that are collected in Lemma 36, and condition (V.1), \(ng^3 \to \infty\). \(\blacksquare\)

**Lemma 37** Under the conditions of Lemma 36,

\[
R^{1,2,1,3}(g_1) = O\left(g_1^4\right),
\]

(173)

\[
R^{1,2,3,1}(g_1) = o(1),
\]

(174)

\[
R^{1,2,3,2}(g_1) = o(1),
\]

(175)

\[
R^{1,2,2,2}(g_1) = o\left(g_1^{-5}\right),
\]

(176)

\[
R^{1,2,1,1}(g_1) = o\left(g_1^{-5}\right),
\]

(177)

\[
R^{1,2,1,2}(g_1) = o\left(g_1^{-5}\right),
\]

(178)

\[
R^{1,2,2,1}(g_1) = o\left(g_1^{-5}\right)
\]

(179)

and

\[
R^{1,1,1,1}(g_1) = O\left(g_1^{-8}\right).
\]

(180)
Proof. Let’s start with the term $R^{1,2,1,3}(g_1)$. Define
\[
R^{1,2,1,3}_1(g_1) = E \left( \int_0^\infty \int_{-\infty}^t z_3(r) K_{g_1}(r-T_1) (\delta_1 - p(r)) \right.
\times (h(r) K_{g_1}'' (r-T_2) - h''(r) K_{g_1} (r-T_2)) \alpha_w(t) dr dt
\times \int_0^\infty \int_{-\infty}^m z_3(s) K_{g_1}(s-T_1) (\delta_1 - p(s)) \left. \times (h(s) K_{g_1}'' (s-T_3) - h''(s) K_{g_1} (s-T_3)) \alpha_w(m) ds dm \right)
\]
and $I_{131}(g_1)$, as in the proof of Lemma 35, to obtain
\[
R^{1,2,1,3}_1(g_1) = R^{1,2,1,3}_1(g_1) - I_{131}(g_1)^2.
\]
After some standard algebra,
\[
R^{1,2,1,3}_{11}(g_1) = R^{1,2,1,3}_{11}(g_1) + R^{1,2,1,3}_{12}(g_1)
\]
where
\[
R^{1,2,1,3}_{11}(g_1) = \int_0^\infty \left( \int_{-\infty}^t \int_{-\infty}^\infty z_3(r) K_{g_1}(r-x) \times \int_0^\infty (h(r) K_{g_1}'' (r-y) - h''(r) K_{g_1} (r-y)) h(y) dy \alpha_w(t) dr dt \right)^2 \psi(x) dx
\]
and
\[
R^{1,2,1,3}_{12}(g_1) = \int_0^\infty \left( \int_{-\infty}^t \int_{-\infty}^\infty z_3(r) K_{g_1}(r-x) \times \int_0^\infty (h(r) K_{g_1}'' (r-y) - h''(r) K_{g_1} (r-y)) h(y) dy \alpha_w(t) dr dt \right)^2 \phi(x) dx.
\]
Using the change of variable $\frac{r_1}{g_1} = r_1$, $\frac{r_2}{g_1} = y_1$,
\[
R^{1,2,1,3}_{11}(g_1) = g_1^{-4} \int_0^\infty \left( \int_{-\infty}^{1-x} \int_{-\infty}^t z_3(x + g_1 r_1) K(r_1) \times \int_{-\infty}^{1-x+g_1 r_1} (h(x + g_1 r_1) K''(y_1) - g_1^2 h''(x + g_1 r_1) K(y_1)) h(x + g_1 (r_1 - y_1)) dy_1 \alpha_w(t) dr_1 dt \right)^2 \psi(x) dx
\]
and assuming $g_1 < \frac{e^2}{L}$,
\[
R^{1,2,1,3}_{11}(g_1) = g_1^{-4} \int_0^\infty \left( \int_{-\infty}^{1-x} \int_{-\infty}^t z_3(x + g_1 r_1) K(r_1) \times \left( h(x + g_1 r_1) K''(y_1) h(x + g_1 (r_1 - y_1)) dy_1 \right. \right. \left. \left. \left. - g_1^2 h''(x + g_1 r_1) K(y_2) h(x + g_1 (r_1 - y_2)) dy_2 \alpha_w(t) dr_1 dt \right)^2 \psi(x) dx.
\]


Some Taylor expansions of $h(x + g_1 (r_1 - y_1), h(x + g_1 (r_1 - y_2), h(x + g_1 r_1)$ and $h''(x + g_1 r_1)$ around $x$ give

$$R^{1,2,3}_{11}(g_1) = \int_0^\infty \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \right) z_31(x + g_1 r_1)K(r_1)$$

$$\times \left( h(x) + g_1 r_1 h'(x) + \frac{1}{2} g_1^2 r_1^2 h''(x) + \frac{1}{3!} g_1^3 r_1^3 h^{(3)}(\theta_3) \right)$$

$$\times \left( h''(x) + g_1 r_1 h^{(3)}(x) + \frac{1}{2} g_1^2 r_1^2 h^{(4)}(x) (r_1^2 + \mu_K) \right)$$

$$+ \frac{1}{3!} g_1^3 \int_{-L}^L K''(y_1) (r_1 - y_1)^{3} h^{(5)}(\theta_1) dy_1$$

$$- \left( h''(x) + g_1 r_1 h^{(3)}(x) + \frac{1}{2} g_1^2 r_1^2 h^{(4)}(x) (r_1^2 + \mu_K) \right)$$

$$\times \left( h(x) + g_1 r_1 h'(x) + \frac{1}{2} g_1^2 r_1^2 h''(x) (r_1^2 + \mu_K) \right)$$

$$+ \frac{1}{3!} g_1^3 \int_{-L}^L K(y_2) (r_1 - y_2)^{3} h^{(3)}(\theta_2) dy_2 \right) \alpha_w(t) dt^2 \psi(x) dx$$

where $\theta_1, \theta_2, \theta_3$ and $\theta_4$ are some intermediate points between $x$ and $x + g_1 (r_1 - y_1)$, $x$ and $x + g_1 (r_1 - y_2)$, $x$ and $x + g_1 r_1$ and $x$ and $x + g_1 r_1$, respectively.

Now, using

$$\int_{-L}^{L} K(y_1) (r_1 - y_1) dy_1 = r_1, \quad \int_{-L}^{L} K(y_1) (r_1 - y_1)^2 dy_1 = r_1^2 + \mu_K,$$

$$\int_{-L}^{L} K''(y_1) (r_1 - y_1) dy_1 = 0, \quad \int_{-L}^{L} K''(y_1) (r_1 - y_1)^2 dy_1 = 2,$$

$$\int_{-L}^{L} K''(y_1) (r_1 - y_1)^3 dy_1 = 6r_1, \quad \int_{-L}^{L} K''(y_1) (r_1 - y_1)^4 dy_1 = 12 (r_1^2 + \mu_K),$$

we have

$$R^{1,2,3}_{11}(g_1) = g_1^4 \frac{1}{4} \mu_K \int_0^\infty \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \right) z_31(x + g_1 r_1)K(r_1) \alpha_w(t) dt^2 \psi(x) dx$$

$$\times \left( h^{(4)}(x) h(x) - h''(x)^2 \right) \psi(x) dx + o(g_1^4)$$

$$= O(g_1^4).$$

Similar arguments lead to

$$R^{1,2,3}_{12}(g_1) = O(g_1^4).$$
and using (170), \( I_{131}(g_1) = O \left( g_1^4 \right) \), we conclude (173).

For the term \( R_{1}^{1.2.3.1}(g_1) \), proceeding as in the proof of the previous lemma and defining

\[
R_{1}^{1.2.3.1}(g_1) = E \left( \int_0^\infty \int_{x'} \int_0^x z_3(r)K_{g_1}(r-T_1)(\delta_1-p(r)) \right. \\
\times \left. \left( h(x-r)K_{g_1}(r-T_1) - h''(r)K_{g_1}(r-T_1) \right) tdrdt \right. \\
\left. \times \int_0^\infty \int_0^m z_3(s)K_{g_1}(s-T_1)(\delta_1-p(s)) \right. \\
\left. \times \left( h(s)K_{g_1}(s-T_1) - h''(s)K_{g_1}(s-T_1) \right) \alpha_t(w)dsdm \right),
\]

we have

\[
R_{1}^{1.2.3.1}(g_1) = R_{1}^{1.2.3.1}(g_1) - I_{131}(g_1)^2.
\]

Now,

\[
R_{1}^{1.2.3.1}(g_1) = \int_0^\infty \int_0^\infty \int_{x'} \int_0^x z_3(r)K_{g_1}(r-x)(p(x)-p(r)) \\
\times \left( h(r)K_{g_1}(r-y) - h''(r)K_{g_1}(r-y) \right) h(y)d\alpha_t(w)drdydt \\
\times \int_0^\infty \int_0^m z_3(s)K_{g_1}(s-u)(p(u)-p(s)) h(u)du \\
\times \left( h(s)K_{g_1}(s-x) - h''(s)K_{g_1}(s-x) \right) \alpha_t(m)h(x)dsdm dx.
\]

The change of variable \( \frac{r-x}{g_1} = r_1, \frac{r-y}{g_1} = y_1, \frac{s-x}{g_1} = s_1, \frac{s-u}{g_1} = u_1 \) and condition \( g_1 < \varepsilon \) imply

\[
R_{1}^{1.2.3.1}(g_1) = g_1^{-4} \int_0^\infty \int_0^\infty \int_{x'} \int_0^x \int_0^L z_3(x+g_1r_1)K(r_1)(p(x)-p(x+g_1r_1)) \\
\times \left( h(x+g_1r_1) \int_{-L}^L K''(y_1)h(x+g_1(r_1-y_1))dy_1 \right) \\
- g_1^2 h''(x+g_1r_1) \int_{-L}^L K(y_2)h(x+g_1(r_1-y_2))dy_2 \right) \alpha_t(w)dr_1dt \\
\times \int_0^\infty \int_0^m \int_{s'} \int_{-L}^L z_3(s)K(u_1)(p(x+g_1(s_1-u_1))-p(x+g_1s_1)) \\
\times h(x+g_1(s_1-u_1))du_1 \left( h(x+g_1s_1)K''(s_1) - g_1^2 h''(x+g_1s_1)K(s_1) \right) \\
\times \alpha_t(m)h(x)ds_1dm dx.
\]
As in the previous lemma, we can analyze the factor

\[ h(x + g_1 r_1) \int_{-L}^{L} K'(y_1) h(x + g_1 (r_1 - y_1))dy_1 \]

\[ -g_1^2 h''(x + g_1 r_1) \int_{-L}^{L} K(y_2) h(x + g_1 (r_1 - y_2))dy_2 \]

using a Taylor expansion. This gives

\[ R_1^{1,2,3,1}(g_1) = \frac{1}{2} \mu K \int_{0}^{\infty} \int_{0}^{\infty} \int_{-\frac{\epsilon}{g_1}}^{\epsilon} z_3(x + g_1 r_1) K(r_1) (p(x) - p(x + g_1 r_1)) \]

\[ \times \left( h^{(4)}(x) h(x) - h''(x)^2 \right) \alpha_\omega(t) dr_1 dt \]

\[ \times \int_{c'}^{\infty} \int_{-\frac{\epsilon}{g_1}}^{\epsilon} z_3(s) \int_{-L}^{L} K(u_1) (p(x + g_1 (s_1 - u_1)) - p(x + g_1 s_1)) \]

\[ \times h(x + g_1 (s_1 - u_1)) du_1 (h(x + g_1 s_1) K''(s_1) - g_1^2 h''(x + g_1 s_1) K(s_1)) \]

\[ \times \alpha_\omega(m) h(x) ds_1 dmdx + o(1) \]

which leads to (174) by just recalling \( I_{131}(g_1) = O \left( g_1^4 \right) \).

To deal with \( R_1^{1,2,3,2}(g_1) \), let’s define

\[ R_1^{1,2,3,2}(g_1) = E \left( \int_{c'}^{\infty} \int_{-\frac{\epsilon}{g_1}}^{\epsilon} z_3(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \right) \]

\[ \times (h(r) K''_{g_1}(r - T_2) - h''(r) K_{g_1}(r - T_2)) \alpha_\omega(t) dr dt \]

\[ \times \int_{c'}^{\infty} \int_{-\frac{\epsilon}{g_1}}^{\epsilon} z_3(s) K_{g_1}(s - T_3) (\delta_3 - p(s)) \]

\[ \times (h(s) K''_{g_1}(s - T_2) - h''(s) K_{g_1}(s - T_2)) \alpha_\omega(m) ds dm \],

to obtain

\[ R_1^{1,2,3,2}(g_1) = R_1^{1,2,3,1}(g_1) - I_{131}(g_1)^2. \]

The term \( R_1^{1,2,3,2}(g_1) \) can be treated as follows

\[ R_1^{1,2,3,2}(g_1) = \int_{0}^{\infty} \left( \int_{c'}^{\infty} \int_{-\frac{\epsilon}{g_1}}^{\epsilon} z_3(r) \int_{0}^{\infty} K_{g_1}(r - y) (p(y) - p(r)) h(y) dy \right) \]

\[ \times (h(r) K''_{g_1}(r - x) - h''(r) K_{g_1}(r - x)) \alpha_\omega(t) dr dt \] \( h(x) \) \] \( dx. \)

Use the change of variable \( \frac{r - x}{g_1} = r_1 \), \( \frac{r - x}{g_1} = y_1 \) and assume \( g_1 < \frac{2}{L} \) to obtain
\[ R_{1}^{1,2,3,2}(g_1) = g_1^{-4} \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_3(x + g_1 r_1) \times \int_{-L}^{L} K(y_1) (p(x + g_1 r_1 - y_1) - p(x + g_1 r_1)) h(x + g_1 r_1 - y_1) dy_1 \right. \\
\times \left. (h(x + g_1 r_1) K''(r_1) - g_1^2 h''(x + g_1 r_1) K(r_1)) \alpha_{w}(t) dr_1 dt \right)^2 h(x) dx. \]

After using Taylor expansions of \( p(x + g_1(r_1 - y_1)), \) \( p(x + g_1 r_1) \) and \( h(x + g_1(r_1 - y_1)) \) around \( x \) we have

\[ R_{1}^{1,2,3,2}(g_1) = \mu_K \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_3(x + g_1 r_1) \left( \frac{1}{2} p''(x) h(x) + p'(x) h'(x) \right) \times \left( h(x + g_1 r_1) K''(r_1) - g_1^2 h''(x + g_1 r_1) K(r_1) \right) \alpha_{w}(t) dr_1 dt \right)^2 h(x) dx \\
+ o(1) = o(1) \]

which, using \( I_{131}(g_1) = O(g_1^4) \), gives (175).

The term \( R_{1}^{1,2,2,2}(g_1) \) can be analyzed similarly. Define

\[ R_{1}^{1,2,2,2}(g_1) = E \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_3(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \times (h(r) K''_{g_1}(r - T_2) - h''(r) K_{g_1}(r - T_2)) \alpha_{w}(t) dr dt \right) \\
\times \left. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_3(s) K_{g_1}(s - T_2) (\delta_2 - p(s)) \times (h(s) K''_{g_1}(s - T_2) - h''(s) K_{g_1}(s - T_2)) \alpha_{w}(m) ds dm \right) \]

and recall \( I_{132}(g_1) \) in (169) to obtain

\[ R_{1}^{1,2,2,2}(g_1) = R_{1}^{1,2,2,2}(g_1) - I_{131}(g_1) I_{132}(g_1). \]

Now,

\[ R_{1}^{1,2,2,2}(g_1) = \int_{0}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_3(r) K_{g_1}(r - x) (p(x) - p(r)) \times \left( h(r) K''_{g_1}(r - y) - h''(r) K_{g_1}(r - y) \right) \alpha_{w}(t) h(x) dr dt dx \\
\times \left. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_3(s) K_{g_1}(s - y) (p(y) - p(s)) \times (h(s) K''_{g_1}(s - y) - h''(s) K_{g_1}(s - y)) \alpha_{w}(m) h(y) ds dm dy. \right) \]
The change of variable \( \frac{r - t_1}{n} = r_1, \frac{r - t_2}{n} = x_1, \frac{r - t_3}{n} = s_1 \) gives

\[
R_{11}^{1,2.2.2}(g_1) = g_1^{-5} \int_0^\infty \int_{t_1}^\infty \int_{t_2}^\infty \int_{t_3}^\infty \int_{t_4}^\infty \int_{t_5}^\infty z_3(y + g_1 r_1) K(r_1 - x_1) (p(y + g_1 x_1) - p(y + g_1 r_1))
\]
\[
\times (h(y + g_1 r_1) K''(r_1) - g_1^2 h''(y + g_1 r_1) K(r_1)) \alpha_w(t) h(y + g_1 x_1)
\]
\[
\times z_3(y + g_1 s_1) K(s_1) (p(y) - p(y + g_1 s_1))
\]
\[
\times (h(y + g_1 s_1) K''(s_1) - g_1^2 h''(y + g_1 s_1) K(s_1)) \alpha_w(m) h(y) ds_1 dmr_1 dt_1 dx_1 dy
\]
\[
= o(g_1^{-5})
\]

which leads to (176) by just using that (170) and (171) imply \( I_{131}(g_1) = O(g_1^4) \) and \( I_{132}(g_1) = o(g_1^{-3}) \).

For \( R_{12}^{1,2.1.1}(g_1) \), as in the proof of the previous results, let’s start by defining

\[
R_{12}^{1,2.1.1}(g_1) = E \left( \int_0^\infty \int_{t_1}^\infty \int_{t_2}^\infty \int_{t_3}^\infty \int_{t_4}^\infty z_3(r) K_{g_1}(r - t_1) (\delta_1 - p(r))
\]
\[
\times (h(r) K''(r - t_2) - h''(r) K_{g_1}(r - t_2)) \alpha_w(t) dr dt
\]
\[
\times \int_0^\infty \int_{t_3}^\infty z_3(s) K_{g_1}(s - T_1) (\delta_1 - p(s))
\]
\[
\times (h(s) K''(s - T_1) - h''(s) K_{g_1}(s - T_1)) \alpha_w(m) ds dm,
\]

which implies

\[
R_{12}^{1,2.1.1}(g_1) = R_{11}^{1,2.1.1}(g_1) - I_{131}(g_1) I_{132}(g_1).
\]

Straightforward algebra leads to

\[
R_{11}^{1,2.1.1}(g_1) = R_{11}^{1,2.1.1}(g_1) + R_{12}^{1,2.1.1}(g_1),
\]

where

\[
R_{11}^{1,2.1.1}(g_1) = \int_0^\infty \int_{t_1}^\infty \int_{t_2}^\infty z_3(r) K_{g_1}(r - x)
\]
\[
\times \int_0^\infty (h(r) K''(r - y) - h''(r) K_{g_1}(r - y)) h(y) dy \alpha_w(t) dr dt
\]
\[
\times \int_0^\infty \int_{t_3}^\infty z_3(s) K_{g_1}(s - x)
\]
\[
\times (h(s) K''(s - x) - h''(s) K_{g_1}(s - x)) \alpha_w(m) ds dm \psi(x) dx
\]

and

\[
R_{12}^{1,2.1.1}(g_1) = \int_0^\infty \int_{t_1}^\infty \int_{t_2}^\infty z_3(r) K_{g_1}(r - x)
\]
\[
\times \int_0^\infty (h(r) K''(r - y) - h''(r) K_{g_1}(r - y)) h(y) dy \alpha_w(t) dr dt
\]
\[
\times \int_0^\infty \int_{t_3}^\infty z_3(s) K_{g_1}(s - x)
\]
\[
\times (h(s) K''(s - x) - h''(s) K_{g_1}(s - x)) \alpha_w(m) ds dm \phi(x) dx.
\]

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The change of variable \( \frac{\xi}{g_1} = r_1, \frac{\eta}{g_1} = y_1, \frac{\zeta}{g_1} = s_1 \) gives

\[
R_{11}^{1,2,1,1}(g_1) = g_1^{-5} \int_0^\infty \int_0^\infty \int_{\frac{\rho}{g_1}} \frac{\rho}{g_1} z_{31}(x + g_1 r_1) K(r_1)
\]
\[
\times \int_0^\infty \frac{h(x + g_1 r_1)}{g_1} \frac{d y_1}{r_1 - y_1} - g_1^2 h''(r_1 - y_1) h(x + g_1 y_1) dy_1 \alpha_w(t) dr_1 dt
\]
\[
\times \int_{\rho}^{\infty} \int_{\frac{\sigma}{g_1}}^{\infty} z_{31}(x + g_1 s_1) K(s_1)
\]
\[
\times (h(x + g_1 s_1) K''(s_1) - g_1^2 h''(x + g_1 s_1) K(s_1)) \alpha_w(m) ds_1 dm \psi(x) dx
\]
\[
= o \left(g_1^{-5}\right)
\]

since, indeed,

\[
\lim_{g_1 \to 0} g_1^{\frac{5}{2}} R_{11}^{1,2,1,1}(g_1) = \int_{-L}^{L} K(r_1) \int_{r_1 - L}^{r_1 + L} K''(r_1 - y_1) dy_1 dr_1 \int_{-L}^{L} K(s_1) K''(s_1) ds_1
\]
\[
\times \int_{\rho}^{\infty} A_w(x)^2 z_{31}(x)^2 \psi(x) h(x)^3 dx = 0.
\]

The same type of arguments lead to

\[
R_{12}^{1,2,1,1}(g_1) = o \left(g_1^{-3}\right).
\]

Now (177) can be directly obtained using the previous results, \( I_{131}(g_1) = O \left(g_1^4\right) \)

and \( I_{132}(g_1) = o \left(g_1^{-3}\right) \).

To deal with the term \( R_{11}^{1,2,1,2}(g_1) \) let’s define

\[
\begin{align*}
R_{11}^{1,2,1,2}(g_1) &= E \left( \int_{\rho}^{\infty} \int_{\sigma}^{\infty} z_3(r) K_{31}(r - T_1) K_2(r - T_2) \alpha_w(t) dr dt \right)
\times (h(r) K''_{31}(r - T_2) - h''(r) K_{31}(r - T_2))
\times \int_{\sigma}^{\infty} \int_{\rho}^{\infty} z_3(s) K_{31}(s - T_1) K_{11}(s - T_2) \alpha_w(m) ds dm
\end{align*}
\]

\[
\times (h(s) K''_{31}(s - T_2) - h''(s) K_{31}(s - T_2)) \alpha_w(m) ds dm
\]

\[
to obtain
\[
\begin{align*}
R_{11}^{1,2,1,2}(g_1) &= R_{11}^{1,2,1,2}(g_1) - I_{131}(g_1)^2.
\end{align*}
\]

Now,

\[
R_{11}^{1,2,1,2}(g_1) = R_{11}^{1,2,1,2}(g_1) + R_{12}^{1,2,1,2}(g_1),
\]

where

\[
R_{11}^{1,2,1,2}(g_1) = \int_0^\infty \int_0^\infty \int_{\rho}^{\infty} z_{31}(r) K_{31}(r - x)
\times (h(r) K''_{31}(r - y) - h''(r) K_{31}(r - y)) \alpha_w(t) dr dt \psi(x) h(y) dx dy
\]

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Now (178) can be proved using the previous results and, according to (170),

\[ R_{12}^{1,2,1,2}(g_1) = \int_0^\infty \int_0^\infty \left( \int_{t_1}^{t_2} z_1(r) K_{g_1}(r-x) \right) \times \left( h(r) K_{g_1''}(r-y) - h''(r) K_{g_1}(r-y) \right) \alpha_w(t) dr dt 1 \times h(x) \alpha(y) dx dy. \]

Using the change of variable \( \frac{x}{g_1} = r_1, \frac{y}{g_1} = y_1 \),

\[ R_{12}^{1,2,1,2}(g_1) = g_1^{-5} \int_0^\infty \int_0^\infty \left( \int_{t_1}^{t_2} \frac{\cdot}{g_1} z_1(x + g_1 r_1) K(r_1) \right) \times \left( h(x + g_1 r_1) K''(r_1 - y_1) - g_1^2 h''(x + g_1 r_1) K(r_1 - y_1) \right) \alpha_w(t) dr_1 dt 2 \times \psi(x) h(x + g_1 y_1) dy_1 dx \]

\[ = o \left( g_1^{-5} \right). \]

The same procedure applied to the term \( R_{12}^{1,2,1,2}(g_1) \) gives

\[ R_{12}^{1,2,1,2}(g_1) = o \left( g_1^{-5} \right). \]

Now (178) can be proved using the previous results and, according to (170),

\[ R_{131}(g_1) = O \left( g_1^4 \right). \]

Let’s deal with \( R_{12}^{1,2,2,1}(g_1) \) and define

\[ R_{12}^{1,2,2,1}(g_1) = E \left( \int_{t_1}^{t_2} z_1(r) K_{g_1}(r - T_1) (\delta_1 - p(r)) \right) \times \left( h(r) K_{g_1''}(r - T_2) - h''(r) K_{g_1}(r - T_2) \right) \alpha_w(t) dr dt \]

\[ \times \int_{t_1}^{t_2} \int_{s_1}^{s_2} z_1(s) K_{g_1}(s - T_1) (\delta_2 - p(s)) \alpha_w(t) dr dt \]

\[ \times \left( h(s) K_{g_1''}(s - T_1) - h''(s) K_{g_1}(s - T_1) \right) \alpha_w(m) ds dm. \]

With this definition it holds

\[ R_{12}^{1,2,2,1}(g_1) = R_{12}^{1,2,2,1}(g_1) - I_{131}(g_1)^2. \]

On the other hand

\[ R_{12}^{1,2,2,1}(g_1) = \int_0^\infty \int_0^\infty \int_{t_1}^{t_2} z_1(r) K_{g_1}(r-x) (p(x) - p(r)) \]

\[ \times \left( h(r) K_{g_1''}(r-y) - h''(r) K_{g_1}(r-y) \right) \alpha_w(t) dr dt \]

\[ \times \int_{t_1}^{t_2} \int_{s_1}^{s_2} z_1(s) K_{g_1}(s-y) (p(y) - p(s)) \]

\[ \times \left( h(s) K_{g_1''}(s-x) - h''(s) K_{g_1}(s-x) \right) \alpha_w(m) ds dm h(x) h(y) dx dy. \]
and the change of variable $\frac{r_1}{y_1} = r_1$, $\frac{r_2}{y_2} = y_1$, $\frac{r_3}{y_3} = s_1$ gives

$$R_1^{1,2,1}(g_1) = g_1^{-5} \int_0^\infty \int_{c'}^{t'} \int_{c'}^{t'} \int_{c'}^{t'} z_3(x + g_1 r_1) K(r_1) (p(x) - p(x + g_1 r_1)) \times (h(x + g_1 r_1) K''(r_1 - y_1) - g_1^2 h''(x + g_1 r_1) K(r_1 - y_1)) \alpha_w(t) dr_1 dt$$

$$\times \int_{c'}^{t'} \int_{c'}^{t'} \int_{c'}^{t'} z_3(x + g_1 s_1) K(s_1 - y_1) (p(x + g_1 y_1) - p(x + g_1 s_1)) \times (h(x + g_1 s_1) K''(s_1) - g_1^2 h''(x + g_1 s_1) K(s_1)) \alpha_w(m) ds_1 dm \times h(x) h(x + g_1 y_1) dy_1 dx$$

$$= o(g_1^{-5}).$$

Now using $I_{131}(g_1) = O(g_1^2)$, we conclude (179).

Finally, we consider the term $R_1^{1,1,1}(g_1)$ and define

$$R_1^{1,1,1}(g_1) = E \left( \int_{c'}^{t'} \int_{c'}^{t'} z_3(r) K_{g_1}(r - T_1) \delta_1 - p(r) \right)$$

$$\times (h(r) K''(r - T_1) - h''(r) K_{g_1}(r - T_1)) \alpha_w(t) dr dt$$

$$\times \int_{c'}^{t'} \int_{c'}^{t'} z_3(s) K_{g_1}(s - T_1) \delta_1 - p(s)$$

$$\times (h(s) K''(s - T_1) - h''(s) K_{g_1}(s - T_1)) \alpha_w(m) ds dm.$$

Thus

$$R_1^{1,1,1}(g_1) = R_1^{1,1,1}(g_1) - I_{132}(g_1)^2.$$

The term $R_1^{1,1,1}(g_1)$ can be directly bounded

$$|R_1^{1,1,1}(g_1)| \leq g_1^{-2} \|K\|_{\infty}^2 \left( g_1^{-6} \|K''\|_{\infty}^2 \left( \int_{c'}^{t'} \int_{c'}^{t'} |z_3(r) h(r) \alpha_w(t)| dr dt \right)^2 \right)$$

$$+ g_1^{-2} \|K\|_{\infty}^2 \left( \int_{c'}^{t'} \int_{c'}^{t'} |z_3(r) h''(r) \alpha_w(t)| dr dt \right)^2$$

and since $\int_{c'}^{t'} \int_{c'}^{t'} |z_3(r) h(r) \alpha_w(t)| dr dt$ and $\int_{c'}^{t'} \int_{c'}^{t'} |z_3(r) h''(r) \alpha_w(t)| dr dt$ are finite,

$$R_1^{1,1,1}(g_1) = O(g_1^{-8}).$$

Now (171) implies $I_{132}(g_1) = o(g_1^{-3})$ and the final result (180) is obtained.

The order of the covariances $\text{Cov} \left( \hat{A}_{12}, \hat{A}_{13} \right)$, $\text{Cov} \left( \hat{A}_{11}, \hat{A}_{12} \right)$ and $\text{Cov} \left( \hat{A}_{11}, \hat{A}_{13} \right)$ is studied in the next lemma.
Lemma 38  Under the conditions of Lemma 17,

\[ \text{Cov} \left( \tilde{A}_{11}, \tilde{A}_{12} \right) = o \left( n^{-2} g_1^{-6} \right), \quad (181) \]

\[ \text{Cov} \left( \tilde{A}_{11}, \tilde{A}_{13} \right) = o \left( n^{-2} g_1^{-6} \right) \quad (182) \]

and

\[ \text{Cov} \left( \tilde{A}_{12}, \tilde{A}_{13} \right) = o \left( n^{-2} g_1^{-6} \right). \quad (183) \]

Proof. Using Lemmas 17, 23 and 36, the variances \( \text{Var} \left( \tilde{A}_{11} \right) \), \( \text{Var} \left( \tilde{A}_{12} \right) \) and \( \text{Var} \left( \tilde{A}_{13} \right) \) are of the following orders \( O(n^{-1}) \), \( o \left( n^{-2} g_1^{-6} \right) \) and \( o \left( n^{-1} \right) \), respectively. Under condition (V.1),

\[ \frac{\text{Var} \left( \tilde{A}_{11} \right)}{n^{-2} g_1^{-6}} = \frac{\text{Var} \left( \tilde{A}_{11} \right)}{n^{-1}} n g_1^6 = O(1) o(1) = o(1) \]

and, similarly,

\[ \frac{\text{Var} \left( \tilde{A}_{13} \right)}{n^{-2} g_1^{-6}} = \frac{\text{Var} \left( \tilde{A}_{13} \right)}{n^{-2} g_1^{-5} + n^{-1}} \frac{n^{-2} g_1^{-5} + n^{-1}}{n^{-2} g_1^{-6}} = \frac{\text{Var} \left( \tilde{A}_{13} \right)}{n^{-2} g_1^{-5} + n^{-1}} (g_1 + n g_1^6) \]

\[ = o(1) o(1) = o(1), \]

which implies that \( \text{Var} \left( \tilde{A}_{11} \right) \) and \( \text{Var} \left( \tilde{A}_{13} \right) \) are of order \( o \left( n^{-2} g_1^{-6} \right) \).

Therefore, Cauchy-Schwarz inequality gives

\[ \left| \text{Cov} \left( \tilde{A}_{11}, \tilde{A}_{12} \right) \right| \leq \text{Var} \left( \tilde{A}_{11} \right)^{1/2} \text{Var} \left( \tilde{A}_{12} \right)^{1/2} = o \left( n^{-2} g_1^{-6} \right), \]

\[ \left| \text{Cov} \left( \tilde{A}_{11}, \tilde{A}_{13} \right) \right| \leq \text{Var} \left( \tilde{A}_{11} \right)^{1/2} \text{Var} \left( \tilde{A}_{13} \right)^{1/2} = o \left( n^{-2} g_1^{-6} \right) \]

and

\[ \left| \text{Cov} \left( \tilde{A}_{12}, \tilde{A}_{13} \right) \right| \leq \text{Var} \left( \tilde{A}_{12} \right)^{1/2} \text{Var} \left( \tilde{A}_{13} \right)^{1/2} = o \left( n^{-2} g_1^{-6} \right) \]

and, therefore (181), (182) and (183). \( \blacksquare \)

Lemma 39  Under the assumptions of Lemma 17,

\[ \tilde{A}_1 - A = O_P \left( g_1^2 + n^{-1} g_1^{-3} \right). \quad (184) \]
Proof. Recall (44). The term $\tilde{A}_1$ is defined as $\tilde{A}_1 = A + \tilde{A}_{11} + \tilde{A}_{12} + \tilde{A}_{13}$. From Lemmas 17, 23, 36 and 38, it is straightforward to conclude that

$$E \left( \tilde{A}_1 - A \right) = C_1 g_1^2 + C_2 n^{-1} g_1^{-3} + O \left( g_1^4 \right) + o \left( n^{-1} g_1^{-3} \right)$$

(185)

and

$$\text{Var} \left( \tilde{A}_1 - A \right) = O(n^{-1}) + o \left( n^{-2} g_1^{-6} \right),$$

(186)

where $C_1$ and $C_2$ where defined in (46) and (47). A direct application of Tchebychev inequality gives (184).

Note that the terms $\tilde{A}_{11}$, $\tilde{A}_{12}$ and $\tilde{A}_{13}$ of $\tilde{A}_1$ have, at most, two factors of the form $(\tilde{h}^{(i)} - h^{(i)})$, $(\tilde{\psi}^{(j)} - \psi^{(j)})$, $i, j = 0, 1, 2$. On the other hand, in the expressions that form the term $\tilde{A}_{14}$ in the representation (44) there are, at least, three factors of the form $(\tilde{p} - p)$, $(\tilde{h}^{(i)} - h^{(i)})$ or $(\tilde{\psi}^{(j)} - \psi^{(j)})$, $i, j = 0, 1, 2$.

> From now on we will establish that the term $\tilde{A}_{14}$ is negligible with respect to $\tilde{A}_1 - A$. To do that term will be bounded in probability at a faster rate than that given in Lemma 184. A useful tool will be the following generalization of the results by Silverman (1978) and Mack-Silverman (1982) to the case of $k$-th order derivatives of the density and the regression functions. More precisely, these results state,

$$\sup_{t \in \mathbb{R}} \left| \tilde{\psi}^{(k)}(t) - \psi^{(k)}(t) \right| = O_P \left( g_1^2 + n^{-\frac{1}{2}} g_1^{-\frac{1}{2} - k} \left( \log \frac{1}{g_1} \right)^{\frac{1}{2}} \right),$$

$$\sup_{t \in \mathbb{R}} \left| \tilde{h}^{(k)}(t) - h^{(k)}(t) \right| = O_P \left( g_1^2 + n^{-\frac{1}{2}} g_1^{-\frac{1}{2} - k} \left( \log \frac{1}{g_1} \right)^{\frac{1}{2}} \right),$$

and

$$\sup_{t \in [\varepsilon', \pi]} \left| \tilde{p}^{(k)}(t) - p^{(k)}(t) \right| = O_P \left( n^{-\frac{1}{2}} g_1^{-\frac{1}{2} - k} \left( \log \frac{1}{g_1} \right)^{\frac{1}{2}} \right).$$

Since the previous results require $\frac{ng_1^{1+1}}{(\log \frac{1}{g_1})} \to \infty$, they will only be used, according to condition (V.1), for the cases $k = 0, 1$.

The following representation of $\tilde{A}_{14}$, obtained using (43), will be useful

$$\tilde{A}_{14} = \tilde{A}_{141} + \tilde{A}_{142} + \tilde{A}_{143},$$

(187)
where

\[ \tilde{A}_{141} = -\frac{1}{2} \int_{c'}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_1(r)z_2(s) \left( (\hat{\psi}'(r) - \psi'(r)) - p(r)(\hat{h}'(r) - h'(r)) \right) \times (\tilde{p}(s) - p(s))(\hat{h}'(s) - h'(s))w(t)drdsdt \]

\[ + \frac{1}{4} \int_{c'}^{\infty} \left( \int_{c'}^{t} z_1(r)(\tilde{p}(r) - p(r))(\hat{h}'(r) - h'(r))dr \right)^2 w(t)dt, \]

\[ \tilde{A}_{142} = -\frac{1}{2} \int_{c'}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_1(r)z_2(s) \left( (\hat{\psi}'(r) - \psi'(r)) - p(r)(\hat{h}'(r) - h'(r)) \right) \times (\tilde{p}(s) - p(s))(\hat{h}'(s) - h'(s))w(t)drdsdt \]

\[ - \frac{1}{2} \int_{c'}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_2(r)z_1(s) \left( (\tilde{\psi}(r) - \psi(r)) - p(r)(\tilde{h}(r) - h(r)) \right) \times (\tilde{p}(s) - p(s))(\hat{h}'(s) - h'(s))w(t)drdsdt \]

\[ + \int_{c'}^{\infty} \int_{c'}^{t} z_1(r)(\tilde{p}(r) - p(r))(\hat{h}(r) - h(r))(\hat{h}''(r) - h''(r))\alpha_w(t)drdt \]

\[ + \frac{1}{2} \int_{c'}^{\infty} \int_{c'}^{t} \int_{c'}^{t} z_1(r)z_2(s)(\tilde{p}(r) - p(r))(\tilde{h}'(r) - h'(r)) \times (\tilde{p}(s) - p(s))(\tilde{h}(s) - h(s))w(t)drdsdt \]

and

\[ \tilde{A}_{143} = \frac{1}{4} \int_{c'}^{\infty} \left( \int_{c'}^{t} z_2(r)(\tilde{p}(r) - p(r))(\tilde{h}(r) - h(r))dr \right)^2 w(t)dt \]

\[ - \frac{1}{2} \int_{c'}^{\infty} \int_{c'}^{t} z_2(r)(\tilde{\psi}(r) - \psi(r)) \times z_2(s)(\tilde{p}(s) - p(s))(\tilde{h}(s) - h(s))w(t)drdsdt \]

\[ + \int_{c'}^{\infty} \int_{c'}^{t} z_2(r)(\tilde{p}(r) - p(r))(\tilde{h}(r) - h(r))^2 \alpha(t)w(t)drdt. \]

**Proof of Lemma 12.** As mentioned above the expressions in the representation \((187)\) of \(\tilde{A}_{14}\) can be classified according to the number of factors \((3 \text{ or } 4)\) of the type \((\tilde{h}^{(i)} - h^{(i)})\), \((\tilde{\psi}^{(j)} - \psi^{(j)})\) or \((\tilde{p} - p)\), with \(i, j = 0, 1, 2\). The general outline of the proof consists in applying the results by Silverman (1978) and Mack and Silverman (1982), and their generalized versions presented after Lemma 39. As pointed out there, the conditions imposed on the bandwidth, collected in \((V.1)\), make these results applicable only in the case when the order of the derivatives appearing in any factor is not larger than one. Consequently, when the maximal order of the derivatives in a term is larger than one, we will try to reduce it using partial integration first.
By proceeding in such a way with the first summand in \( \hat{A}_{141} \), decomposing it as a sum of two terms and using

\[
\begin{align*}
z_1(r) &= u_1, \hat{\psi}''(r) - \psi''(r) dr = dv_1, \\
z_1(s)(\hat{p}(s) - p(s)) &= u_2, \hat{h}''(s) - h''(s) ds = dv_2,
\end{align*}
\]

for the first one and

\[
\begin{align*}
z_{12}(r) &= u_1, \hat{h}''(r) - h''(r) dr = dv_1, \\
z_1(s)(\hat{p}(s) - p(s)) &= u_2, \hat{h}''(s) - h''(s) ds = dv_2,
\end{align*}
\]

for the second, we get

\[
\begin{align*}
&-\frac{1}{2} \int_{\epsilon_1}^{t} \int_{\epsilon_2}^{t} \int_{\epsilon_3}^{t} z_1(r)z_1(s) \left( \left( \hat{\psi}''(r) - \psi''(r) \right) - p(r) \left( \hat{h}''(r) - h''(r) \right) \right) \\
&\times (\hat{p}(s) - p(s)) (\hat{h}''(s) - h''(s)) w(t) dr ds dt \\
&= -\frac{1}{2} \int_{\epsilon_1}^{t} \int_{\epsilon_2}^{t} \int_{\epsilon_3}^{t} \left( z_1(t) (\hat{\psi}'(t) - \psi'(t)) - z_1(\epsilon') (\hat{\psi}'(\epsilon') - \psi'(\epsilon')) \right) \\
&\times (\hat{p}(s) - p(s)) (\hat{h}'(t) - h'(t)) - z_1(\epsilon') (\hat{p}(\epsilon') - p(\epsilon')) (\hat{h}'(\epsilon') - h'(\epsilon')) \\
&- \int_{\epsilon_2}^{t} (z_1'(s)(\hat{p}(s) - p(s)) + z_1(s)(\hat{p}'(s) - p'(s))) (\hat{h}'(s) - h'(s)) ds w(t) dt \\
&+ \frac{1}{2} \int_{\epsilon_2}^{t} \int_{\epsilon_3}^{t} \left( z_{12}(t) (\hat{h}'(t) - h'(t)) - z_{12}(\epsilon') (\hat{h}'(\epsilon') - h'(\epsilon')) \right) \\
&\times (\hat{p}(s) - p(s)) (\hat{h}'(t) - h'(t)) - z_1(\epsilon') (\hat{p}(\epsilon') - p(\epsilon')) (\hat{h}'(\epsilon') - h'(\epsilon')) \\
&- \int_{\epsilon_3}^{t} (z_1'(s)(\hat{p}(s) - p(s)) + z_1(s)(\hat{p}'(s) - p'(s))) (\hat{h}'(s) - h'(s)) ds w(t) dt
\end{align*}
\]
Now, since $\epsilon = 1$ and $\psi(\epsilon') = h'(\epsilon')$ and for small enough bandwidths we have $\hat{p}(\epsilon') = 1$ and $\hat{\psi}(\epsilon') = \hat{h}'(\epsilon')$, with probability 1, the previous expression gives

\[
- \frac{1}{2} \int_{\epsilon'}^{\epsilon} \left( z_1(t) \left( \hat{\psi}'(t) - \psi'(t) \right) - p(t)(\hat{h}'(t) - h'(t)) \right) \, dt
\]

For the second summand of $\hat{A}_{141}$ a simple comparison with the first one gives

\[
= \frac{1}{4} \int_{\epsilon'}^{\epsilon} \left( \int_{\epsilon'}^{t} z_1(r)(\hat{p}(r) - p(r))(\hat{h}''(r) - h''(r)) \, dr \right)^2 w(t) \, dt
\]

Expressions (188) and (189) and the generalized Silverman and Mack-Silverman results lead to

\[
\hat{A}_{141} = O_P \left( n^{-\frac{1}{2}} g_1^{\frac{5}{2}} \left( \log \frac{1}{g_1} \right)^{\frac{1}{2}} + n^{-\frac{3}{2}} g_1^{-\frac{2}{3}} \left( \log \frac{1}{g_1} \right)^{\frac{5}{2}} \right)
\]
and, using condition (V.1),

$$\hat{A}_{141} = O_P \left( n^{-\frac{3}{2}g_1^{-\frac{3}{2}} \left( \log \frac{1}{g_1} \right)^{\frac{3}{2}}} \right).$$ \quad (190)$$

The term $\hat{A}_{142}$ can be handled using similar algebra to that already used for $\hat{A}_{141}$.

$$\hat{A}_{142} = -\frac{1}{2} \int_{c'}^{\infty} \left( z_1(t) \left( \psi'(t) - \psi'(t) - p(t)(\hat{h}'(t) - h'(t)) \right) 
- \int_{c'}^{t} \left( z_1'(r)(\psi'(r) - \psi'(r)) - z_{12}(r)(\hat{h}'(r) - h'(r)) \right) dr \right) 
\times \int_{c'}^{\infty} z_2(s)(p(s) - p(s))(\hat{h}(s) - h(s))dw(t)dt 
- \frac{1}{2} \int_{c'}^{\infty} z_2(r) \left( \psi'(r) - p(r)(\hat{h}(r) - h(r)) \right) 
\times \left( z_1(t)(p(t) - p(t))(\hat{h}'(t) - h'(t)) \right) 
- \frac{1}{2} \int_{c'}^{\infty} z_{12}(r)(\hat{h}'(r) - h'(r)) \right) \right) \alpha_w(t)dt 
+ \frac{1}{2} \int_{c'}^{\infty} \left( z_1(t)(p(t) - p(t))(\hat{h}'(t) - h'(t)) \right) 
- \frac{1}{2} \int_{c'}^{\infty} \left( z_1(t)(p(t) - p(t))(\hat{h}'(t) - h'(t)) \right) \alpha_w(t)dt 
+ \frac{1}{2} \int_{c'}^{\infty} \left( z_1(t)(p(t) - p(t))(\hat{h}'(t) - h'(t)) \right) 
\times \int_{c'}^{\infty} z_2(s)(p(s) - p(s))(\hat{h}(s) - h(s))dw(t)dt$$

and a further application of the results by Silverman and Mack-Silverman, together with condition (V.1), give

$$\hat{A}_{142} = O_P \left( n^{-\frac{3}{2}g_1^{-\frac{3}{2}} \left( \log \frac{1}{g_1} \right)^{\frac{3}{2}}} \right).$$ \quad (191)$$

Finally, the results by Silverman and Mack-Silverman can be directly applied for the term $\hat{A}_{143}$ to obtain, under (V.1),

$$\hat{A}_{143} = O_P \left( n^{-\frac{1}{2}g_1^{-\frac{1}{2}} \left( \log \frac{1}{g_1} \right)^{\frac{1}{2}}} \right).$$ \quad (192)$$
Collecting expressions (190), (191) and (192) and using (V.1) it is straightforward to conclude

\[ \hat{A}_{14} = O_P \left( n^{-\frac{3}{2}} g_1^{-\frac{3}{2}} \left( \log \frac{1}{g_1} \right)^{\frac{3}{2}} \right). \] (193)

The final result in the lemma is a direct consequence of (193) and condition (V.1). \( \Box \)

**Proof of Theorem 11.** From expressions (185) and (186) for the expectation and the variance of \( A_1 \), obtained in the proof of Lemma 39, it is straightforward to derive its asymptotic mean squared error, as a function of \( g_1 \),

\[ MSE (\hat{A}_1) = \left( C_1 g_1^2 + C_2 n^{-1} g_1^{-3} \right)^2 + O \left( g_1^6 \right) + o \left( n^{-1} g_1^{-1} \right) + o \left( n^{-2} g_1^{-6} \right). \]

The value of \( g_1 \) that minimizes the mean squared error is, up to negligible terms, that one minimizing its dominant part. Since \( C_2 > 0 \), two different cases have to be considered according to the sign of \( C_1 \). The particular case \( C_1 = 0 \) would lead to a deeper analysis of second order terms in the bias and the variance. For this reason it will not be considered here. If \( C_1 < 0 \), the minimum is attained for \( g_1 \) satisfying \( C_1 g_1^2 + C_2 n^{-1} g_1^{-3} = 0 \), i.e.,

\[ g_1 = \left( -\frac{C_2}{C_1} \right)^{\frac{1}{5}} n^{-\frac{1}{5}}. \]

If \( C_1 > 0 \) the minimum is attained at some point where the derivative of \( C_1 g_1^2 + C_2 n^{-1} g_1^{-3} \), with respect to \( g_1 \), is zero. This gives

\[ g_1 = \left( \frac{3C_2}{2C_1} \right)^{\frac{1}{5}} n^{-\frac{1}{5}}, \]

an the proof of the theorem is concluded. \( \Box \)

### 6.3 Auxiliary results for the term \( \hat{Q} \)

**Proof of Lemma 13.** The estimator \( \hat{Q} \) can be represented as

\[ \hat{Q} = \hat{Q}_1 + \hat{Q}_2 \]

where

\[ \hat{Q}_1 = \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} \hat{p}(T_i) (1 - \hat{p}(T_i)) w(T_i) \]
and

\[
\hat{Q}_2 = \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} \tilde{p}(T_i) (1 - \tilde{p}(T_i)) w(T_i) \\
\times \left( \left( 1 - H_n(T_i) + \frac{1}{n} \right)^{-2} (1 - H(T_i))^2 - 1 \right)^{1/2},
\]

(194)

Now,

\[
\hat{Q}_2 = \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} \tilde{p}(T_i) (1 - \tilde{p}(T_i)) w(T_i) \left( 1 - H_n(T_i) + \frac{1}{n} \right)^{-2} \\
\times \left( (1 - H(T_i))^2 - \left( 1 - H_n(T_i) + \frac{1}{n} \right)^2 \right) \\
= \frac{1}{n} \sum_{i=1}^{n} (1 - H(T_i))^{-2} \tilde{p}(T_i) (1 - \tilde{p}(T_i)) w(T_i) \left( 1 - H_n(T_i) + \frac{1}{n} \right)^{-2} \\
\times \left( 1 - H(T_i) - (H_n(T_i) - H(T_i)) + \frac{1}{n} \right)^{-2} \\
\times \left( H_n(T_i) - H(T_i) - \frac{1}{n} \right) \left( 2 - H(T_i) - H_n(T_i) + \frac{1}{n} \right)
\]

and, since using (W.1) the support of the weight function, \(w\), is \((\varepsilon', t_0)\),

\[
\left| \hat{Q}_2 \right| \leq \frac{3}{n} \left( \sup_{\varepsilon' < t < t_0} |H_n(t) - H(t)| + \frac{1}{n} \right) \\
\times \sum_{i=1}^{n} (1 - H(T_i))^{-2} \tilde{p}(T_i) (1 - \tilde{p}(T_i)) w(T_i) \left( \frac{1}{n} - H(T_i) \right)^{-2}.
\]

There exist \(n_0 \in \mathbb{N}\) and \(\delta > 0\) such that for all \(n \geq n_0\) and \(t > \varepsilon'\) it holds \(\left| \frac{1}{n} - H(t) \right| > \delta\), and therefore,

\[
\left| \hat{Q}_2 \right| \leq \frac{3}{\delta^2} \left( \sup_{\varepsilon' < t < t_0} |H_n(t) - H(t)| + \frac{1}{n} \right) \hat{Q}_1.
\]

Now using well known empirical processes results,

\[
\sup_{\varepsilon' < t < t_0} |H_n(t) - H(t)| = O_P \left( n^{-\frac{1}{2}} \right),
\]

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and it is straightforward to conclude that
\[ \hat{Q}_2 = O_P \left( n^{-\frac{1}{2}} \hat{Q}_1 \right) = o_P \left( \hat{Q}_1 \right), \]
which proves (48).

**Lemma 40** Assume (K.1), (P.1), (H.1), (W.1) and (V.2). Then,
\[ E(\hat{Q}_{11}) = Q \]
and
\[ \text{Var}(\hat{Q}_{11}) = n^{-1} \left( \int_0^\infty p(x)^2 (1 - p(x))^2 z_5(x)^2 h(x) dx - Q^2 \right). \]

**Proof.** Since, by definition, \( q = (1 - H)^{-2} p(1 - p) h \),
\[ E(\hat{Q}_{11}) = E(p(T_1)(1 - p(T_1)) z_5(T_1)) = \int_0^\infty q(x) w(x) dx = Q \]
and, on the other hand,
\[ \text{Var}(\hat{Q}_{11}) = n^{-1} \text{Var}(p(T_1)(1 - p(T_1)) z_5(T_1)) \]
\[ = n^{-1} \left( \int_0^\infty p(x)^2 (1 - p(x))^2 z_5(x)^2 h(x) dx - Q^2 \right). \]

**Lemma 41** Under conditions (K.1), (P.1), (H.1), (W.1) and (V.2),
\[ E\left( \hat{Q}_{12} \right) = g_2^2 \mu_K \int_e^\infty z_6(x) \left( p'(x) h'(x) + \frac{1}{2} p''(x) h(x) \right) dx + o \left( g_2^2 \right) + O \left( n^{-1} g_2^2 \right) \]
and
\[ \text{Var} \left( \hat{Q}_{12} \right) = n^{-1} \int_e^\infty p(x) (1 - p(x)) z_6(x)^2 h(x) dx + O \left( n^{-1} g_2^2 \right) + O \left( n^{-2} g_2^1 \right). \]

**Proof.** For the expectation it holds
\[ E \left( \hat{Q}_{12} \right) = \frac{1}{n} \sum_{i=1}^n E \left( (\hat{\psi}(T_i) - p(T_i) \tilde{h}(T_i)) z_6(T_i) \right) \]
\[ = \frac{1}{n} \sum_{i=1}^n E \left( \frac{1}{n} \sum_{j=1}^n K_{g_2} (T_i - T_j) (\delta_j - p(T_i)) z_6(T_i) \right) \]
\[ = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n E(K_{g_2} (T_i - T_j) (p(T_j) - p(T_i)) z_6(T_i)) \]
\[ = \frac{n(n-1)}{n^2} E(K_{g_2} (T_1 - T_2) (p(T_2) - p(T_1)) z_6(T_1)), \]
\[ \text{Var} \left( \hat{Q}_{12} \right) = n^{-1} \int_e^\infty p(x) (1 - p(x)) z_6(x)^2 h(x) dx + O \left( n^{-1} g_2^2 \right) + O \left( n^{-2} g_2^1 \right). \]
and therefore
\[ E\left(\hat{Q}_{12}\right) = \frac{n(n-1)}{n^2} \int_{\varepsilon'}^\infty \int_0^\infty K_{g_2}(x-y) \left(p(y) - p(x)\right) z_{61}(x)h(x)h(y)dydx, \]

where it has been used that \( w, \) and thus \( z_{61}, \) vanish for \( x < \varepsilon'. \)

Define
\[ I_{12}(g_2) = \int_{\varepsilon'}^\infty \int_0^\infty K_{g_2}(x-y) \left(p(y) - p(x)\right) z_{61}(x)h(x)h(y)dydx. \]  

(198)

The change of variable \( \frac{g_2 y}{y_1} = y_1 \) gives
\[ I_{12}(g_2) = \int_{\varepsilon'}^\infty \int_0^\infty K(y_1) \left(p(x - g_2 y_1) - p(x)\right) h(x - g_2 y_1)z_6(x)g_2y_1dx \]

and, for \( g_2 < \frac{\varepsilon'}{L}, \)
\[ I_{12}(g_2) = \int_{\varepsilon'}^\infty \int_{-L}^{L} K(y_1) \left(p(x - g_2 y_1) - p(x)\right) h(x - g_2 y_1)z_6(x)dy_1dx. \]

Using Taylor expansions of \((p(x - g_2 y_1) - p(x))\) and \(h(x - g_2 y_1)\) around \(x,\) it is easy to prove that
\[ I_{12}(g_2) = g_2^2 \mu_K \int_{\varepsilon'}^\infty z_6(x) \left(p'(x)h'(x) + \frac{1}{2} p''(x)h(x) \right) dx + o\left(g_2^2\right) \]

which directly implies (196).

The variance of \( \hat{Q}_{12} \) is
\[ \text{Var} \left( \hat{Q}_{12} \right) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \text{Cov} \left( \left(\hat{\psi}(T_i) - p(T_i) \hat{h}(T_i)\right)z_{61}(T_i), \right. \]
\[ \left. \left(\hat{\psi}(T_j) - p(T_j) \hat{h}(T_j)\right)z_{61}(T_j) \right) \]
\[ = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \text{Cov} \left( \frac{1}{n} \sum_{k=1}^n K_{g_2}(T_i - T_k) \left(\delta_k - p(T_i)\right) z_{61}(T_i), \right. \]
\[ \left. \frac{1}{n} \sum_{i=1}^n K_{g_2}(T_j - T_i) \left(\delta_i - p(T_j)\right) z_{61}(T_j) \right) \]
\[ = \frac{1}{n^4} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n M_{i,j,k,l}(g_2) \]
\[ = \frac{1}{n^4} \left( n M^{1,1,1,1}(g_2) + 2n(n-1)M^{1,2,2,2}(g_2) \right. \]
\[ + 2n(n-1)M^{1,1,2,1}(g_2) + n(n-1)M^{1,1,1,2}(g_2) \]
\[ + n(n-1)M^{1,2,2,1}(g_2) + n(n-1)(n-2)M^{1,2,3,1}(g_2) \]
\[ + n(n-1)(n-2)M^{1,2,3,1}(g_2) + n(n-1)(n-2)M^{1,2,2,3}(g_2) \]
\[ + n(n-1)(n-2)M^{1,1,2,3}(g_2) \]  

(199)

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where, for $1 \leq i, j, k, l \leq n$,
\[
M^{i,j,k,l}(g_2) = \text{Cov}(K_{g_2}(T_i - T_k)(\delta_k - p(T_i)) z_{61}(T_i), K_{g_2}(T_j - T_l)(\delta_l - p(T_l)) z_{61}(T_j)).
\]

The thesis of the lemma is a direct consequence of condition (V.2) and the following lemma, that is concerned with the orders of the covariances $M^{i,j,k,l}(g_2)$.

\begin{lemma}
Assume the conditions of Lemma 41. Then
\[
M^{1,1,1,1}(g_2) = O(g_2^{-2}),
\]
\[
M^{1,2,2,2}(g_2) = O(g_2^{-1}),
\]
\[
M^{1,1,2,1}(g_2) = 0,
\]
\[
M^{1,1,2,2}(g_2) = O(g_2^{-1}),
\]
\[
M^{1,2,2,1}(g_2) = o(g_2^{-1}),
\]
\[
M^{1,2,3,3}(g_2) = \int_{c'}^{\infty} p(s) (1 - p(s)) z_6(s)^2 h(s) ds + O(g_2^2),
\]
\[
M^{1,2,3,1}(g_2) = o(g_2^2),
\]
\[
M^{1,2,2,3}(g_2) = o(g_2^2),
\]
and
\[
M^{1,1,2,3}(g_2) = O(g_2^2).
\]
\end{lemma}

\begin{proof}
Starting from $M^{1,1,1,1}(g_2)$
\[
M^{1,1,1,1}(g_2) = E \left( K_{g_2} (0)^2 (\delta_1 - p(T_1))^2 z_{61}(T_1)^2 \right)
\]
\[
- E \left( K_{g_2} (0) (\delta_1 - p(T_1)) z_{61}(T_1) \right)^2
\]
\[
= E \left( K_{g_2} (0)^2 (\delta_1 - p(T_1))^2 z_{61}(T_1)^2 \right)
\]
\[
= g_2^{-2} K(0) \int_{c'}^{\infty} p(x) (1 - p(x)) z_{61}(x)^2 h(x) dx,
\]
\end{proof}
which gives (200).

The term $M_{1.2.2.2}(g_2)$ can be easily analyzed

\[
M_{1.2.2.2}(g_2) = K_{g_2}(0)E(K_{g_2}(T_1 - T_2) (\delta_2 - p(T_1)) z_{61}(T_1) (\delta_2 - p(T_2)) z_{61}(T_2))
- E(K_{g_2}(T_1 - T_2) (\delta_2 - p(T_1)) z_{61}(T_1)) E(K_{g_2}(0) (\delta_1 - p(T_1)) z_{61}(T_1))
= K_{g_2}(0)E(K_{g_2}(T_1 - T_2) (\delta_2 - p(T_1)) z_{61}(T_1) (\delta_2 - p(T_2)) z_{61}(T_2)).
\]

Thus,

\[
M_{1.2.2.2}(g_2) = g_2^{-1}K(0) \int_{\mathbb{R}} \int_{\mathbb{R}} K_{g_2}(x - y)p(y) (1 - p(y)) z_{61}(x) z_{61}(y) h(y) h(x) dy dx
= g_2^{-1}K(0) \int_{\mathbb{R}} \int_{\mathbb{R}} K(y_1) p(x - g_2 y_1) (1 - p(x - g_2 y_1)) z_6(x) z_6(x - g_2 y_1) dy_1 dx
= g_2^{-1}K(0) \int_{\mathbb{R}} p(x) (1 - p(x)) z_6(x)^2 dx + o(g_2^{-1}),
\]

after using the change of variable $\frac{x - g_2 y_1}{g_2} = y_1$. Expression (201) is then straightforward.

Concerning $M_{1.1.2.1}(g_2)$, it is easy to prove (202). Indeed,

\[
M_{1.1.2.1}(g_2) = E(K_{g_2}(T_1 - T_2) K_{g_2}(0) (\delta_2 - p(T_1)) (\delta_1 - p(T_1)) z_{61}(T_1)^2)
- E(K_{g_2}(T_1 - T_2) (\delta_2 - p(T_1)) z_{61}(T_1)) E(K_{g_2}(0) (\delta_1 - p(T_1)) z_{61}(T_1))
= 0.
\]

Standard algebra gives for $M_{1.1.2.2}(g_2)$:

\[
M_{1.1.2.2}(g_2) = E(K_{g_2}(T_1 - T_2)^2 (\delta_2 - p(T_1))^2 z_{61}(T_1)^2)
- E(K_{g_2}(T_1 - T_2) (\delta_2 - p(T_1)) z_{61}(T_1))^2
= M_{1.1.2.2}(g_2) - I_{12}(g_2)^2
\]

where $I_{12}(g_2)$ was defined in (198). Now

\[
M_{1.1.2.2}(g_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} K_{g_2}(x - y)^2 p(y) (1 - 2p(x)) + p(x)^2 z_6(x)^2 h(y) h(x) dy dx
= g_2^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} K(y_1)^2 (p(x - g_2 y_1) (1 - 2p(x)) + p(x))
\times z_6(x)^2 h(x - g_2 y_1) h(x) dy_1 dx
= g_2^{-1} c_K \int_{\mathbb{R}} p(x) (1 - p(x)) z_6(x)^2 dx + o(g_2^{-1})
\]

after using the change of variable $\frac{x - g_2 y_1}{g_2} = y_1$. This gives

\[
M_{1.1.2.2}(g_2) = g_2^{-1} c_K \int_{\mathbb{R}} p(x) (1 - p(x)) z_6(x)^2 dx + o(g_2^{-1}),
\]

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and, therefore, (203).

The term $M^{1.2.2.1}(g_2)$ can be treated similarly

$$M^{1.2.2.1}(g_2) = E \left( K_{g_2} (T_1 - T_2)^2 (\delta_2 - p(T_1)) (\delta_1 - p(T_2)) z_{61}(T_1) z_{61}(T_2) \right)$$

$$- E \left( K_{g_2} (T_1 - T_2) (\delta_2 - p(T_1)) z_{61}(T_1) \right)^2$$

$$= M^{1.2.2.1}_1(g_2) - I_{12}(g_2)^2$$

and

$$M^{1.2.2.1}_1(g_2) = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{g_2} (x - y)^2 (p(y) - p(x))^2 z_{61}(x) z_{61}(y) h(y) h(x) dy dx$$

$$= -g_2^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(y_1)^2 (p(x - g_2 y_1) - p(x))^2 z_6(x) z_6(x - g_2 y_1) dy_1 dx$$

$$= o \left( g_2^{-1} \right)$$

where the change of variable $\frac{y - y_1}{g_2} = y_1$ has been used. Thus (204) is a straightforward consequence of the preceding results.

Let’s consider $M^{1.2.3.3}(g_2)$,

$$M^{1.2.3.3}(g_2) = E \left( K_{g_2} (T_1 - T_3) (\delta_3 - p(T_1)) z_{61}(T_1) K_{g_2} (T_2 - T_3) (\delta_3 - p(T_2)) z_{61}(T_2) \right)$$

$$- E \left( K_{g_2} (T_1 - T_3) (\delta_3 - p(T_1)) z_{61}(T_1) \right)^2$$

$$= M^{1.2.3.3}_1(g_2) - I_{12}(g_2)^2$$

with

$$M^{1.2.3.3}_1(g_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{g_2}(x - s) K_{g_2}(y - s) (p(s) (1 - p(x) - p(y)) + p(x) p(y))$$

$$\times z_6(x) z_6(y) h(s) ds dy dx$$

$$= \int_{0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x_1) K(y_1)$$

$$\times (p(s) (1 - p(s + g_2 x_1) - p(s + g_2 y_1)) + p(s + g_2 x_1) p(s + g_2 y_1))$$

$$\times z_6(s + g_2 x_1) z_6(s + g_2 y_1) h(s) ds dy_1 dx_1 ds,$$

after using the change of variable $\frac{x - x_1}{g_2} = x_1$, $\frac{y - y_1}{g_2} = y_1$.

It is easy to check that

$$\lim_{g_2 \to 0} M^{1.2.3.3}_1(g_2) = \int_{-\infty}^{\infty} p(s) (1 - p(s)) z_6(s)^2 h(s) ds.$$
On the other hand, the derivative of $M^{1,2,3,3}_{1}(g_2)$ is

$$
\frac{dM^{1,2,3,3}_{1}(g_2)}{dg_2} = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} K(x_1)K(y_1) \times (p(s) (1 - p(s + g_2x_1) - p(s + g_2y_1)) + p(s + g_2x_1)p(s + g_2y_1)) \times z_6(s + g_2x_1)z_6(s + g_2y_1)h(s)dy_1dx_1ds
$$

$$
= \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} K(x_1)K(y_1) \times (p(s) (1 - p(x') - p(s + g_2y_1)) + p(x')p(s + g_2y_1)) \times z_6(x')z_6(s + g_2y_1)h(s)dy_1dx_1ds
$$

$$
+ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} K(x_1)K(y_1)\Psi(g_2, x_1, y_1, s)h(s)dy_1dx_1ds
$$

$$
= \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} K(x_1)K(y_1)\Psi(g_2, x_1, y_1, s)h(s)dy_1dx_1ds
$$

where

$$
\Psi(g_2, x_1, y_1, s) = (x_1p'(s + g_2x_1) (p(s + g_2y_1) - p(s)) + y_1p'(s + g_2y_1) (p(s + g_2x_1) - p(s)) \times z_6(s + g_2x_1)z_6(s + g_2y_1)
$$

$$
+ (p(s) (1 - p(s + g_2x_1) - p(s + g_2y_1)) + p(s + g_2x_1)p(s + g_2y_1))
$$

$$
\times (x_1z_6(s + g_2x_1)z_6(s + g_2y_1) + y_1z_6(s + g_2x_1)z_6(s + g_2y_1)),
$$

and we have used that $z_6(x') = 0$. Consequently,

$$
\lim_{g_2 \to 0} \frac{dM^{1,2,3,3}_{1}(g_2)}{dg_2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x_1)K(y_1)(x_1 + y_1)dy_1dx_1
$$

$$
\times \int_{\varepsilon'}^{\infty} p(s)(1 - p(s))z_6(s)z_6'(s)h(s)ds
$$

$$
= 0.
$$

Therefore, we have the following Taylor expansion of $M^{1,2,3,3}_{1}(g_2)$ around 0:

$$
M^{1,2,3,3}_{1}(g_2) = \int_{\varepsilon'}^{\infty} p(s)(1 - p(s))z_6(s)^2h(s)ds + O(g_2^2)
$$

which leads to (205).
The term $M_{1,2,3}^{1}(g_2)$ can be analyzed in a parallel way to the previous ones:

$$M_{1,2,3}^{1}(g_2) = E(K_{g_2}(T_1 - T_3)(\delta_3 - p(T_1)) z_6(T_1)$$
$$\times K_{g_2}(T_2 - T_1)(\delta_1 - p(T_2)) z_6(T_2))$$
$$- E(K_{g_2}(T_1 - T_3)(\delta_3 - p(T_1)) z_6(T_1))^2$$

and

$$M_{1,2,3}^{1}(g_2) = \int_{-C}^{C} \int_{-C}^{C} \int_{0}^{\infty} K_{g_2}(x - s)(p(s) - p(x)) z_6(x)$$
$$\times K_{g_2}(y - x)(p(x) - p(y)) z_6(y) h(s) h(x) ds dy dx$$

$$= \int_{-C}^{C} \int_{-C}^{C} \int_{-\infty}^{\infty} K(s_1)(p(x - g_2 s_1) - p(x)) z_6(x)$$
$$\times K(y_1)(p(x) - p(x - g_2 y_1))$$
$$\times z_6(x - g_2 y_1) h(x - g_2 s_1) ds_1 dy_1 dx$$

$$= \int_{-C}^{C} \int_{-C}^{C} \int_{-\infty}^{L} K(s_1)(p(x - g_2 s_1) - p(x)) z_6(x)$$
$$\times K(y_1)(p(x) - p(x - g_2 y_1))$$
$$\times z_6(x - g_2 y_1) h(x - g_2 s_1) ds_1 dy_1 dx,$$

where we have used the change of variable $\frac{x - y}{g_2} = y_1$, $\frac{x - s}{g_2} = s_1$ and, in the last equation, the condition $g_2 < \frac{C}{g_2}$.

Using Taylor expansions of $p(x - g_2 s_1) - p(x)$ and $h(x - g_2 s_1)$ around $x$ it is easy to check that

$$M_{1,2,3}^{1,2,3}(g_2) = g_2^3 \mu_L \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{g_2 L}{x}} \left( p'(x) h'(x) + \frac{1}{2} p''(x) h(x) \right) z_6(x)$$
$$\times K(y_1)(p(x) - p(x - g_2 y_1)) z_6(x - g_2 y_1) dy_1 dx$$
$$+ o(g_2^3)$$

$$= o(g_2^3),$$

which leads to (206).

The next term is

$$M_{1,2,3}^{1,2,3}(g_2) = E(K_{g_2}(T_1 - T_2)(\delta_2 - p(T_1)) z_6(T_1)$$
$$\times K_{g_2}(T_2 - T_3)(\delta_3 - p(T_2)) z_6(T_2))$$
$$- E(K_{g_2}(T_1 - T_2)(\delta_2 - p(T_1)) z_6(T_1))^2$$

$$= M_{1,2,3}^{1,2,3}(g_2) - I_{12}(g_2)^2,$$

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Now, Taylor expansions of $M$ and $g$ with $M_1(x) = \frac{1}{x}$, $M_2(x) = \frac{x}{x^2 + 1}$, and $M_3(x) = \sqrt{x}$, give

$$M_1^{1,2,3}(g_2) = \int_{x'}^{x} \int_{x'}^{x'} \int_{x'}^{x} K_{g_2}(x-y)(p(y) - p(x)) z_61(x) K_{g_2}(y-s) \times (p(s) - p(y)) z_61(y) h(s) h(y) h(x) ds dy dx$$

$$= \int_{x'}^{x} \int_{x'}^{x} \int_{x'}^{x} K(y_1) (p(y) - p(y + g_2 x_1)) z_6(y + g_2 x_1) \times K(s_1) (p(y - g_2 s_1) - p(y)) z_6(y) h(y + g_2 s_1) ds_1 dy_1 dx_1,$$

after the change of variable $\frac{s-y}{x} = x_1$, $\frac{s-y}{x} = s_1$. Assuming $g_2 < \frac{\epsilon}{x'}$.

$$M_1^{1,2,3}(g_2) = g_2^3 \mu_K \int_{x'}^{x} \int_{x'}^{x} \int_{x'}^{x} K(x_1) (p(y) - p(y + g_2 x_1)) z_6(y + g_2 x_1)$$

$$\times K(s_1) (p(y + g_2 s_1) - p(y)) z_6(y) h(y + g_2 s_1) ds_1 dy_1 dx_1,$$

Now, Taylor expansions of $p(y - g_2 s_1) - p(y)$ and $h(y + g_2 s_1)$ around $y$ give

$$M_1^{1,2,3}(g_2) = g_2^3 \mu_K \int_{x'}^{x} \int_{x'}^{x} \int_{x'}^{x} K(x_1) (p(y) - p(y + g_2 x_1)) z_6(y + g_2 x_1)$$

$$\times \left( p'(y) h'(y) + \frac{1}{2} p''(y) h(y) \right) z_6(y) dy_1 dx_1 + o(g_2^2)$$

$$= o(g_2^2),$$

which implies expression (207).

Finally,

$$M_1^{1,2,3}(g_2) \quad = \quad E \left( K_{g_2}(T_1 - T_2) (\delta_2 - p(T_1)) z_6(T_1) K_{g_2}(T_1 - T_3) (\delta_3 - p(T_1)) \right)$$

$$- E \left( K_{g_2}(T_1 - T_2) (\delta_2 - p(T_1)) z_6(T_1) \right)^2$$

$$= M_1^{1,2,3}(g_2) - I_{12}(g_2) \mu_K^2$$

and

$$M_1^{1,2,3}(g_2) = \int_{x'}^{x} \int_{x'}^{x} \int_{x'}^{x} K_{g_2}(x-y) K_{g_2}(x-s) (p(y) - p(x)) (p(s) - p(x))$$

$$\times z_6(x) h(s) h(y) h(x) ds dy dx$$

$$= \int_{x'}^{x} \int_{x'}^{x} \int_{x'}^{x} K(y_1) K(s_1) (p(x - g_2 y_1) - p(x))$$

$$\times (p(x - g_2 s_1) - p(x)) z_6(x) h(x - g_2 s_1) h(x - g_2 y_1) h(x) ds dy_1 dx_1,$$

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which needs of the change of variable \(\frac{x}{g_2} = y_1, \frac{x}{g_2} = s_1\). For \(g_2 < \frac{\epsilon}{h}\),

\[
M_1^{1,2,3}(g_2) = \int_{\epsilon'}^{\infty} \int_{-\epsilon'}^{\infty} \int_{-\epsilon'}^{\infty} K(y_1)K(s_1) (p(x - g_2 y_1) - p(x)) \\
\times (p(x - g_2 s_1) - p(x)) z_{61}(x)^2 h(x - g_2 s_1) \\
\times h(x - g_2 y_1) h(x) ds_1 dy_1 dx
\]

\[
= g_2^4 \mu^2 K \int_{\epsilon'}^{\infty} \left( p'(x) h'(x) + \frac{1}{2} p''(x) h(x) \right)^2 z_{61}(x)^2 h(x) dx
\]

+ o \left( g_2^2 \right),

as can be checked by considering Taylor expansions of \(p(x - g_2 y_1) - p(x), \)

\(p(x - g_2 s_1) - p(x), h(x - g_2 y_1) and h(x - g_2 s_1)\) around \(x\) As a straightforward consequence expression (208) holds. 

**Lemma 43** Under conditions (K.1), (P.1), (H.1), (W.1), and (V.2),

\[
E \left( \hat{Q}_{13} \right) = -\frac{1}{2} g_2^4 \mu^2 K \int_{\epsilon'}^{\infty} \left( p'(x) h'(x) + \frac{1}{2} p''(x) h(x) \right) h''(x) z_{61}(x) dx
\]

\[
= -g_2^4 \mu^2 K \int_{\epsilon'}^{\infty} \left( p'(x) h'(x) + \frac{1}{2} p''(x) h(x) \right)^2 z_{52}(x) dx
\]

\[
- n^{-1} g_2^{-3} c K \int_{\epsilon'}^{\infty} p(x) (1 - p(x)) z_5(x) dx
\]

\[
+ o \left( g_2^4 \right) + o \left( n^{-1} g_2^{-1} \right)
\]

and

\[
Var \left( \hat{Q}_{13} \right) = O \left( n^{-1} g_2^2 \right) + O \left( n^{-2} g_2^{-1} \right) + O \left( n^{-3} g_2^{-4} \right).
\]

**Proof.** Define

\[
\hat{Q}_{131} = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\psi}(T_i) - p(T_i) \hat{h}(T_i) \right)^2 z_{51}(T_i)
\]

and

\[
\hat{Q}_{132} = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\psi}(T_i) - p(T_i) \hat{h}(T_i) \right) (\hat{h}(T_i) - h(T_i)) z_{62}(T_i),
\]

to obtain the representation

\[
\hat{Q}_{13} = -\hat{Q}_{131} - \hat{Q}_{132}. \tag{209}
\]

The thesis of the lemma is a straightforward consequence of Lemmas 44, 45, 46, 47 and 48, that are stated below. 

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Lemma 44 Under the conditions in Lemma 43,

\[
E \left( \tilde{Q}_{131} \right) = g_2^4 \mu_K^2 \int_{-\infty}^{\infty} \left( p'(x)h'(x) + \frac{1}{2} p''(x)h(x) \right)^2 z_{52}(x) dx \\
+ n^{-1} g_2^{-1} \epsilon \int_{-\infty}^{\infty} p(x) (1 - p(x)) z_{55}(x) dx \\
+ o \left( g_2^2 \right) + o \left( n^{-1} g_2^{-1} \right). 
\] (210)

Proof. Indeed,

\[
E \left( \tilde{Q}_{131} \right) = \frac{1}{n} \sum_{i=1}^{n} E \left( \frac{1}{n} \sum_{j=1}^{n} K_{g_2} (T_i - T_j) (\delta_j - p(T_i)) \right) \\
+ \frac{1}{n} \sum_{k=1}^{n} K_{g_2} (T_i - T_k) (\delta_k - p(T_i)) z_{51}(T_i) \\
= \frac{1}{n^2} \sum_{i,j,k=1}^{n} E \left( K_{g_2} (T_i - T_j) K_{g_2} (T_i - T_k) \right) \\
(\delta_j - p(T_i)) (\delta_k - p(T_i)) z_{51}(T_i) \\
= \frac{1}{n^2} I_{1311}(g_2) + \frac{2(n-1)}{n^2} I_{1312}(g_2) \\
+ \frac{(n-1)(n-2)}{n^2} I_{1313}(g_2) + \frac{(n-1)(n-2)}{n^2} I_{1314}(g_2) 
\] (211)

where

\[
I_{1311}(g_2) = E \left( K_{g_2} (0)^2 (\delta_1 - p(T_1))^2 z_{51}(T_1) \right), \\
I_{1312}(g_2) = E \left( K_{g_2} (0) K_{g_2} (T_1 - T_2) (\delta_1 - p(T_1)) (\delta_2 - p(T_1)) z_{51}(T_1) \right), \\
I_{1313}(g_2) = E \left( K_{g_2} (T_1 - T_2)^2 (\delta_2 - p(T_1))^2 z_{51}(T_1) \right) \\
\]

and

\[
I_{1314}(g_2) = E \left( K_{g_2} (T_1 - T_2) K_{g_2} (T_1 - T_3) (\delta_2 - p(T_1)) (\delta_3 - p(T_1)) z_{51}(T_1) \right). 
\]

We now deal with these four terms. The first of them is

\[
I_{1311}(g_2) = g_2^2 K(0)^2 \int_{-\infty}^{\infty} p(x)(1 - p(x))z_{52}(x) dx = O \left( g_2^{-2} \right). 
\] (212)

On the other hand, it is straightforward to prove that

\[
I_{1312}(g_2) = 0. 
\] (213)

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The term \( I_{1313}(g_2) \) can be analyzed as follows:

\[
I_{1313}(g_2) = \int_{c'}^{\infty} \int_{0}^{\infty} K_{g_2}(x-y)^2 (p(y)(1-2p(x)) + p(x)^2) \ z_{51}(x)h(y)h(x) dy dx \\
= \int_{c'}^{\infty} \int_{0}^{\infty} h K(y_1)^2 \ (p(x-g_2y_1)(1-2p(x)) + p(x)^2) \\
\times \ z_{52}(x)h(x-g_2y_1) dy_1 dx \\
= \int_{c'}^{\infty} \int_{0}^{\infty} hK \ z_{52}(x)h(x-g_2y_1) dy_1 dx + o (g_2^{-3}) , \quad (214)
\]

where the following change of variable \( \frac{x-y}{g_2} = y_1 \) has been used.

Finally,

\[
I_{1314}(g_2) = \int_{c'}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} K_{g_2}(x-y)K_{g_2}(x-s) \ (p(y) - p(x)) \\
\times (p(s) - p(x)) \ z_{51}(x)h(s)h(y)h(x) ds dy dx \\
= \int_{c'}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} K(y_1)K(s_1) (p(x-g_2y_1) - p(x)) \\
\times (p(x-g_2s_1) - p(s)) \ z_{52}(x)h(x-g_2s_1)h(x-g_2y_1)ds_1 dy_1 dx \\
= \int_{c'}^{\infty} \int_{0}^{\infty} \int_{-L}^{L} K(y_1)K(s_1) (p(x-g_2y_1) - p(x)) \\
\times (p(x-g_2s_1) - p(s)) \ z_{52}(x)h(x-g_2s_1)h(x-g_2y_1)ds_1 dy_1 dx
\]

where the change of variable \( \frac{x-y}{g_2} = y_1, \frac{x-s}{g_2} = s_1 \) have been used and condition \( g_2 < \frac{c'}{2} \) was also used in the last equation. Some Taylor expansions of \( h(x-g_2s_1), h(x-g_2y_1), p(x-g_2y_1) - p(x) \) and \( p(x-g_2s_1) - p(x) \) around \( x \) give

\[
I_{1314}(g_2) = \int_{c'}^{\infty} \int_{0}^{\infty} \left( \frac{1}{2}p''(x)h(x) + \frac{1}{2}p''(x)h(x) \right) \ z_{52}(x) dx + o (g_2^4) . \quad (215)
\]

Using (212)-(215) in (211) it is immediate to check (210).

Lemma 45  Under the conditions of Lemma 43, we have

\[
E \left( Q_{132} \right) = \frac{1}{2} g_2^4 \mu_k^2 \int_{c'}^{\infty} \left( p'(x)h'(x) + \frac{1}{2} p''(x)h(x) \right) h''(x) z_{61}(x) dx \\
+ o (g_2^4) + o (n^{-1}g_2^{-1}) . \quad (216)
\]

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Proof. Clearly,

$$
E(\hat{Q}_{132}) = \frac{1}{n} \sum_{i=1}^{n} E \left( \frac{1}{n} \sum_{j=1}^{n} K_{g_2}(T_i - T_j) \left( \delta_j - p(T_i) \right) \right)
$$

$$
\times \frac{1}{n} \sum_{k=1}^{n} \left( K_{g_1}(T_i - T_k) - h(T_i) \right) z_{62}(T_i)
$$

$$
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} E \left( K_{g_2}(T_i - T_j) \left( \delta_j - p(T_i) \right) \right)
\times \left( K_{g_2}(T_i - T_k) - h(T_i) \right) z_{62}(T_i)
$$

$$
= \frac{1}{n^2} I_{1321}(g_2) + \frac{n-1}{n^2} I_{1312}(g_2) + \frac{n-1}{n^2} I_{1323}(g_2)
$$

$$
+ \frac{n-1}{n^2} I_{1324}(g_2) + \frac{(n-1)(n-2)}{n^2} I_{1325}(g_2)
$$

(217)

where

$$
I_{1321}(g_2) = E \left( K_{g_2}(0) \left( \delta_1 - p(T_1) \right) (K_{g_2}(0) - h(T_1)) z_{62}(T_1) \right),
$$

$$
I_{1322}(g_2) = E \left( K_{g_2}(0) \left( \delta_1 - p(T_1) \right) (K_{g_2}(T_1 - T_2) - h(T_1)) z_{62}(T_1) \right)
$$

$$
I_{1323}(g_2) = E \left( K_{g_2}(T_1 - T_2) \left( \delta_2 - p(T_1) \right) (K_{g_2}(0) - h(T_1)) z_{62}(T_1) \right),
$$

$$
I_{1324}(g_2) = E \left( K_{g_2}(T_1 - T_2) \left( \delta_2 - p(T_1) \right) (K_{g_2}(T_1 - T_2) - h(T_1)) z_{62}(T_1) \right),
$$

and

$$
I_{1325}(g_2) = E \left( K_{g_2}(T_1 - T_2) \left( \delta_2 - p(T_1) \right) (K_{g_2}(T_1 - T_3) - h(T_1)) z_{62}(T_1) \right).
$$

Obviously,

$$
I_{1321}(g_2) = I_{1322}(g_2) = 0.
$$

(218)

On the other hand,

$$
I_{1323}(g_2) = g_2^{-1} \int_{c_1}^{c_2} \int_{0}^{\infty} K_{g_2}(x - y) \left( p(y) - p(x) \right)
\times (K(0) - g_2 h(x)) z_{62}(x) h(y) h(x) dy dx
$$

$$
= g_2^{-1} \int_{c_1}^{c_2} \int_{-\infty}^{\infty} K(y_1) \left( p(x - g_2 y_1) - p(x) \right)
\times (K(0) - g_2 h(x)) z_{61}(x) h(x - g_2 y_1) dy_1 dx
$$

$$
= o \left( g_2^{-1} \right),
$$

(219)
which has been obtained after using the change of variable \( \frac{-x}{y} = y_1 \).

Similar arguments give

\[
I_{1324}(g_2) = \int_{\epsilon'}^{\infty} \int_{0}^{\infty} K_{g_2}(x-y) \left( p(y) - p(x) \right) \\
\times \left( K_{g_2}(x-y) - h(x) \right) z_{62}(x) h(y) dy dx
\]

\[
= g_2^{-1} \int_{\epsilon'}^{\infty} \int_{-\infty}^{\infty} K(y_1) \left( p(x - g_2 y_1) - p(x) \right) \\
\times \left( K(y_1) - g_2 h(x) \right) z_{61}(x) h(x - g_2 y_1) dy_1 dx
\]

\[
= o \left( g_2^{-1} \right). \tag{220}
\]

Finally,

\[
I_{1325}(g_2) = \int_{\epsilon'}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} K_{g_2}(x-y) \left( p(y) - p(x) \right) \left( K_{g_2}(x-s) - h(x) \right) z_{62}(x) \\
\times h(s) h(y) h(x) ds dy dx
\]

\[
= \int_{\epsilon'}^{\infty} \int_{0}^{\infty} K_{g_2}(x-y) \left( p(y) - p(x) \right) \\
\times \left( \int_{0}^{\infty} K_{g_2}(x-s) h(s) ds - h(x) \right) z_{61}(x) h(y) dy dx
\]

\[
= \int_{\epsilon'}^{\infty} \int_{-\infty}^{\infty} K(y_1) \left( p(x - g_2 y_1) - p(x) \right) \\
\times \left( \int_{-\infty}^{\infty} K(s_1) h(x - g_2 s_1) ds_1 - h(x) \right) z_{61}(x) h(x - g_2 y_1) dy_1 dx
\]

\[
= \int_{\epsilon'}^{\infty} \int_{-L}^{L} K(y_1) \left( p(x - g_2 y_1) - p(x) \right) \\
\times \left( \int_{-L}^{L} K(s_1) h(x - g_2 s_1) ds_1 - h(x) \right) z_{61}(x) h(x - g_2 y_1) dy_1 dx,
\]

which has been obtained with the change of variable \( \frac{-x}{y} = y_1, \frac{-x}{y} = s_1 \) using condition \( g_2 < \frac{\epsilon'}{\epsilon} \). Further Taylor expansions of \( h(x - g_2 s_1), p(x - g_2 y_1) - p(x) \) and \( h(x - g_2 y_1) \) around \( x \) give

\[
I_{1325}(g_2) = \frac{1}{2} g_2^2 \mu''_K \int_{\epsilon'}^{\infty} \left( p'(x) h'(x) + \frac{1}{2} \mu''(x) h(x) \right) h''(x) z_{61}(x) dx + o \left( g_2^4 \right). \tag{221}
\]

Plugging (218)-(221) in (217) gives (216). \( \Box \)

**Lemma 46** Under the conditions of Lemma 43,

\[
Var \left( \hat{Q}_{131} \right) = o \left( n^{-1} g_2^3 \right) + O \left( n^{-2} g_2^{-1} \right) + O \left( n^{-3} g_2^{-4} \right). \tag{222}
\]

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Proof. Standard calculations show that

\[
Var \left( \hat{Q}_{131} \right) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Cov \left( \hat{\psi}(T_i) - p(T_i) \hat{h}(T_i) \right)^2 z_{51}(T_i),
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Cov \left( \frac{1}{n} \sum_{i=3}^{n} K_{g2}(T_i) \left( \delta_{i3} - p(T_i) \right) \right.
\]

\[
\times \left. \frac{1}{n} \sum_{i=4}^{n} K_{g2}(T_i) \left( \delta_{i4} - p(T_i) \right) z_{51}(T_i), \right. \]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{i=1}^{n} M_{i1,i2,i3,i4,i5,i6}(g2)
\]

where, for \(1 \leq i_1, i_2, i_3, i_4, i_5, i_6 \leq n\), we define

\[
M_{i1,i2,i3,i4,i5,i6}(g2) = Cov \left( K_{g2}(T_i - T_{i3}) \left( \delta_{i3} - p(T_i) \right) \right)
\]

\[
\times K_{g2}(T_i - T_{i4}) \left( \delta_{i4} - p(T_i) \right) z_{51}(T_i), \]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} M_{i1,i2,i3,i4,i5,i6}(g2)
\]

The order of these covariances is discussed depending on the cardinal of the set of indices \(I = \{i_1, i_2, i_3, i_4, i_5, i_6\}\). For \(Card(I) = 5\), there are 6 possible cases, using symmetry arguments they can be reduced to only 3.

- \(i_1 = i_2\). The covariance is

\[
M_{1,2,3,4,5}^{1,2,3,4,5}(g2) = M_{1,2,3,4,5}(g2) - I_{1314}(g2)^2.
\]

where

\[
M_{1,2,3,4,5}(g2) = E \left( K_{g2}(T_1 - T_2) \left( \delta_2 - p(T_1) \right) \right)
\]

\[
\times K_{g2}(T_1 - T_3) \left( \delta_3 - p(T_1) \right) z_{51}(T_1), \]

\[
\times K_{g2}(T_1 - T_4) \left( \delta_4 - p(T_1) \right) K_{g2}(T_1 - T_5) \left( \delta_5 - p(T_1) \right) z_{51}(T_1)). \]

Now,

\[
M_{1,2,3,4,5}^{1,2,3,4,5}(g2) = \int_{c}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} K_{g2}(x_1 - x_2) \left( p(x_2) - p(x_1) \right) K_{g2}(x_1 - x_3)
\]

\[
\times (p(x_3) - p(x_1)) K_{g2}(x_1 - x_4) \left( p(x_4) - p(x_1) \right) K_{g2}(x_1 - x_5)
\]

\[
\times (p(x_5) - p(x_1)) z_{51}(x_1)^2 h(x_3) h(x_4) h(x_3) h(x_2) h(x_1) dx_3 dx_4 dx_3 dx_2 dx_1
\]

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and the change of variable \( \frac{x_1 - x_2}{g_2} = x_{21}, \ \frac{x_1 - x_3}{g_2} = x_{31}, \ \frac{x_1 - x_4}{g_2} = x_{41}, \ \frac{x_1 - x_5}{g_2} = x_{51} \)
gives

\[
M_1^{1,2,3,4,5}(g_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x_{21}) (p(x_{1} - g_2 x_{21}) - p(x_1)) \ K(x_{31})
\times (p(x_1 - g_2 x_{31}) - p(x_1)) \ K(x_{41}) (p(x_1 - g_2 x_{41}) - p(x_1)) \ K(x_{51})
\times (p(x_1 - g_2 x_{51}) - p(x_1)) \ z_5_1(x_1)^2 h(x_1 - g_2 x_{51}) h(x_1 - g_2 x_{41})
\times h(x_1 - g_2 x_{31}) h(x_1 - g_2 x_{21}) h(x_{1}) dx_{21} dx_{31} dx_{41} dx_{51} dx_{1}
\]

which, for \( g_2 < \frac{2}{L} \), leads to

\[
M_1^{1,2,3,4,5}(g_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{L} \int_{-\infty}^{L} \int_{-\infty}^{L} K(x_{21}) (p(x_{1} - g_2 x_{21}) - p(x_1)) \ K(x_{31})
\times (p(x_1 - g_2 x_{31}) - p(x_1)) \ K(x_{41}) (p(x_1 - g_2 x_{41}) - p(x_1)) \ K(x_{51})
\times (p(x_1 - g_2 x_{51}) - p(x_1)) \ z_5_1(x_1)^2 h(x_1 - g_2 x_{51}) h(x_1 - g_2 x_{41})
\times h(x_1 - g_2 x_{31}) h(x_1 - g_2 x_{21}) h(x_{1}) dx_{21} dx_{31} dx_{41} dx_{51} dx_{1}.
\]

Now Taylor expansions of \( p(x_1 - g_2 x_{21}) - p(x_1), \ h(x_1 - g_2 x_{21}), \ h(x_1 - g_2 x_{31}), \ h(x_1 - g_2 x_{41}) \) and \( h(x_1 - g_2 x_{51}) \) around \( x_1 \) give

\[
M_1^{1,2,3,4,5}(g_2) = g_2^8 K \int_{-\infty}^{\infty} \left( p'(x_1) h'(x_1) + \frac{1}{2} p''(x_1) h(x_1) \right)^4 z_5_1(x_1)^2 h(x_1) dx_1
\]

and, finally,

\[
M_1^{1,2,3,4,5}(g_2) = O \left( g_2^8 \right). \tag{223}
\]

The cases \( i_3 = i_5 \) and \( i_3 = i_6 \) are equal, by symmetry, and it suffices to consider only the first one of these two.Proceeding as in the previous case,

\[
M_1^{1,2,3,4,5,15}(g_2) = M_1^{1,2,3,4,5,15}(g_2) - I_{1314}(g_2)^2.
\]

where

\[
M_1^{1,2,3,4,5,15}(g_2) = E \left( K_{g_2}(T_1 - T_3) (\delta_3 - p(T_1)) K_{g_2}(T_1 - T_4) (\delta_4 - p(T_1)) z_5_1(T_1) \right)
\times K_{g_2}(T_2 - T_1) (\delta_1 - p(T_2)) K_{g_2}(T_2 - T_5) (\delta_5 - p(T_2)) z_5_1(T_2).
\]

On the other hand,

\[
M_1^{1,2,3,4,5,15}(g_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} K_{g_2}(x_1 - x_3) (p(x_3) - p(x_1))
\times K_{g_2}(x_1 - x_4) (p(x_4) - p(x_1)) z_5_1(x_1)
\times K_{g_2}(x_2 - x_1) (p(x_1) - p(x_2))
\times K_{g_2}(x_2 - x_5) (p(x_5) - p(x_2)) z_5_1(x_2)
\times h(x_3) h(x_4) h(x_5) h(x_2) h(x_1) dx_3 dx_4 dx_5 dx_2 dx_1.
\]

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The change of variable $\frac{x_{1} - x_{2}}{y_{2}} = x_{21}$, $\frac{x_{1} - x_{3}}{y_{2}} = x_{31}$, $\frac{x_{1} - x_{4}}{y_{2}} = x_{41}$, $\frac{x_{1} - x_{5}}{y_{2}} = x_{51}$ and condition $g_{2} < 0$ give

$$M_{1.2.3.4.1.5}^{1.2.3.4.1.5.5}(g_{2}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-L}^{L} \int_{-L}^{L} K(x_{31}) (p(x_{1} - g_{2} x_{31}) - p(x_{1}))$$

$$\times K(x_{41}) (p(x_{1} - g_{2} x_{41}) - p(x_{1})) z_{52}(x_{1})$$

$$\times K(x_{51}) (p(x_{1} - g_{2} x_{51}) - p(x_{1} - g_{2} x_{21})) z_{52}(x_{1} - g_{2} x_{21})$$

$$\times h(x_{1} - g_{2} x_{51}) h(x_{1} - g_{2} x_{41}) h(x_{1} - g_{2} x_{31}) dx_{51} dx_{41} dx_{31} dx_{21} dx_{1}.$$  

Now, Taylor expansions of $p(x_{1} - g_{2} x_{31}) - p(x_{1})$, $h(x_{1} - g_{2} x_{31}) - p(x_{1})$ and $h(x_{1} - g_{2} x_{41})$ around $x_{1}$ are used to prove

$$M_{1.2.3.4.1.5}^{1.2.3.4.1.5.5}(g_{2}) = g_{2}^{4} \mu_{K} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-L}^{L} K(x_{51} - x_{21}) K(x_{21})$$

$$\times (p(x_{1} - g_{2} x_{51}) - p(x_{1} - g_{2} x_{21})) (p(x_{1}) - p(x_{1} - g_{2} x_{21}))$$

$$\times h(x_{1} - g_{2} x_{51}) z_{52}(x_{1} - g_{2} x_{21}) \left( p'(x_{1}) h'(x_{1}) + \frac{1}{2} p''(x_{1}) h(x_{1}) \right)^{2}$$

$$z_{52}(x_{1}) dx_{51} dx_{21} dx_{1} + o \left( g_{2}^{4} \right)$$

and, therefore,

$$M_{1.2.3.4.1.5}^{1.2.3.4.1.5.5}(g_{2}) = o \left( g_{2}^{4} \right). \quad (224)$$

The cases $i_{3} = i_{5}$, $i_{3} = i_{6}$ and $i_{4} = i_{6}$ are also symmetric. The first covariance is

$$M_{1.2.3.4.3.5}^{1.2.3.4.3.5}(g_{2}) = E \left( K_{g_{2}}(T_{1} - T_{3}) (\delta_{3} - p(T_{1})) K_{g_{2}}(T_{1} - T_{4}) (\delta_{4} - p(T_{1})) z_{51}(T_{1}) \right.$$

$$\times K_{g_{2}}(T_{2} - T_{3}) (\delta_{3} - p(T_{2})) K_{g_{2}}(T_{2} - T_{5}) (\delta_{5} - p(T_{2})) z_{51}(T_{2}))$$

$$- E \left( K_{g_{2}}(T_{1} - T_{2}) (\delta_{2} - p(T_{1})) K_{g_{2}}(T_{1} - T_{3}) (\delta_{3} - p(T_{1})) z_{51}(T_{1}) \right)^{2}$$

$$= M_{1.2.3.4.3.5}^{1.2.3.4.3.5}(g_{2}) - I_{1314}(g_{2})^{2},$$

with

$$M_{1.2.3.4.3.5}^{1.2.3.4.3.5}(g_{2}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} K_{g_{2}}(x_{1} - x_{3}) K_{g_{2}}(x_{1} - x_{4}) (p(x_{4}) - p(x_{1}))$$

$$\times z_{51}(x_{1}) (p(x_{3}) (1 - p(x_{1}) - p(x_{2})) + p(x_{1}) p(x_{2}))$$

$$\times K_{g_{2}}(x_{2} - x_{3}) K_{g_{2}}(x_{2} - x_{5}) (p(x_{5}) - p(x_{2})) z_{51}(x_{2})$$

$$\times h(x_{5}) h(x_{4}) h(x_{3}) h(x_{2}) h(x_{1}) dx_{5} dx_{4} dx_{3} dx_{2} dx_{1}.$$
\[ M^{1,2,3,4,5}_{1}(g_2) = \int_{c'}^{\infty} \int_{-\infty}^{L} K(x_1) K(x_2) (p(x_1 - g_2 x_1) - p(x_1)) z_5(x_1) \]
\[ \times (p(x_1 - g_2 x_1)(1 - p(x_1) - p(x_1 - g_2 x_1)) + p(x_1) p(x_1 - g_2 x_1)) \]
\[ \times K(x_1 - x_1) K(x_1 - x_1) (p(x_1 - g_2 x_1) - p(x_1 - g_2 x_1)) \]
\[ \times z_5(x_1 - g_2 x_1) h(x_1 - g_2 x_1) h(x_1 - g_2 x_1) h(x_1 - g_2 x_1) \]
\[ \times dx_1 dy_1 dz_1. \]

Now, Taylor expansions of \( p(x_1 - g_2 x_1) - p(x_1) \) and \( h(x_1 - g_2 x_1) \) around \( x_1 \) directly prove

\[ M^{1,2,3,4,5}_{1}(g_2) = g_2^2 \mu K \int_{c'}^{\infty} \int_{-\infty}^{L} K(x_1) z_5(x_1) \]
\[ \times (p'(x_1) h'(x_1) + \frac{1}{2} p''(x_1) h(x_1)) \]
\[ \times (p(x_1 - g_2 x_1)(1 - p(x_1) - p(x_1 - g_2 x_1)) + p(x_1) p(x_1 - g_2 x_1)) \]
\[ \times K(x_1 - x_1) K(x_1 - x_1) (p(x_1 - g_2 x_1) - p(x_1 - g_2 x_1)) \]
\[ \times z_5(x_1 - g_2 x_1) h(x_1 - g_2 x_1) h(x_1 - g_2 x_1) dx_1 dy_1 dz_1 \]
\[ + o(g_2^2) \]
\[ = o(g_2^2). \]

which gives

\[ M^{1,2,3,4,5}_{1}(g_2) = o(g_2^2). \] (225)

\( Card(I) = 4 \) may occur if two pairs of indices are equal (and the other two are different each other and different to the two previous pairs) or if three of the six indices are equal (and, hence, the other three are different each other and different to the previous three). Under the first of these possibilities, the number of different cases can be reduced, by symmetry to 8:

\[ i_1 = i_2, \ i_3 = i_4, \ i_1 = i_2, \ i_3 = i_5, \ i_1 = i_4, \ i_3 = i_5, \ i_1 = i_5, \ i_2 = i_3, \ i_1 = i_5, \ i_3 = i_4, \ i_1 = i_5, \ i_2 = i_6, \ i_1 = i_5, \ i_3 = i_6, \ i_3 = i_5, \ i_4 = i_6. \]

To avoid repetitive calculations we only deal with the last case in detail. It is not difficult to prove that the rest of the terms are of the same order to be obtained for the last one. Let’s consider the covariance

\[ M^{1,2,3,4,3,4}_{1}(g_2) = E (K_{g_2}(T_1 - T_3) (\delta_3 - p(T_1)) K_{g_2}(T_2 - T_4) (\delta_4 - p(T_1)) z_5(T_1) \]
\[ \times K_{g_2}(T_2 - T_3) (\delta_3 - p(T_2)) K_{g_2}(T_2 - T_4) (\delta_4 - p(T_2)) z_5(T_2)) \]
\[ - E (K_{g_2}(T_1 - T_3) (\delta_3 - p(T_1)) K_{g_2}(T_1 - T_4) (\delta_4 - p(T_1)) z_5(T_1))^2 \]
\[ = M^{1,2,3,4,3,4}_{1}(g_2) - I_{1314}(g_2)^2. \]
Now,

\[ M_{1,2,3,4,4}^{1,2,3,4,4}(g_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{g_2}(x_1 - x_3)K_{g_2}(x_1 - x_4) \]
\[ \times (p(x_3)(1-p(x_1)-p(x_2)) + p(x_1)p(x_2))z_{51}(x_1)K_{g_2}(x_2 - x_3) \]
\[ \times K_{g_2}(x_2 - x_4)(p(x_4)(1-p(x_1)-p(x_2)) + p(x_1)p(x_2))z_{51}(x_2) \]
\[ \times h(x_4)h(x_2)h(x_1)dx_4dx_3dx_2dx_1 \]

and the change of variable \( \frac{x_1-x_2}{g_2} = x_{21}, \frac{x_1-x_3}{g_2} = x_{31}, \frac{x_1-x_4}{g_2} = x_{41} \) gives

\[ M_{1,2,3,4,4}^{1,2,3,4,4}(g_2) = g_2^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x_{31})K(x_{41})K(x_{31} - x_{21}) \]
\[ \times K(x_{41} - x_{21})z_{52}(x_1) \]
\[ \times (p(x_1 - g_2x_{31})(1-p(x_1)-p(x_1 - g_2x_{21}))) + p(x_1)p(x_1 - g_2x_{21})) \]
\[ \times (p(x_1 - g_2x_{41})(1-p(x_1)-p(x_1 - g_2x_{21}))) + p(x_1)p(x_1 - g_2x_{21})) \]
\[ \times z_{52}(x_1 - g_2x_{21})h(x_1 - g_2x_{41})h(x_1 - g_2x_{31})dx_{41}dx_{31}dx_{21}dx_1 \]
\[ = O(g_2^{-1}) \]

or, equivalently,

\[ M_{1,2,3,4,4}^{1,2,3,4,4}(g_2) = O(g_2^{-1}). \] (226)

For the second possibility, symmetry arguments reduce the number of possible cases to 4:

\[ i_1 = i_2 = i_3, \quad i_1 = i_3 = i_5, \quad i_1 = i_5 = i_6, \quad i_3 = i_4 = i_5. \]

Only the last one will be analyzed in detail. As usually, the order that will be found for this term will be also valid for the rest of the terms. Thus, the covariance is

\[ M_{1,2,3,4,4}^{1,2,3,4,4}(g_2) = E(K_{g_2}(T_1 - T_3)^2(\delta_3 - p(T_1))^2z_{51}(T_1)) \]
\[ \times K_{g_2}(T_2 - T_3)(\delta_3 - p(T_2))K_{g_2}(T_2 - T_4)(\delta_4 - p(T_2))z_{51}(T_2)) \]
\[ - E(K_{g_2}(T_1 - T_3)^2(\delta_3 - p(T_1))^2z_{51}(T_1)) \]
\[ \times E(K_{g_2}(T_2 - T_3)(\delta_3 - p(T_2))K_{g_2}(T_2 - T_4)(\delta_4 - p(T_2))z_{51}(T_2)) \]
\[ = M_{1,2,3,3,4}^{1,2,3,3,4}(g_2) - I_{1313}(g_2)I_{1314}(g_2) \]

and

\[ M_{1,2,3,3,4}^{1,2,3,3,4}(g_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{g_2}(x_1 - x_3)^2 \]
\[ \times (p(x_3)(1-2p(x_1))(1-p(x_2)) + p(x_1)^2(p(x_3) - p(x_2))) \]
\[ \times z_{51}(x_1)K_{g_2}(x_2 - x_3)K_{g_2}(x_2 - x_4)(p(x_4) - p(x_2))z_{51}(x_2) \]
\[ \times h(x_4)h(x_2)h(x_1)dx_4dx_3dx_2dx_1 \]
Now, the change of variable $\frac{t_1 - t_2}{g_2} = x_{21}$, $\frac{t_3 - t_4}{g_2} = x_{31}$, $\frac{t_5 - t_6}{g_2} = x_{41}$ gives

$$M_{1,2,3,4}^{1,2,3,4}(g_2) = g_2^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{x_{21}} \int_{-\infty}^{x_{31}} \int_{-\infty}^{x_{41}} K(x_{31})^2$$

$$\times (p(x_1 - g_2 x_{31}) (1 - 2p(x_1)) (1 - p(x_1 - g_2 x_{21}))$$

$$+ p(x_1)^2 (p(x_1 - g_2 x_{31}) - p(x_1 - g_2 x_{21})))$$

$$\times z_{52}(x_1) K(x_{31} - x_{21}) K(x_{41} - x_{21}) (p(x_1 - g_2 x_{41}) - p(x_1 - g_2 x_{21}))$$

$$\times z_{52}(x_1 - g_2 x_{21}) h(x_1 - g_2 x_{41}) h(x_1 - g_2 x_{31}) dx_{41} dx_{31} dx_{21} dx_{1}$$

$$= o \left( g_2^{-1} \right),$$

which implies

$$M_{1,2,3,4}^{1,2,3,4}(g_2) = o \left( g_2^{-1} \right) + O \left( g_2^{-1} \right) O \left( g_2^4 \right) = o \left( g_2^{-1} \right). \quad (227)$$

If the cardinal of the set $I$ is smaller than 4 a somewhat more direct argument can be used to obtain the order of the covariance $M_{1,2,3,4}^{1,2,3,4}(g_2)$. Taking into account the results obtained when studying the expectation of $\hat{Q}_{131}$,

$$M_{i_1, i_2, i_3, i_4, i_5, i_6}^{1,2,3,4}(g_2) = E \left( K_{g_2} (T_{i_1} - T_{i_3}) (\delta_{i_3} - p(T_{i_1})) \right)$$

$$\times K_{g_2} (T_{i_2} - T_{i_4}) (\delta_{i_4} - p(T_{i_2})) \times z_{51}(T_{i_1})$$

$$\times K_{g_2} (T_{i_2} - T_{i_5}) (\delta_{i_5} - p(T_{i_2}))$$

$$\times K_{g_2} (T_{i_3} - T_{i_6}) (\delta_{i_6} - p(T_{i_3})) \times z_{51}(T_{i_2}) - O \left( g_2^4 \right)$$

and, for indices $1 \leq i_1, i_2, i_3, i_4, i_5, i_6 \leq n,$

$$|M_{i_1, i_2, i_3, i_4, i_5, i_6}^{1,2,3,4}(g_2)| = g_2^{-4} \|K\|_\infty^4 \|z_{51}\|_\infty^2 + O \left( g_2^{-4} \right) = O \left( g_2^{-4} \right). \quad (228)$$

Now expressions (223)-(228) imply (222).

**Lemma 47** Under the conditions of Lemma 43,

$$\text{Var} \left( \hat{Q}_{132} \right) = O \left( n^{-1} g_2^{-2} \right) + O \left( n^{-2} g_2^{-1} \right) + O \left( n^{-3} g_2^{-4} \right). \quad (229)$$
Proof. Straightforward calculations give

\[
\text{Var} \left( \hat{Q}_{132} \right) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov} \left( (\hat{\psi}(T_i) - p(T_i))\hat{h}(T_i) \right) \left( (\hat{\psi}(T_j) - p(T_j))\hat{h}(T_j) - h(T_j) \right) z_{22}(T_j)
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov} \left( \frac{1}{n} \sum_{i_3=1}^{n} K_{g_2}(T_{i_1} - T_{i_3}) (\delta_{i_3} - p(T_{i_1})) \right)
\]

\[
\times \frac{1}{n} \sum_{i_4=1}^{n} (K_{g_2}(T_{i_1} - T_{i_4}) - h(T_{i_1})) z_{22}(T_{i_1}),
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i_5=1}^{n} (K_{g_2}(T_{i_2} - T_{i_5}) (\delta_{i_5} - p(T_{i_2}))
\]

\[
\times \frac{1}{n} \sum_{i_6=1}^{n} (K_{g_2}(T_{i_2} - T_{i_6}) - h(T_{i_2})) z_{22}(T_{i_2})
\]

where, for \(1 \leq i_1, i_2, i_3, i_4, i_5, i_6 \leq n\),

\[
N_{i_1, i_2, i_3, i_4, i_5, i_6}^{g_2} = \text{Cov} \left( K_{g_2}(T_{i_1} - T_{i_3}) (\delta_{i_3} - p(T_{i_1})) \right)
\]

\[
\times (K_{g_2}(T_{i_1} - T_{i_4}) - h(T_{i_1})) z_{22}(T_{i_1}),
\]

\[
K_{g_2}(T_{i_2} - T_{i_5}) (\delta_{i_5} - p(T_{i_2}))
\]

\[
\times (K_{g_2}(T_{i_2} - T_{i_6}) - h(T_{i_2})) z_{22}(T_{i_2})
\]

As already done for \(\hat{Q}_{131}\), the order of these covariances is studied according to the cardinal of the set \(I = \{i_1, i_2, i_3, i_4, i_5, i_6\}\).

If \(\text{Card}(I) = 5\) it is not difficult but tedious to prove that a standard analysis of the six possible cases give the order \(O(g_2^2)\). This is illustrated by considering the case \(i_3 = i_5\) Its pertaining covariance is

\[
N_{1,2,3,4,5}^{g_2} = E \left( K_{g_2}(T_1 - T_3) (\delta_3 - p(T_1)) \right) \left( K_{g_2}(T_1 - T_4) - h(T_1) \right) z_{22}(T_1)
\]

\[
\times K_{g_2}(T_2 - T_3) (\delta_2 - p(T_2)) \left( K_{g_2}(T_2 - T_5) - h(T_2) \right) z_{22}(T_2)
\]

\[
- E \left( K_{g_2}(T_1 - T_2) (\delta_2 - p(T_1)) \right) \left( K_{g_2}(T_1 - T_3) - h(T_1) \right) z_{22}(T_1)
\]

\[
= N_{1,2,3,4,5}^{g_2} - I_{1325}(g_2)^2.
\]
Now,

\[
N_{1,2,3,4,5}(g_2) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty K_{g_2}(x_1 - x_3)(K_{g_2}(x_1 - x_4) - h(x_1)) \\
\times z_{62}(x_1)K_{g_2}(x_2 - x_3) \\
\times (p(x_3)(1 - p(x_1) - p(x_2)) + p(x_1)p(x_2)) \\
\times (K_{g_2}(x_2 - x_5) - h(x_2))z_{e_2}(x_2) \\
\times h(x_5)h(x_4)h(x_3)(h(x_2)h(x_1)dx_5dx_4dx_3dx_2dx_1) \\
= \int_0^{\xi'} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left( \int_0^\infty K_{g_2}(x_1 - x_4)h(x_4)dx_4 - h(x_1) \right) \\
\times K_{g_2}(x_1 - x_3)z_{61}(x_1)K_{g_2}(x_2 - x_3) \\
\times (p(x_3)(1 - p(x_1) - p(x_2)) + p(x_1)p(x_2)) \\
\times (K_{g_2}(x_2 - x_5) - h(x_2))z_{61}(x_2)h(x_5)h(x_3)dx_5dx_3dx_2dx_1.
\]

Using the change of variable \( \frac{x_1 - x_3}{g_2} = x_{21}, \frac{x_1 - x_4}{g_2} = x_{31}, \frac{x_1 - x_5}{g_2} = x_{41}, \frac{x_1 - x_6}{g_2} = x_{51} \) and condition \( g_2 < \frac{\xi'}{T} \),

\[
N_{1,2,3,4,5}(g_2) = \int_0^{\xi'} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left( \int_0^L K_{g_2}(x_{41})h(x_1 - g_2x_{41})dx_{41} - h(x_1) \right) \\
\times K_{g_2}(x_{31} - x_{21})(K_{g_2}(x_{51} - x_{21}) - g_2h(x_1 - g_2x_{21})) \\
\times z_{61}(x_1)z_{61}(x_1 - g_2x_{21}) \\
\times (p(x_1 - g_2x_{31})(1 - p(x_1) - p(x_1 - g_2x_{21})) + p(x_1)p(x_1 - g_2x_{21})) \\
\times h(x_1 - g_2x_{51})h(x_1 - g_2x_{31})dx_{51}dx_{31}dx_{21}dx_1
\]

and a Taylor expansion of \( h(x_1 - g_2x_{41}) \) around \( x_1 \) gives

\[
N_{1,2,3,4,5}(g_2) = \frac{1}{2}g_2^2\mu_R
\int_0^{\xi'} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^L \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left( \int_0^L K_{g_2}(x_{31})K_{g_2}(x_{31} - x_{21}) \\
\times (K_{g_2}(x_{51} - x_{21}) - g_2h(x_1 - g_2x_{21}))z_{61}(x_1)z_{61}(x_1 - g_2x_{21}) \\
\times (p(x_1 - g_2x_{31})(1 - p(x_1) - p(x_1 - g_2x_{21})) + p(x_1)p(x_1 - g_2x_{21})) \\
\times h(x_1 - g_2x_{51})h(x_1 - g_2x_{31})h''(x_1)dx_{51}dx_{31}dx_{21}dx_1 \\
+ o(g_2^2) \\
= O(g_2^2)
\]

or, alternatively,

\[
N_{1,2,3,4,5}(g_2) = O(g_2^2).
\] (230)

Instead of studying in detail all the cases related to Card \((I) = 4\) we will restrict ourselves to the cases where there are two pairs of indices equal and where three of the six indices are equal. These two are representative cases in
the sense that the order to be obtained is also valid for any of the remainder cases.

The first possibility corresponds to \( i_3 = i_5, i_4 = i_6 \), whose covariance is

\[
N_{1,2,3,4,4}(g_2) = E(K_{g_2}(T_1 - T_3)(\delta_3 - p(T_1))(K_{g_2}(T_1 - T_4) - h(T_1))\varepsilon_2(T_1)
\times K_{g_2}(T_2 - T_3)(\delta_3 - p(T_3))(K_{g_2}(T_2 - T_4) - h(T_2))\varepsilon_2(T_2)
- E(K_{g_2}(T_1 - T_2)(\delta_2 - p(T_1))(K_{g_2}(T_1 - T_3) - h(T_1))\varepsilon_2(T_1))^2
= N_{1,2,3,4,4}(g_2) - I_{1325}(g_2)^2.
\]

Now,

\[
N_{1,2,3,4,4}(g_2) = \int_{c_1}^{\infty} \int_{c_1}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} (p(x_3)(1 - p(x_1) - p(x_2)) + p(x_1)p(x_2))
\times K_{g_2}(x_1 - x_3)(K_{g_2}(x_1 - x_4) - h(x_1))\varepsilon_2(x_1)
\times K_{g_2}(x_2 - x_3)(K_{g_2}(x_2 - x_4) - h(x_2))\varepsilon_2(x_2)
\times h(x_4)h(x_3)h(x_2)h(x_1)dx_4dx_3dx_2dx_1
\]

and the change of variable \( \frac{x_1 - x_2}{g_2} = x_{21}, \frac{x_1 - x_4}{g_2} = x_{31}, \frac{x_1 - x_4}{g_2} = x_{41} \) gives

\[
N_{1,2,3,4,4}(g_2) = g_2^{-1} \int_{c_1}^{\infty} \int_{c_1}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} (p(x_1 - g_2x_{31})(1 - p(x_1) - p(x_1 - g_2x_{21}))
+ p(x_1)p(x_1 - g_2x_{21}))K(x_{31})(K(x_{41} - g_2h(x_{41}))K(x_{31} - x_{21})
\times (K(x_{41} - x_{21}) - g_2h(x_1 - g_2x_{21}))\varepsilon_2(x_1)\varepsilon_1(x_1 - g_2x_{21})
\times h(x_1 - g_2x_{41})h(x_1 - g_2x_{31})dx_{41}dx_{31}dx_{21}dx_{1}
= O(g_2^{-1}).
\]

Therefore,

\[
N_{1,2,3,4,4}(g_2) = O(g_2^{-1}).
\]

The second possibility covers the case \( i_3 = i_4 = i_5 \), for which

\[
N_{1,2,3,4,4}(g_2) = E(K_{g_2}(T_1 - T_3)(\delta_3 - p(T_1))(K_{g_2}(T_1 - T_3) - h(T_1))\varepsilon_2(T_1)
\times K_{g_2}(T_2 - T_3)(\delta_3 - p(T_3))(K_{g_2}(T_2 - T_4) - h(T_2))\varepsilon_2(T_2)
- E(K_{g_2}(T_1 - T_2)(\delta_2 - p(T_1))(K_{g_2}(T_1 - T_2) - h(T_1))\varepsilon_2(T_1))
\times E(K_{g_2}(T_1 - T_2)(\delta_2 - p(T_1))(K_{g_2}(T_1 - T_3) - h(T_1))\varepsilon_2(T_1))
= N_{1,2,3,4,4}(g_2) - I_{1324}(g_2)I_{1325}(g_2).
\]

Standard algebra gives

\[
N_{1,2,3,4,4}(g_2) = \int_{c_1}^{\infty} \int_{c_1}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} (p(x_3)(1 - p(x_1) - p(x_2)) + p(x_1)p(x_2))
\times K_{g_2}(x_1 - x_3)(K_{g_2}(x_1 - x_4) - h(x_1))\varepsilon_2(x_1)
\times K_{g_2}(x_2 - x_3)(K_{g_2}(x_2 - x_4) - h(x_2))\varepsilon_2(x_2)
\times h(x_4)h(x_3)h(x_2)h(x_1)dx_4dx_3dx_2dx_1
\]

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Lemma 48

Under the conditions of Lemma 43, and, for any indices $N$\textsuperscript{1}, $g$\textsubscript{2} is a condition $g_2 < \frac{1}{L}$ turns out to be

\begin{align*}
N_{1,2,3,3,4}^{1,2,3,3,4}(g_2) &= g_2^{-1} \int_{c'} \int_{c''} \int_{-L}^{L} (p(x_2 - g_2 x_{31}) (1 - p(x_2 - g_2 x_{31})) - p(x_2) \\
&+ p(x_2 - g_2 x_{31}) p(x_2)) K(x_{31} - x_{11})(K(x_{31} - x_{11}) - g_2 h(x_2 - g_2 x_{11})) \\
&\times z_{61}(x_2 - g_2 x_{11}) K(x_{31}) \left( \int_{-L}^{L} K(x_{41}) h(x_2 - g_2 x_{41}) dx_{41} - h(x_2) \right) \\
&\times z_{61}(x_2) h''(x_2) h(x_2 - g_2 x_{31}) dx_{31} dx_{11} dx_2.
\end{align*}

A Taylor expansion of $h(x_2 - g_2 x_{41})$ around $x_2$ can be used to conclude

\begin{align*}
N_{1,2,3,3,4}^{1,2,3,3,4}(g_2) &= g_2^{-\frac{1}{2}} \mu K \int_{c'} \int_{c''} \int_{-L}^{L} (p(x_2 - g_2 x_{31}) (1 - p(x_2 - g_2 x_{31})) - p(x_2) \\
&+ p(x_2 - g_2 x_{31}) p(x_2)) K(x_{31} - x_{11})^2 K(x_{31}) z_{61}(x_2 - g_2 x_{11}) \\
&\times z_{61}(x_2) h''(x_2) h(x_2 - g_2 x_{31}) dx_{31} dx_{11} dx_2 \\
&+ o(g_2) \\
&= O(g_2).
\end{align*}

which implies

\begin{equation}
N_{1,2,3,3,4}^{1,2,3,3,4}(g_2) = O(g_2).
\end{equation}

To handle the cases with $\text{Card}(I) < 4$ it is sufficient to observe that

\begin{align*}
N_{i_1, i_2, i_3, i_4, i_5, i_6}^{1,2,3,3,4}(g_2) &= E(K_{g_2}(T_{i_1} - T_{i_3})(\delta_{i_3} - p(T_{i_1})) \\
&\times (K_{g_2}(T_{i_1} - T_{i_4}) - h(T_{i_1})) z_{62}(T_{i_1}) \\
&\times K_{g_2}(T_{i_2} - T_{i_4})(\delta_{i_5} - p(T_{i_2})) \\
&\times (K_{g_2}(T_{i_2} - T_{i_6}) - h(T_{i_2})) z_{62}(T_{i_2}) \\
&- o(g_2^2)
\end{align*}

and, for any indices $1 \leq i_1, i_2, i_3, i_4, i_5, i_6 \leq n$,

\begin{align*}
|N_{1,2,3,3,4}^{1,2,3,3,4}(g_2)| &= g_2^{-4} \| K \|_\infty^2 \| (\| K \|_\infty^2 + \| h \|_\infty^2) \|_\infty^2 \| z_{62} \|_\infty^2 + o(g_2^{-2}) \\
&= O(g_2^{-4}).
\end{align*}

Finally (230)-(232) can be used to conclude (229).

\begin{lemma}
Under the conditions of Lemma 43,

\begin{equation}
\text{Cov} \left( Q_{131}, \hat{Q}_{132} \right) = O(n^{-1} g_2^2) + O(n^{-2} g_2^{-1}) + O(n^{-3} g_2^{-4}).
\end{equation}
\end{lemma}
Proof. Cauchy-Schwarz inequality gives
\[ \left| \text{Cov} \left( \hat{Q}_{131}, \hat{Q}_{132} \right) \right| \leq \text{Var} \left( \hat{Q}_{131} \right)^{\frac{1}{2}} \text{Var} \left( \hat{Q}_{132} \right)^{\frac{1}{2}} \]
and expression (233) follows using (222) and (229).

The covariances \( \text{Cov} \left( Q_{11}, \hat{Q}_{12} \right) \), \( \text{Cov} \left( Q_{11}, \hat{Q}_{13} \right) \) and \( \text{Cov} \left( \hat{Q}_{12}, \hat{Q}_{13} \right) \) need of a more careful study. This is done in Lemmas 49, 50 and 51.

Lemma 49 Under the conditions of Lemma ??, \( \text{Cov} \left( Q_{11}, \hat{Q}_{12} \right) = O \left( n^{-1} g_2^2 \right) \).

Proof. Straightforward calculations give
\[ \text{Cov} \left( Q_{11}, \hat{Q}_{12} \right) = \text{Cov} \left( \frac{1}{n} \sum_{i=1}^{n} p(T_i)(1 - p(T_i))z_5(T_i), \right. \]
\[ \left. \frac{1}{n} \sum_{j=1}^{n} \frac{1}{n} \sum_{k=1}^{n} K_{g_2}(T_j - T_k) (\delta_k - p(T_j)) z_61(T_j) \right) \]
\[ = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} M^{i,j,k}(g_2) \]
\[ = \frac{1}{n^2} \left( n(n-1)M^{1,1,2}(g_2) + n(n-1)M^{1,2,1}(g_2) \right) + nM^{1,1,1}(g_2) \] (235)

where

\[ M^{i,j,k}(g_2) = \text{Cov} \left( p(T_i)(1 - p(T_i))z_5(T_i), K_{g_2}(T_j - T_k) (\delta_k - p(T_j)) z_61(T_j) \right). \]

The first of these three covariances is
\[ M^{1,1,2}(g_2) = M^{1,1,2}_1(g_2) - QI_{12}(g_2) \]
where
\[ M^{1,1,2}_1(g_2) = E \left( p(T_1)(1 - p(T_1))z_5(T_1)K_{g_2}(T_1 - T_2) (\delta_2 - p(T_1)) z_61(T_1) \right). \]

On the other hand,
\[ M^{1,1,2}_1(g_2) = \int_{0}^{\infty} \int_{0}^{\infty} p(x)(1 - p(x))z_5(x)K_{g_2}(x-y)(p(y) - p(x)) \]
\[ \times z_61(x) h(y)h(x) dy dx \]
\[ = O \left( g_2^2 \right), \]
as proved for the expression similar to (198), defined in the term $I_{12}(g_2)$. Therefore,

$$M^{1,1,2}(g_2) = O\left(g_2^2\right). \quad (236)$$

Similarly,

$$M^{1,2,1}(g_2) = M_1^{1,2,1}(g_2) - QI_{12}(g_2),$$

where

$$M_1^{1,2,1}(g_2) = E(p(T_1)(1 - p(T_1))z_5(T_1)K_{g_2} (T_2 - T_1) (\delta_1 - p(T_2)) z_{61}(T_2)).$$

Now

$$M_1^{1,2,1}(g_2) = \int_{\varepsilon'}^{\infty} \int_{x_1}^{\infty} p(x)(1 - p(x))z_5(x)K_{g_2} (y - x) (p(x) - p(y))$$

$$\times z_{61}(y)h(y)h(x)dydx$$

$$= \int_{\varepsilon'}^{\infty} \int_{x_1}^{\infty} p(x)(1 - p(x))z_5(x)K(y_1) (p(x) - p(x + g_2 y_1))$$

$$\times z_6(x + g_2 y_1)h(y_1)dy_1dx$$

after using the change of variable $\frac{y - x}{g_2} = y_1$.

Obviously,

$$\lim_{g_2 \to 0} M_1^{1,2,1}(g_2) = 0.$$

Standard calculations lead to

$$\frac{dM_1^{1,2,1}(g_2)}{dg_2} = \int_{\varepsilon'}^{\infty} p(x)(1 - p(x))z_5(x)K\left(\frac{\varepsilon' - x}{g_2}\right) (p(x) - p(\varepsilon'))$$

$$\times z_6(\varepsilon') \frac{\varepsilon' - x}{g_2} h(x)dy_1dx$$

$$+ \int_{\varepsilon'}^{\varepsilon} \int_{x_1}^{\infty} ((p(x) - p(x + g_2 y_1)) z_6(x + g_2 y_1)$$

$$- p'(x + g_2 y_1)z_6(x + g_2 y_1))$$

$$\times p(x)(1 - p(x))z_5(x)h(x)K(y_1) dy_1dx$$

which, using $z_6(\varepsilon') = 0$, as a consequence of the assumptions on $w$, implies

$$\lim_{g_2 \to 0} \frac{dM_1^{1,2,1}(g_2)}{dg_2} = 0.$$
Similarly,

\[
\frac{d^2 M^{1,2,1}_1(g_2)}{dg_2^2} = \int_{\varepsilon'}^\infty \left( (p(x) - p(x')) z_6' (x') - p' (x') z_6 (x') \right)
\times K \left( \frac{x'}{g_2} \right) \left( \frac{x'}{g_2} \right)^2 p(x) (1 - p(x)) z_5 (x) h(x) dx 
+ \int_{\varepsilon'}^\infty \int_{\varepsilon'-x'}^\infty \left( (p(x) - p(x + g_2 y_1)) z_6'' (x + g_2 y_1) 
- 2p'(x + g_2 y_1) z_6' (x + g_2 y_1) - p''(x + g_2 y_1) z_6 (x + g_2 y_1) 
\times K(y_1) y_1^2 dy_1 p(x) (1 - p(x)) z_5 (x) h(x) dx \right.
\]

and using \( z_6 (x') = z_6' (x') = 0 \),

\[
\lim_{g_2 \to 0} \frac{dM^{1,2,1}_1(g_2)}{dg_2} = - \int_{\varepsilon'}^\infty \left( 2p'(x) z_6 (x) + p''(x) z_6 (x) \right) p(x) (1 - p(x)) z_5 (x) h(x) dx.
\]

Now, a Taylor expansion of \( M^{1,2,1}_1(g_2) \) around 0 gives

\[
M^{1,2,1}_1(g_2) = O (g_2^2)
\]
and, finally,

\[
M^{1,2,1}_1(g_2) = O (g_2^2).
\] (237)

For the third covariance of (235), it is easy to observe that

\[
M^{1,1,1}_1(g_2) = 0.
\] (238)

Collecting (236), (237) and (238), the final expression (234) easily follows.

Lemma 50 Assume the conditions of Lemma ?? Then,

\[
\text{Cov} \left( Q_{11}, \hat{Q}_{13} \right) = o \left( n^{-1} g_2^2 \right) + O \left( n^{-2} g_2^{-2} \right).
\] (239)

Proof. Recall representation (209) and let's start with the covariances \( \text{Cov} \left( Q_{11}, \hat{Q}_{131} \right) \).
For $\text{Card}_g N$ where
\[ \begin{aligned}
& 2 < i_1 \leq \epsilon \\
\end{aligned} \]

Again, these covariances will be studied depending on the cardinal of $I = \{ i_1, i_2, i_3, i_4 \}$. To do this, for conciseness, we will select only some relevant cases. For $\text{Card}(I) = 3$, we will only study the case $i_1 = i_3$.

\[ \begin{aligned}
N^{1,2,1,3}_{g_2} &= N^{1,2,1,3}_{g_1} - Q_{1314}(g_2)
\end{aligned} \]

where
\[ \begin{aligned}
N^{1,2,1,3}_{g_2} &= E(p(T_1)(1 - p(T_1))z_3(T_1)K_{g_2}(T_2 - T_1)
\times (\delta - p(T_2)) K_{g_2}(T_2 - T_3) (\delta_3 - p(T_2)) z_5(T_2)).
\end{aligned} \]

Now,
\[ \begin{aligned}
N^{1,2,1,3}_{g_2} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x_1)(1 - p(x_1))z_5(x_1)
\times K_{g_2}(x_2 - x_1) (p(x_1) - p(x_2)) K_{g_2}(x_2 - x_3)
\times (p(x_3) - p(x_2)) z_5(x_2) h(x_3) h(x_2) h(x_1) dx_3 dx_2 dx_1
\end{aligned} \]

after using the change of variable $\frac{x_2 - x_1}{g_2} = x_1$, $\frac{x_2 - x_3}{g_2} = x_3$ and condition $g_2 < \frac{1}{\epsilon}$. As usually, Taylor expansions of $p(x_2 - g_2 x_3) - p(x_2)$ and $b(x_2 - g_2 x_3)$.
around \( x_2 \), give

\[
N_{11213}^1(g_2) = g_2^2 \mu K \int_{c'} \int_{-\infty}^{\infty} p(x_2 - g_2 x_{11}) (1 - p(x_2 - g_2 x_{11})) z_5(x_2 - g_2 x_{11})
\times K(x_{11}) (p(x_2 - g_2 x_{11}) - p(x_2)) \left( p'(x_2) h'(x_2) + \frac{1}{2} p''(x_2) h(x_2) \right)
\times z_5(x_2) h(x_2 - g_2 x_{11}) dx_{11} dx_2 + o(g_2^2)
= o(g_2^2)
\]

or, alternatively,

\[
N_{11213}^1(g_2) = o(g_2^2). \tag{240}
\]

For \( \text{Card}(I) = 2 \) we will study a couple of different situations. The first of these corresponds to \( i_1 = i_2, i_3 = i_4 \). Its covariance is

\[
N_{11122}^1(g_2) = N_{11122}^1(g_2) - QI_{1313}(g_2).
\]

where

\[
N_{11122}^1(g_2) = E \left( p(T_1)(1 - p(T_1)) z_5(T_1) K_{g_2}(T_1 - T_2)^2 (T_2 - p(T_1))^2 z_51(T_1) \right).
\]

On the other hand,

\[
N_{11122}^1(g_2) = \int_{c'} \int_{0}^{\infty} p(x_1)(1 - p(x_1)) z_5(x_1) K_{g_2}(x_1 - x_2)^2
\times (p(x_2)(1 - 2p(x_1)) + p(x_1)^2) z_51(x_1) h(x_2) h(x_1) dx_2 dx_1
= g_2^{-1} \int_{c'} \int_{-\infty}^{\infty} p(x_1)(1 - p(x_1)) z_51(x_1) K(x_{21})^2
\times (p(x_2)(1 - 2p(x_1)) + p(x_1)^2) z_52(x_1) h(x_1 - g_2 x_{21}) dx_{21} dx_1
= O(g_2^{-1})
\]

where the change of variable \( \frac{x_1 - x_2}{g_2} = x_{21} \) has been used. Therefore,

\[
N_{11122}^1(g_2) = O(g_2^{-1}). \tag{241}
\]

The second case deals with \( i_1 = i_3 = i_4 \), for which the covariance is

\[
N_{11211}^1(g_2) = N_{11211}^1(g_2) - QI_{1313}(g_2).
\]

where

\[
N_{11211}^1(g_2) = E \left( p(T_1)(1 - p(T_1)) z_5(T_1) K_{g_2}(T_2 - T_1)^2 (T_2 - p(T_1))^2 z_51(T_2) \right).
\]

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Now,  

\[
N^{1,2,1,1}_{1}(g_2) = \int_{e'}^{\infty} \int_{e'}^{\infty} p(x_1)(1 - p(x_1))z_5(x_1)K_{g_2}(x_2 - x_1)^2 \times (p(x_1)(1 - 2p(x_2)) + p(x_2)^2) z_51(x_2)h(x_2)h(x_1)dx_2dx_1
\]

\[
= g_2^{-1} \int_{e'}^{\infty} \int_{-\infty}^{\infty} \frac{d}{dx} p(x_1)(1 - p(x_1))z_5(x_1)K(x_21)^2 \times (p(x_1)(1 - 2p(x_2)) + p(x_2)^2) z_52(x_1 - g_2x_21)h(x_1)dx_21dx_1
\]

\[
= O\left(g_2^{-1}\right),
\]

after using the same change of variable as before. Thus, 

\[
N^{1,2,1,1}_{1}(g_2) = O\left(g_2^{-1}\right). \tag{242}
\]

Finally, if \( \text{Card}(I) = 1 \), the only covariance is 

\[
N^{1,1,1,1}_{1,1}(g_2) = N^{1,1,1,1}_{1}(g_2) - QI_{1311}(g_2).
\]

where 

\[
N^{1,1,1,1}_{1,1}(g_2) = E\left( p(T_1)(1 - p(T_1))z_5(T_1)K_{g_2}(0)^2 (\delta_1 - p(T_1))^2 z_51(T_1) \right).
\]

As a consequence, 

\[
N^{1,1,1,1}_{1}(g_2) = g_2^{-2}K(0)^2 \int_{e'}^{\infty} z_5(x)p(x)^2(1 - p(x))^2z_51(x)h(x)dx
\]

\[
= O\left(g_2^{-2}\right),
\]

that leads to 

\[
N^{1,1,1,1}_{1}(g_2) = O\left(g_2^{-2}\right). \tag{243}
\]

Using (240)-(243) it is easy to obtain 

\[
Cov\left( Q_{11}, \hat{Q}_{131} \right) = o\left(n^{-1}g_2^2\right) + O\left(n^{-2}g_2^{-1}\right) + O\left(n^{-3}g_2^{-2}\right)
\]

and, condition (V.2), gives 

\[
Cov\left( Q_{11}, \hat{Q}_{131} \right) = o\left(n^{-1}g_2^2\right) + O\left(n^{-2}g_2^{-1}\right). \tag{244}
\]

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Similar arguments can be used to reach to

\[
\text{Cov} \left( Q_{11}, \hat{Q}_{132} \right) = \frac{1}{n^2} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \text{Cov} (p(T_{i_1})(1 - p(T_{i_1}))z_5(T_{i_1}),
\]

\[
\frac{1}{n} \sum_{i_3=1}^{n} K_{g_2} (T_{i_2} - T_{i_3}) \left( \delta_{i_3} - p(T_{i_2}) \right)
\times \frac{1}{n} \sum_{i_4=1}^{n} \left( K_{g_2} (T_{i_2} - T_{i_4}) - h(T_{i_2}) \right) z_{62}(T_{i_2})
\]

\[
= \frac{1}{n^4} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \sum_{i_4=1}^{n} R_{i_1,i_2,i_3,i_4}^{g_2},
\]

where,

\[
R_{i_1,i_2,i_3,i_4}^{g_2} = \text{Cov} (p(T_{i_1})(1 - p(T_{i_1}))z_5(T_{i_1}),
\]

\[K_{g_2} (T_{i_2} - T_{i_3}) \left( \delta_{i_3} - p(T_{i_2}) \right) \left( K_{g_2} (T_{i_2} - T_{i_4}) - h(T_{i_2}) \right) z_{62}(T_{i_2}).\]

Let’s consider one by one the different possibilities for the cardinal of the set 
\( I = \{i_1, i_2, i_3, i_4\} \). If Card(\( I \)) = 3, we will only consider the case \( i_1 = i_3 \). Its covariance is

\[
R_{1,2,1,3}^{1,2,1,3}(g_2) = R_{1,2,1,3}^{1,2,1,3}(g_2) - Q_{1325}(g_2)
\]

where

\[
R_{1,2,1,3}^{1,2,1,3}(g_2) = E \left( p(T_1)(1 - p(T_1))z_5(T_1) \right)
\]

\[K_{g_2} (T_2 - T_1) \left( \delta_1 - p(T_2) \right) \left( K_{g_2} (T_2 - T_3) - h(T_2) \right) z_{62}(T_2).
\]

and

\[
R_{1,2,1,3}^{1,2,1,3}(g_2) = \int_{c'}^{c'} \int_{c}^{c} \int_{0}^{\infty} p(x_1)(1 - p(x_1))z_5(x_1)K_{g_2} (x_2 - x_1) \left( p(x_1) - p(x_2) \right)
\times (K_{g_2} (x_2 - x_3) - h(x_2)) z_{62}(x_2) h(x_3) h(x_2) h(x_1) dx_3 dx_2 dx_1
\]

\[
= \int_{c}^{c} \int_{c}^{c} \int_{0}^{\infty} p(x_1)(1 - p(x_1))z_5(x_1)K_{g_2} (x_2 - x_1) \left( p(x_1) - p(x_2) \right)
\times \left( \int_{0}^{\infty} K_{g_2} (x_2 - x_3) h(x_3) dx_3 - h(x_2) \right) z_{62}(x_2) h(x_1) dx_2 dx_1
\]

\[
= \int_{c}^{c} \int_{c}^{c} \int_{0}^{\infty} p(x_2 - g_2 x_1)(1 - p(x_2 - g_2 x_1))z_5(x_2 - g_2 x_1)
\times K(x_1) \left( p(x_2 - g_2 x_1) - p(x_2) \right)
\times \left( \int_{-L}^{L} K(x_3) h(x_2 - g_2 x_3) dx_3 - h(x_2) \right)
\times z_{62}(x_2) h(x_2 - g_2 x_1) dx_1 dx_2,
\]

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after using the change of variable \( \frac{x - x_1}{g_2} = x_{11}, \frac{x - x_3}{g_2} = x_{31} \) and condition \( g_2 < \frac{\epsilon}{T} \). A Taylor expansion of \( h(x_2 - g_2 x_{31}) \) around \( x_2 \) gives

\[
R_{11214}^{1.2.1.3}(g_2) = g^2 \frac{1}{2} \mu K \int_{x'}^\infty \int_{-\infty}^{x_1} p(x_2 - g_2 x_{11})(1 - p(x_2 - g_2 x_{11}))z_5(x_2 - g_2 x_{11})
\times K(x_{11}) (p(x_2 - g_2 x_{11}) - p(x_2)) h''(x_2)z_6_1(x_2)h(x_2 - g_2 x_{11})dx_{11}dx_2
+ o(g_2^2),
\]

which implies

\[ R_{11214}^{1.2.1.3}(g_2) = o(g_2^2). \] (245)

If \( Card(I) = 2 \) only two cases will be considered. The first one corresponds to \( i_1 = i_2, i_3 = i_4 \), whose covariance is

\[ R_{11214}^{1.1.2.2}(g_2) = R_{11214}^{1.1.2.2}(g_2) - Q I_{12214}(g_2), \]

where

\[ R_{11214}^{1.1.2.2}(g_2) = E(p(T_1)(1 - p(T_1)) z_5(T_1) \]

\[ K_{g_2}(T_1 - T_2)(\delta_2 - p(T_1)) (K_{g_2}(T_1 - T_2) - h(T_1)) z_{6_2}(T_1). \]

Standard algebra gives

\[
R_{11214}^{1.1.2.2}(g_2) = \int_{x'}^\infty \int_{0}^{x_1} p(x_1)(1 - p(x_1))z_5(x_1)K_{g_2}(x_1 - x_2)(p(x_2) - p(x_1))
\times (K_{g_2}(x_1 - x_2) - h(x_1)) z_{6_2}(x_1)h(x_2)h(x_1)dx_2dx_1
\]

\[ = g_2^{-1} \int_{x'}^\infty \int_{-\infty}^{x_1} p(x_1)(1 - p(x_1))z_5(x_1)K(x_{21})
\times (p(x_1 - g_2 x_{21}) - p(x_1)) (K(x_{21}) - g_2 h(x_1)) z_{6_1}(x_1)
\times h(x_1 - g_2 x_{21})dx_{21}dx_1
\]

\[ = o(g_2^2), \]

after using the change of variable \( \frac{x_1 - x_2}{g_2} = x_{21} \). Therefore,

\[ R_{11214}^{1.1.2.2}(g_2) = o(g_2^{-1}). \] (246)

The second possibility is concerned with the case \( i_1 = i_3 = i_4 \). Its covariance is

\[
R_{11214}^{1.2.1.1}(g_2) = E(p(T_1)(1 - p(T_1)) z_1(T_1)w(T_1)
K_{g_2}(T_2 - T_1)(\delta_1 - p(T_2)) (K_{g_2}(T_2 - T_1) - h(T_2)) z_{2_2}(T_2)
-E(p(T_1)(1 - p(T_1)) z_1(T_1)w(T_1))
\times E(K_{g_2}(T_2 - T_1)(\delta_1 - p(T_2)) (K_{g_2}(T_2 - T_1) - h(T_2)) z_{2_2}(T_2))
\]

\[ = R_{11214}^{1.2.1.1}(g_2) - Q I_{12214}(g_2). \]
Now,

\[ R_{1,1}^{1,2,1,1}(g_2) = \int_{e^{-\epsilon}}^{e^\infty} \int_{e^{-\epsilon}}^{e^\infty} p(x_1)(1 - p(x_1))z_1(x_1)w(x_1)K_{g_2}(x_2 - x_1)(p(x_1) - p(x_2)) \]
\[ \times (K_{g_2}(x_2 - x_1) - h(x_2)) z_{22}(x_2) h(x_2) h(x_1) dx_2 dx_1 \]
\[ = g_2^{-1} \int_{e^{-\epsilon}}^{e^\infty} \int_{e^{-\epsilon}}^{e^\infty} p(x_1)(1 - p(x_1))z_1(x_1)w(x_1)K(x_{21}) \]
\[ \times (p(x_1) - p(x_1 - g_2 x_{21})) (K(x_{21}) - g_2 h(x_1 - g_2 x_{21})) z_{22}(x_1 - g_2 x_{21}) \]
\[ \times h(x_1 - g_2 x_{21}) h(x_1) dx_{21} dx_1 \]
\[ = o(g_2^{-1}), \]

after applying the same change of variable of the previous case. This gives

\[ R_{1,1}^{1,2,1,1}(g_2) = o(g_2^{-1}). \] (247)

Finally, if \( \text{Card}(I) = 1 \) it is straightforward to check that

\[ R_{1,1}^{1,1,1,1}(g_2) = 0. \] (248)

Now, expressions (245)-(248) can be used to obtain

\[ \text{Cov} \left( \hat{Q}_{11}, \hat{Q}_{132} \right) = o \left( n^{-1} g_2^{-1} \right) + o \left( n^{-2} g_2^{-1} \right). \] (249)

The final result (239) can be obtained using (244), (249) and the representation (209).

**Lemma 51** Under the conditions of Lemma ??,

\[ \text{Cov} \left( \hat{Q}_{12}, \hat{Q}_{13} \right) = O \left( n^{-1} g_2^{-1} \right) + O \left( n^{-2} g_2^{-1} \right) + O \left( n^{-3} g_2^{-1} \right). \] (250)

**Proof.** Using the representation (209), we first consider the covariance \( \text{Cov} \left( \hat{Q}_{12}, \hat{Q}_{131} \right) \).

\[ \text{Cov} \left( \hat{Q}_{12}, \hat{Q}_{131} \right) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov} \left( (\tilde{\psi}(T_j) - p(T_j)) h(T_1) z_{61}(T_1), \right. \]
\[ \left. (\tilde{\psi}(T_i) - p(T_i)) h(T_1) z_{51}(T_1) \right) \]
\[ = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov} \left( \frac{1}{n} \sum_{i=1}^{n} K_{g_2}(T_{i_1} - T_{i_3}) (\delta_{i_3} - p(T_{i_1})) z_{61}(T_{i_1}), \right. \]
\[ \left. \frac{1}{n} \sum_{i=1}^{n} K_{g_2}(T_{i_2} - T_{i_4}) (\delta_{i_4} - p(T_{i_2})) \right) \]
\[ \times \frac{1}{n^2} \sum_{i=1}^{n} \sum_{i=1}^{n} K_{g_2}(T_{i_2} - T_{i_3}) (\delta_{i_5} - p(T_{i_2})) z_{51}(T_{i_2}) \]
\[ = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} M^{i_1,i_2,i_3,i_4,i_5}(g_2), \]
where, for $1 \leq i_1, i_2, i_3, i_4, i_5 \leq n$,

\[ M^{i_1,i_2,i_3,i_4,i_5}(g_2) = \text{Cov}(K_{g_2}(T_{i_1} - T_{i_3}) (\delta_{i_1} - p(T_{i_1})), K_{g_2}(T_{i_2} - T_{i_4}) (\delta_{i_4} - p(T_{i_4}))) z_{61}(T_{i_5}) \]

Different cases for these covariances will be discussed according to the cardinal of the set $I = \{i_1, i_2, i_3, i_4, i_5\}$. If $\text{Card}(I) = 4$, all the possible cases can be reduced, by symmetry, to only 4: $i_1 = i_2$, $i_1 = i_4$, $i_3 = i_2$, $i_3 = i_4$. Only the last one will be considered in detail, since it will give the dominant order. Thus,

\[ M^{1,2,3,4,4}(g_2) = M^{1,2,3,3,4}(g_2) - I_{12}(g_2)I_{1314}(g_2) \]

where

\[ M^{1,2,3,4,4}(g_2) = E(K_{g_2}(T_1 - T_3) (\delta_3 - p(T_1))) z_{61}(T_1) \]

\[ \times K_{g_2}(T_2 - T_3) (\delta_3 - p(T_2)) K_{g_2}(T_2 - T_4) (\delta_4 - p(T_2)) z_{51}(T_2) \]

Now,

\[ M^{1,2,3,4,4}(g_2) = \int_{x'}^{\infty} \int_{x'}^{\infty} \int_{-\infty}^{\infty} (p(x_3) (1 - p(x_1) - p(x_2)) + p(x_1)p(x_2)) \]

\[ \times K_{g_2}(x_1 - x_3)K_{g_2}(x_2 - x_3)K_{g_2}(x_2 - x_4) (p(x_4) - p(x_2)) \]

\[ \times z_{61}(x_1)z_{51}(x_2)h(x_4)h(x_3)h(x_2)h(x_1)dx_4dx_3dx_2dx_1 \]

and the change of variable $\frac{x_2 - x_3}{g_2} = x_{11}$, $\frac{x_2 - x_3}{g_2} = x_{31}$, $\frac{x_2 - x_4}{g_2} = x_{41}$ and condition $g_2 < \frac{L}{2}$ gives

\[ M^{1,2,3,4,4}(g_2) = \int_{x'}^{\infty} \int_{-\infty}^{\frac{x_2 - x_3}{g_2}} \int_{-L}^{L} \int_{-L}^{L} (p(x_2 - g_2x_{31}) (1 - p(x_2 - g_2x_{11}) - p(x_2)) \]

\[ + p(x_2 - g_2x_{11})p(x_2)) K(x_{31} - x_{11})K(x_{31}) \times K(x_{41}) (p(x_2 - g_2x_{41}) - p(x_2)) z_{6}(x_2 - g_2x_{11})z_{52}(x_2) \]

\[ \times h(x_2 - g_2x_{41})h(x_2 - g_2x_{31})h(x_2 - g_2x_{11})dx_{41}dx_{31}dx_{11}dx_2 \]

\[ = g_2^{\mu K} \int_{x'}^{\infty} \int_{-\infty}^{\frac{x_2 - x_3}{g_2}} \int_{-L}^{L} (p(x_2 - g_2x_{31}) (1 - p(x_2 - g_2x_{11}) - p(x_2)) \]

\[ + p(x_2 - g_2x_{11})p(x_2)) K(x_{31} - x_{11})K(x_{31}) \times \left( \frac{1}{2} p'(x_2)h(x_2) + p'(x_2)h'(x_2) \right) z_{6}(x_2 - g_2x_{11})z_{52}(x_2) \]

\[ \times h(x_2 - g_2x_{31})dx_{31}dx_{11}dx_2 + o(g_2^2) \]

\[ = O(g_2^2) \]

after using Taylor expansions of $p(x_2 - g_2x_{41}) - p(x_2)$ and $h(x_2 - g_2x_{41})$ around $x_2$. Therefore,

\[ M^{1,2,3,4,4}(g_2) = O(g_2^2). \]
If \( \text{Card}(I) = 3 \) there may be two equal pairs of indices or three indices equal. Using symmetry arguments the first possibility reduces the following 7 cases:

\[
\begin{align*}
&i_1 = i_2, \ i_3 = i_4 \quad i_1 = i_2, \ i_4 = i_5 \quad i_1 = i_4, \ i_2 = i_3 \\
&i_1 = i_4, \ i_3 = i_5 \quad i_1 = i_4, \ i_2 = i_5 \quad i_2 = i_3, \ i_4 = i_5 \\
&i_2 = i_5, \ i_3 = i_4.
\end{align*}
\]

For conciseness only the case \( i_1 = i_4, \ i_2 = i_5 \) will be considered. The order of this term will be also valid for the rest of the terms, not studied here. Its covariance is

\[
M^{1,2,3,1,3}(g_2) = M^{1,2,3,1,3}_{\text{II}}(g_2) - J_{12}(g_2)I_{1314}(g_2)
\]

where

\[
M^{1,2,3,1,3}_{\text{I}}(g_2) = E \left( K_{g_2}(T_1 - T_3) (\delta_3 - p(T_1)) z_{61}(T_1) \right)
\times K_{g_2}(T_2 - T_1) (\delta_1 - p(T_2)) K_{g_2}(T_2 - T_3) (\delta_3 - p(T_2)) z_{51}(T_2)).
\]

Now,

\[
M^{1,2,3,1,3}_{\text{I}}(g_2) = \int_{c_1}^{\infty} \int_{c_1}^{\infty} \int_{0}^{\infty} \left( p(x_3) (1 - p(x_1) - p(x_2)) + p(x_1) p(x_2) \right)
\times K_{g_2}(x_1 - x_3) K_{g_2}(x_2 - x_1) (p(x_1) - p(x_2)) \times K_{g_2}(x_2 - x_3) \\
\times z_{61}(x_1) z_{51}(x_2) h(x_3) b(x_2) h(x_1) dx_3 dx_2 dx_1.
\]

The change of variable \( \frac{x_1 - x_2}{g_2} = x_{21}, \ \frac{x_1 - x_3}{g_2} = x_{31} \) gives

\[
M^{1,2,3,1,3}_{\text{I}}(g_2) = g_2^{-1} \int_{c_1}^{\infty} \int_{-\infty}^{x_{21}} \int_{-\infty}^{x_{31}} \left( p(x_1 - g_2 x_{31}) (1 - p(x_1) - p(x_1 - g_2 x_{21})) + p(x_1) p(x_1 - g_2 x_{21})) \right)
\times K(x_{31} - x_{21}) z_{61}(x_1) z_{52}(x_1 - g_2 x_{21}) h(x_1 - g_2 x_{31}) dx_{31} dx_{21} dx_1
\]

which is \( o \left( g_2^{-1} \right) \).

On the other hand, again symmetry arguments lead to only the 6 following cases:

\[
\begin{align*}
&i_1 = i_2 = i_3 \quad i_1 = i_2 = i_4 \quad i_1 = i_3 = i_4 \quad i_1 = i_4 = i_5 \quad i_2 = i_3 = i_4 \\
&i_3 = i_4 = i_5.
\end{align*}
\]

For the last of them,

\[
M^{1,2,3,3,3}_{\text{I}}(g_2) = M^{1,2,3,3,3}_{\text{I}}(g_2) - J_{12}(g_2)I_{1313}(g_2)
\]

where

\[
M^{1,2,3,3,3}_{\text{I}}(g_2) = E \left( K_{g_2}(T_1 - T_3) (\delta_3 - p(T_1)) z_{61}(T_1) \right)
\times K_{g_2}(T_2 - T_3) (\delta_3 - p(T_2))^2 z_{51}(T_2)).
\]

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Standard algebra gives
\[
M_{1}^{1,2,3,3,3}(g_2) = \int_{\epsilon'}^{\infty} \int_{\epsilon'}^{\infty} \int_{0}^{\infty} (p(x_3) (1 - (p(x_1) + p(x_2))(1 - p(x_2))) - p(x_1)p(x_2)^2)
\times K_N(x_1 - x_3)K_N(x_2 - x_3)^2 \times 61(x_1) \times 51(x_2)
\times h(x_3)h(x_2)h(x_1)dx_3dx_2dx_1
\]
\[
= g_2^{-1} \int_{\epsilon'}^{\infty} \int_{-\infty}^{\epsilon'} \int_{-\infty}^{\epsilon'} (p(x_1 - g_2x_3)(1 - (p(x_1) + p(x_1 - g_2x_2)))
\times (1 - p(x_1 - g_2x_2)) - p(x_1)p(x_1 - g_2x_2)^2)
\times K(x_31)K(x_31 - x_21)^2 \times 62(x_1) \times 52(x_1 - g_2x_2)
\times h(x_1 - g_2x_3)dx_3dx_2dx_1
\]

after using the change of variable \( \frac{x_1 - x_2}{g_2} = x_21 \), \( \frac{x_1 - x_3}{g_2} = x_31 \). Therefore,
\[
M_{1}^{1,2,3,3,3}(g_2) = O(g_2^{-1})
\]
which implies
\[
M_{1}^{1,2,3,3,3}(g_2) = O(g_2^{-1}).
\]

If \( Card(I) < 3 \) it is sufficient to check that
\[
M_{1,12,13,14,15}(g_2) = E(K_{g_2}(T_{i_1} - T_{i_3})(\delta_{i_3} - p(T_{i_3})) z_{61}(T_{i_1})
\times K_{g_2}(T_{i_2} - T_{i_4})(\delta_{i_4} - p(T_{i_4})) K_{g_2}(T_{i_3} - T_{i_5})
\times (\delta_{i_5} - p(T_{i_5})) z_{51}(T_{i_5})) - O(g_2^2) O(g_2^{-2}),
\]

since, for any indices \( 1 \leq i_1, i_2, i_3, i_4, i_5 \leq n \),
\[
|M_{1,12,13,14,15}(g_2)| \leq g_2^{-3} \| K \|_\infty^3 \| z_{61} \|_\infty \| z_{51} \|_\infty + O(1).
\]
Collecting the previous result gives
\[
Cov (\hat{Q}_{12}, \hat{Q}_{131}) = O(n^{-1}g_2^2) + O(n^{-2}g_2^{-1}) + O(n^{-3}g_2^{-3}). \quad (251)
\]

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The covariance $Cov\left(\hat{Q}_{12}, \hat{Q}_{132}\right)$ can be studied in a parallel way:

$$
Cov\left(\hat{Q}_{12}, \hat{Q}_{132}\right) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Cov\left(\hat{\psi}(T_i) - p(T_i)\hat{h}(T_i), \hat{\psi}(T_j) - p(T_j)\hat{h}(T_j)\right) z_6(T_i) z_6(T_j),
$$

$$
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Cov\left(\frac{1}{n} \sum_{i=3}^{n} K_{g_2}(T_{i_1} - T_{i_2}) (\delta_{i_3} - p(T_{i_1})) z_6(T_{i_1}),
K_{g_2}(T_{i_2} - T_{i_4}) (\delta_{i_4} - p(T_{i_2}))
\times \frac{1}{n} \sum_{i=1}^{n} (K_{g_2}(T_{i_1} - T_{i_3}) - h(T_{i_1})) z_6(T_{i_2}) \right)
$$

$$
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} N^{i_1,i_2,i_3,i_4,i_5}(g_2),
$$

where, for $1 \leq i_1, i_2, i_3, i_4, i_5 \leq n$,

$$
N^{i_1,i_2,i_3,i_4,i_5}(g_2) = Cov(K_{g_2}(T_{i_1} - T_{i_3}) (\delta_{i_3} - p(T_{i_1})) z_6(T_{i_1}),
K_{g_2}(T_{i_2} - T_{i_4}) (\delta_{i_4} - p(T_{i_2}))
\times (K_{g_2}(T_{i_1} - T_{i_3}) - h(T_{i_1})) z_6(T_{i_2})).
$$

We now discuss the order of these covariances in terms of the cardinal of $I = \{i_1, i_2, i_3, i_4, i_5\}$ as done with the covariances $M^{i_1,i_2,i_3,i_4,i_5}(g_2)$. There are 6 possible cases for $Card(I) = 4$: $i_3 = i_2$, $i_1 = i_4$, $i_3 = i_4$, $i_3 = i_4$, but only the case $i_3 = i_4$ will be examined. First of all,

$$
N^{1,2,3,4,3,4}(g_2) = N^{1,2,3,4}(g_2) - I_{12}(g_2)I_{1325}(g_2)
$$

where

$$
N^{1,2,3,4}(g_2) = E(K_{g_2}(T_1 - T_3) (\delta_3 - p(T_1)) z_6(T_1)
\times K_{g_2}(T_2 - T_3) (\delta_3 - p(T_2)) (K_{g_2}(T_2 - T_4) - h(T_2)) z_6(T_2))
$$

and

$$
N^{1,2,3,4}(g_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p(x_3)(1-p(x_1)) - p(x_2)) + p(x_1)p(x_2) dx_1 dx_2 dx_3 dx_4
\times K_{g_2}(x_1 - x_3)K_{g_2}(x_2 - x_3) (K_{g_2}(x_2 - x_4) - h(x_2))
\times z_6(x_1)z_6(x_2)h(x_4)h(x_3)h(x_2)h(x_1)dx_4dx_3dx_2dx_1
$$

$$
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p(x_3)(1-p(x_1)) - p(x_2)) + p(x_1)p(x_2) dx_1 dx_2 dx_3
\times K_{g_2}(x_1 - x_3)K_{g_2}(x_2 - x_3) \left(\int_{-\infty}^{\infty} K_{g_2}(x_2 - x_4)h(x_4)dx_4 - h(x_2)\right)
\times z_6(x_1)z_6(x_2)h(x_3)dx_3dx_2dx_1.
$$

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The change of variable $\frac{x-x_0}{x_2} = x_{11}, \frac{x-x_1}{x_2} = x_{31}, \frac{x-x_2}{x_2} = x_{41}$ and condition $g_2 < \frac{x}{T}$ give

$$
N_{1,2,3,4} (g_2) = \int_0^\infty \int_{-\infty}^{x_2} L \int_{-\infty}^{x_2} (p(x_2 - g_2 x_31) (1 - p(x_2 - g_2 x_11) - p(x_2))
+p(x_2 - g_2 x_11) p(x_2)) K(x_31 - x_11) K(x_31)
\times z_6(x_2 - g_2 x_11) \left( \int_{-\infty}^{L} K_{g_2} (x_2 - x_4) h(x_4) dx_4 - h(x_2) \right)
\times z_61(x_2) h(x_2 - g_2 x_31) dx_{31} dx_{11} dx_2
= \frac{g_2^2}{2} h K \int_0^\infty \int_{-\infty}^{x_2} \int_{-\infty}^{x_2} (p(x_2 - g_2 x_31) (1 - p(x_2 - g_2 x_11) - p(x_2))
+p(x_2 - g_2 x_11) p(x_2)) K(x_31 - x_11) K(x_31) z_6(x_2 - g_2 x_11)
\times h^3(x_2) z_61(x_2) h(x_2 - g_2 x_31) dx_{31} dx_{11} dx_2 + O(g_2^2)
= O(g_2^2),
$$

after using a Taylor expansion of $h(x_2 - g_2 x_{41})$ around $x_2$. Therefore,

$$
N_{1,2,3,4} (g_2) = O(g_2^2).
$$

It is straightforward to conclude that the same order is valid for the other remaining five covariances.

For $\text{Card}(I) = 3$ it is already know that there are two possible situations.

The first one is concerned with two equal pairs of indices and the second one with three indices equal. The twelve cases included in the first possibility are

- $i_1 = i_2, i_3 = i_4, i_1 = i_2, i_3 = i_5, i_1 = i_2, i_4 = i_5, i_1 = i_4$,
- $i_2 = i_3$
- $i_1 = i_3, i_3 = i_5, i_1 = i_4, i_2 = i_5, i_1 = i_5, i_2 = i_3$
- $i_1 = i_5, i_3 = i_4, i_2 = i_3, i_4 = i_5, i_2 = i_4, i_3 = i_5, i_2 = i_5$,
- $i_3 = i_4$

but only the case $i_1 = i_5, i_3 = i_4$ will be considered. Its covariance is

$$
N_{1,2,3,3,1} (g_2) = N_{1,2,3,3,1}^1 (g_2) - I_{12} (g_2) I_{1325} (g_2)
$$

where

$$
N_{1,2,3,3,1} (g_2) = E(K_{g_2} (T_1 - T_3) (\delta_3 - p(T_1)) z_61(T_1)
\times K_{g_2} (T_2 - T_3) (\delta_3 - p(T_2)) (K_{g_2} (T_2 - T_1) - h(T_2)) z_62(T_2)).
$$

Now,

$$
N_{1,2,3,3,1}^1 (g_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} (p(x_3) (1 - p(x_1) - p(x_2)) + p(x_1) p(x_2))
\times K_{g_2} (x_1 - x_3) K_{g_2} (x_2 - x_3) (K_{g_2} (x_2 - x_1) - h(x_2))
\times z_61(x_1) z_62(x_2) h(x_3) h(x_2) h(x_1) dx_3 dx_2 dx_1
$$

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and the change of variable $\frac{x_1 - x_2}{y_2} = x_{21}$, $\frac{x_1 - x_4}{y_2} = x_{31}$ gives

$$N^{1,2,3,3,1}_1(g_2) = g_2^{-1} \int_{c'}^{c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p(x_1 - g_2x_3)(1 - p(x_1) - p(x_1 - g_2x_{21}))$$

$$+ p(x_1)p(x_1 - g_2x_{21})) K(x_31)K(x_{31} - x_{21})$$

$$\times (K(x_{21}) - g_2 h(x_1 - g_2x_{21})) \zeta_6(x_1)\zeta_{61}(x_1 - g_2x_{21})$$

$$\times h(x_1 - g_2x_{31}) dx_{31} dx_{21} dx_1$$

$$= O(g_2^{-1}).$$

Therefore,

$$N^{1,2,3,3,1}_1(g_2) = O(g_2^{-1}).$$

This order is also valid for the remainder eleven cases.

For the second possibility there are nine cases:

$$N^{1,2,3,3,3}_1(g_2) = N^{1,2,3,3,3}_1(g_2) - l_{12}(g_2) l_{1324}(g_2)$$

where

$$N^{1,2,3,3,3}_1(g_2) = E(K_{92}(T_1 - T_3)(\delta_3 - p(T_1)) \zeta_{61}(T_1)$$

$$\times K_{92}(T_2 - T_3)(\delta_3 - p(T_2)) (K_{92}(T_2 - T_3) - h(T_2)) \zeta_{62}(T_2)).$$

Standard algebra gives

$$N^{1,2,3,3,3}_1(g_2) = \int_{c'}^{c} \int_{c'}^{c} \int_{0}^{\infty} (p(x_3)(1 - p(x_1) - p(x_2)) + p(x_1)p(x_2))$$

$$\times K_{92}(x_1 - x_3)K_{92}(x_2 - x_3)(K_{92}(x_2 - x_3) - h(x_2))$$

$$\times \zeta_{61}(x_1) \zeta_{62}(x_2) h(x_3) h(x_2) h(x_1) dx_3 dx_2 dx_1$$

$$= g_2^{-1} \int_{c'}^{c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p(x_1 - g_2x_3)(1 - p(x_1) - p(x_1 - g_2x_{21}))$$

$$+ p(x_1)p(x_1 - g_2x_{21})) K(x_31)K(x_{31} - x_{21})$$

$$\times (K(x_{31} - x_{21}) - g_2 h(x_1 - g_2x_{21})) \zeta_6(x_1) \zeta_{61}(x_1 - g_2x_{21})$$

$$\times h(x_1 - g_2x_{31}) dx_{31} dx_{21} dx_1,$$

after using the change of variable $\frac{x_1 - x_2}{y_2} = x_{21}$, $\frac{x_1 - x_3}{y_2} = x_{31}$. Therefore,

$$N^{1,2,3,3,3}_1(g_2) = O(g_2^{-1}),$$

and consequently

$$N^{1,2,3,3,3}_1(g_2) = O(g_2^{-1}).$$
Finally, to deal with the case \( \text{Card}(I) < 3 \), observe that

\[
N^{i_1,i_2,i_3,i_4,i_5}(g_2) = E(K_{g_2}(T_{i_1} - T_{i_3}) (\delta_{i_1} - p(T_{i_3})) z_{61}(T_{i_3}) \\
\quad \times K_{g_2}(T_{i_2} - T_{i_4}) (\delta_{i_4} - p(T_{i_2})) (K_{g_2}(T_{i_2} - T_{i_4}) - h(T_{i_2})) z_{62}(T_{i_2})) \nonumber
\]

and the following, valid for any indices \( 1 \leq i_1, i_2, i_3, i_4, i_5 \leq n \),

\[
|N^{i_1,i_2,i_3,i_4,i_5}(g_2)| \leq g_2^{-3} \|K\|_\infty^2 (\|K\|_\infty + \|h\|_\infty) \|z_{61}\|_\infty \|z_{62}\|_\infty + o(g_2).
\]

can be used to conclude

\[
\text{Cov} \left( \tilde{Q}_{12}, \tilde{Q}_{132} \right) = O \left( n^{-1}g_2^2 + O \left( n^{-2}g_2^{-1} \right) + O \left( n^{-3}g_2^{-3} \right) \right) .
\] (252)

Finally (251), (252) and the representation (209) lead to (250).

Lemma 52 Assume conditions (K.1), (P.1), (H.1), (W.1).and (V.2). Then

\[
\tilde{Q}_1 - Q = O_P \left( n^{-\frac{3}{2}} + g_2^2 + n^{-1}g_2^{-1} \right)
\] (253)

Proof. From (54), \( \tilde{Q}_1 = Q_{11} + \tilde{Q}_{12} + \tilde{Q}_{13} \). From Lemmas ??, 41, 43, 49, 50 and 51, it is straightforward to conclude that

\[
E \left( \tilde{Q}_1 - Q \right) = D_1g_2^2 + D_2n^{-1}g_2^{-1} + o \left( g_2^2 \right) + o \left( n^{-1}g_2^{-1} \right)
\] (254)

after some easy algebra,

\[
\text{Var} \left( \tilde{Q}_1 - Q \right) = D_3n^{-1} + O \left( n^{-1}g_2^2 \right) + O \left( n^{-2}g_2^{-1} \right) + O \left( n^{-3}g_2^{-4} \right),
\] (255)

where \( D_1 \) and \( D_2 \) were defined in (56) and (57) and

\[
D_3 = \int_{x'}^\infty (1 - H(x))^{-4}p(x)(1-p(x))(1-3p(x)(1-p(x)))h(x)w(x)^2 dx - Q^2.
\] (256)

Condition (V.2) and Tchebychev inequality can be used to prove, for \( \tilde{Q}_1 - Q \),
the in probability bound given in (253).
Proof of Lemma 15. Define

\[ \hat{Q}_{141} = \frac{1}{n} \sum_{i=1}^{n} (\hat{p}(T_i) - p(T_i)) (\hat{h}(T_i) - h(T_i))^2 z_{62}(T_i), \]

\[ \hat{Q}_{142} = \frac{2}{n} \sum_{i=1}^{n} (\hat{\psi}(T_i) - p(T_i)) (\hat{p}(T_i) - p(T_i)) (\hat{h}(T_i) - h(T_i))^2 z_{51}(T_i), \]

and

\[ \hat{Q}_{143} = -\frac{1}{n} \sum_{i=1}^{n} (\hat{p}(T_i) - p(T_i))^2 (\hat{h}(T_i) - h(T_i))^2 z_{51}(T_i), \]

to obtain, according to (53),

\[ \hat{Q}_{14} = \hat{Q}_{141} + \hat{Q}_{142} + \hat{Q}_{143}. \]

Using the results by Silverman (1978) and Mack and Silverman (1982) and conditions (V.2), (W.1) and (H.1), it is easy to obtain

\[ \hat{Q}_{141} = O_P \left( n^{-\frac{3}{2}} g_2^{-\frac{1}{2}} \log \left( \frac{1}{g_2} \right)^{\frac{3}{2}} \right). \]

Similarly, let’s write

\[ \hat{Q}_{142} = \frac{2}{n} \sum_{i=1}^{n} (\hat{\psi}(T_i) - \psi(T_i) - p(T_i) \left( \hat{h}(T_i) - h(T_i) \right)) \times (\hat{p}(T_i) - p(T_i)) (\hat{h}(T_i) - h(T_i))^2 z_{51}(T_i), \]

to obtain

\[ \hat{Q}_{142} = O_P \left( n^{-\frac{3}{2}} g_2^{-\frac{1}{2}} \log \left( \frac{1}{g_2} \right)^{\frac{3}{2}} \right). \]

Finally, it is also easy to check that, under the same conditions,

\[ \hat{Q}_{143} = O_P \left( n^{-2} g_2^{-2} \log \left( \frac{1}{g_2} \right)^2 \right) \]

Therefore,

\[ \hat{Q}_{14} = O_P \left( n^{-\frac{1}{2}} g_2^{-\frac{1}{2}} \log \left( \frac{1}{g_2} \right)^{\frac{3}{2}} + n^{-2} g_2^{-2} \log \left( \frac{1}{g_2} \right)^2 \right) \]
and (58) follows from a further application of condition (V.2).

**Proof of 14.** Using expressions (254) and (255) for the expectation and the variance of $\bar{Q}_1 - Q$, given in the proof of Lemma 52, and condition (V.2), the mean squared error can be easily obtained:

$$
MSE(\bar{Q}_1) = \left( D_1 g^2_2 + D_2 n^{-1} g^{-1}_2 \right)^2 + D_3 n^{-1}
+ o \left( n^{-1} g_2 \right) + o \left( n^{-2} g^2_2 \right)
$$

where $D_1$, $D_2$ and $D_3$ were defined in (56), (57) and (256), respectively.

The value of $g_2$ that minimizes the dominant part of (257) will depend on the sign of $D_1$ (the case $D_1 = 0$ is not considered here). Indeed, if $D_1 < 0$ the minimum is attained in the point at which the derivative of $D_1 g^2_2 + D_2 n^{-1} g^{-1}_2$ with respect to $g_2$ vanish. This means

$$
g_2 = \left( \frac{D_2}{2D_1} \right)^{\frac{1}{2}} n^{-\frac{1}{4}}.
$$

On the other hand, if $D_1 > 0$ the minimum is attained when $D_1 g^2_2 + D_2 n^{-1} g^{-1}_2 = 0$, i.e.,

$$
g_2 = \left( -\frac{D_2}{D_1} \right)^{\frac{1}{2}} n^{-\frac{1}{4}},
$$

and the thesis of the theorem is proved.

**Proof of 16** We use the definitions given in (68), (43), (44), (194), (53) and (54), as well as

$$
\hat{A} = \hat{A}_1 + \hat{A}_{14} + \hat{A}_2
$$

and

$$
\hat{Q} = \hat{Q}_1 + \hat{Q}_{14} + \hat{Q}_2.
$$

It has been proved, in lemma 39, that

$$
\hat{A}_1 - A = O_P \left( g_1^2 + n^{-1} g_1^{-3} \right).
$$

and, in lemma 12,

$$
\hat{A}_{14} = o_P \left( n^{-1} g_1^{-3} \right).
$$

On the other hand, as noted in the end of the proof of 10,

$$
\hat{A}_2 = O_P \left( n^{-\frac{1}{2}} \right).
$$
Now, condition (V.1) gives
\[ \tilde{A} = A + O_P \left( g_1^2 + n^{-1} g_1^{-3} \right) \]
and for \( g_1 = g_{1,\text{AMSE}} \),
\[ \tilde{A} = A + O_P \left( n^{-\frac{1}{2}} \right). \] (258)

Similarly, using Lemmas 52 and 15,
\[ \tilde{Q}_1 - Q = O_P \left( n^{-\frac{1}{2}} + g_2^2 + n^{-1} g_2^{-1} \right) \]
and
\[ \tilde{Q}_2 = O_P \left( n^{-\frac{1}{2}} \right). \]

On the other hand, expression (195) gives
\[ \tilde{Q}_2 = O_P \left( n^{-\frac{1}{2}} \tilde{Q}_1 \right), \]
which, under condition (V.2), leads to
\[ \tilde{Q} = Q + O_P \left( n^{-\frac{1}{2}} + g_2^2 + n^{-1} g_2^{-1} \right) \]
and for \( g_2 = g_{2,\text{AMSE}} \),
\[ \tilde{Q} = Q + O_P \left( n^{-\frac{1}{2}} + n^{-\frac{1}{2}} \right) = Q + O_P \left( n^{-\frac{1}{2}} \right). \] (259)

Now,
\[ \hat{b} - b_{OPT} = \left( \frac{e_k \tilde{Q}}{2 d_k^2 n A} \right)^{1/3} - \left( \frac{e_k Q}{2 d_k^2 n A} \right)^{1/3} \]
\[ = \left( \frac{Q}{A} \right)^{1/3} - \left( \frac{Q}{A} \right)^{1/3} \left( \frac{e_k}{2 d_k^2 n} \right)^{\frac{1}{3}} \]
and a Taylor expansion for the function \( g(x) = x^{\frac{1}{2}} \) around \( x_0 \geq 0 \),
\[ g(x) = g(x_0) + O(x - x_0), \]
gives
\[ \hat{b} - b_{OPT} = O_P \left( \left( \frac{Q}{A} - \frac{Q}{A} \right) n^{-\frac{1}{2}} \right). \] (260)
Some straightforward algebra leads to

\[
\frac{\hat{Q}}{A} - \frac{Q}{A} = \frac{\hat{Q} - Q}{A} - \frac{\hat{Q} - A}{A}
\]

and, using (258) and (259),

\[
\frac{\hat{Q}}{A} - \frac{Q}{A} = O_P\left(n^{-\frac{1}{2}}\right) + O_P(1)O_P\left(n^{-\frac{1}{2}}\right) = O_P\left(n^{-\frac{1}{2}}\right).
\]

The result in (59) can be obtained substituting this order in (260), while (60) can be proved using the fact that \(b_{OPT}\) is of precise order \(n^{-\frac{1}{2}}\). \(\square\)

7 Acknowledgments

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