STRONG REPRESENTATION OF A CONDITIONAL QUANTILE FUNCTION ESTIMATOR WITH TRUNCATED AND CENSORED DATA

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Strong representation of a conditional quantile function estimator with truncated and censored data

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Abstract

We consider lifetime data with covariables which are subject to both left truncation and right censorship (LTRC). In this context, it is interesting to study the conditional distribution function of the lifetime and the corresponding conditional quantile function. A generalized product-limit estimator (GPLE) of the conditional distribution function has been studied in Iglesias-Pérez and González-Manteiga (1999). In the present paper we define a conditional quantile function estimator via the mentioned GPLE and we derive an almost sure representation for this quantile estimator. This result extends strong quantile representations studied on conditional survival analysis for censored data (Dabrowska (1992) and Van Keilegom and Veraverbeke (1998)). As a consequence of this representation we establish the asymptotic normality of the conditional quantile estimator.

Keywords and phrases: censored data, truncated data, generalized product-limit estimator, conditional quantile estimator, almost sure representation.

Running head: Strong conditional quantile representation for LTRC data.

1. Introduction.

In this paper we study lifetime data with covariables which are subject to both left truncation and right censorship. Let \((X; Y; T; S)\) be a random vector, where \(Y\) is the lifetime, \(T\) is the random left truncation time, \(S\) denotes the random right censoring time and \(X\) is a covariable related with \(Y\). It is assumed that \(Y; T; S\) are conditionally independent given \(X = x\) and \(@P(x) = P(T \cdot Z < T|X = x)\).
\( X = x \) \( > 0 \), where \( Z = \min fY; Sg \): In this model, one observes \((X; Z; T; \pm) \) if \( Z \leq T \); where \( \pm = 1 \) \( Y \). When \( Z > T \) nothing is observed.

Let \((X_1; Z_1; T_1; \pm_1); ...; n, \) be an i.i.d. random sample from \((X; Z; T; \pm) \) which one observes (then \( T_i \leq Z_i \); for all \( i \)). If \( F(y \mid x) = P(Y \leq y \mid X = x) \) denotes the conditional distribution function of \( Y \) when \( X = x \); a nonparametric estimator of \( F(\cdot \mid x) \); called generalized product-limit estimator (GPLE), \( F_h^n(\cdot \mid x) \), is defined in Iglesias-Pérez and González-Manteiga (1999), as follows:

\[
F_h^n(y \mid x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{T_i - Z_i} B_{hi}(x) \#
\]

where \( B_{hi}(x) \) is a sequence of nonparametric weights (specifically, Nadaraya and Watson weights) and \( h = h_n \) is the bandwidth parameter. Also, some important properties about this estimator are provided: an almost sure asymptotic representation of \( F_h^n(\cdot \mid x) \), the asymptotic normality of \( (nh^{1/2} F_h^n(y \mid x) - F(y \mid x)) \) and the weak convergence of the corresponding process.

Moreover, a bootstrap procedure for approximating the distribution of \( (nh^{1/2} F_h^n(y \mid x) - F(y \mid x)) \) is defined and validated in Iglesias-Pérez and González-Manteiga (2001).

In this paper, we study the estimation of quantiles of the conditional distribution function \( F(y \mid x) \) for left truncation and right censored data (LTRC data). The \( p \)-th conditional quantile \( F_q^{1}(p \mid x) \) is defined as

\[
F^{1}(p \mid x) = \inf y : F(y \mid x) \geq p, \quad 0 < p < 1:
\]

Then, if we replace \( F(y \mid x) \) in (1.2) by the GPLE, \( F_h^n(y \mid x) ; \) given by (1.1), we obtain a natural estimator of \( F^{1}(p \mid x) \), given by

\[
F_h^n(1)(p \mid x) = \inf y : F_h^n(y \mid x) \geq p
\]

which will be called generalized product-limit quantile estimator (GPLQE).

For this GPLQE we derive an almost sure representation and, as consequence, we establish its asymptotic normality. These properties are shown in section 4. Before to do so, we introduce, in section 2, the notation and the necessary assumptions and we prove, in section 3, some preliminary lemmas (two of them about the strong consistency of the GPLQE).

Asymptotic representations for different quantile estimators with censored and/or truncated data have been studied by many authors. In absence of covariables, representations of the product-limit quantile estimator are obtained by Lo and Singh (1986) for censored data, by Gürler, Stute and Wang (1993) for truncated data and by Zhou (1997) for jointly censored and truncated data; representations of smooth quantile estimators can be seen in Zhou and An (1996), Liu and Zhu.
(1996) and in Sun and Zheng (1999) for censored data and for censored and truncated data, respectively. In the presence of covariables we cite the representations derived by Dabrowska (1992) and Van Keilegom and Veravebeke (1998) for conditional quantile estimators with censored data. Asymptotic representations for conditional quantile estimators with truncated data are not available in the literature so far.

2. Notation and assumptions.

Before deriving the a.s. representation of the GPLQE for LTRC data given by (1.3), we need to introduce some notation and assumptions. The following curves are defined:

(i) \( M (x) = P(X < x); \) represents the distribution function of \( X \);

(ii) \( L(y \mid x) = P(T < y \mid X = x); \) is the conditional distribution function of \( T \) at \( X = x \).

(iii) \( H(y \mid x) = P(Z < y \mid X = x); \) is the conditional distribution function of \( Z \) at \( X = x \).

(iv) \( H_1(y \mid x) = P(Z < y; \pm = 1 \mid X = x); \) is the conditional subdistribution function (when \( Z = Y \) of \( Z \) at \( X = x \))

(v) \( C(y \mid x) = P(T < y; \ Z \mid X = x; T \cdot Z). \)

(vi) \( F(y \mid x) = P(Y < y \mid X = x); \) the conditional distribution function of \( Y \) when \( X = x \); and

(vii) \( @x) = P(T \cdot (Z \mid X = x); \) the conditional probability of absence of truncation at \( X = x \).

Moreover, for any distribution function \( W(t) = P(\cdot \mid t) \), we denote the left and right support endpoints by \( a_W = \inf t: W(t) > 0 \) and \( b_W = \inf t: W(t) = 1 \), respectively. Specifically, we will use the notation: \( a_{\cdot \mid x}(y), a_{+ \mid x}(y), b_{- \mid x}, \) and \( b_{+ \mid x} \) for the support endpoints of functions \( L(y \mid x) \) and \( H(y \mid x); \) considering \( L \) and \( H \) as functions of the variable \( y \) for a fixed \( x \) value.

Finally, for a distribution function \( W; \) we denote \( W(t) = P(\cdot \mid t; T \cdot Z); \) So, we can consider: \( M^\#(x) = P(X \cdot \pm = 1 \mid T \cdot Z) \) and \( H_1^\#(y \mid x) = P(Z \cdot y; \pm = 1 \mid X = x; T \cdot Z) \).

To formulate our results, we will use some of the hypotheses listed below:

\(^2\) The model assumptions:

(H1) \( X; Y; T \) and \( S \) are absolutely continuous random variables.

(H2) a) The variable \( X \) takes values in an interval \( I = [x_1; x_2] \) contained in the support of \( m^\# \) (density of \( M^\# \) (see Remark 1 in Iglesias-Pérez and González-Manteiga (1999))), such that

\[ 0 < \gamma = \inf \frac{\mathcal{E}}{m^\#(x)} : x \in I^- < \sup \frac{\mathcal{E}}{m^\#(x)} : x \in I^- = i < 1 \]

for some \( I^- = [x_1; \ x_2] \) with \( > 0 \) and \( 0 < \gamma < 1 \).

And for all \( x \in I^- \) the r.v. \( Y; T; S \) are conditionally independent at \( X = x \):

b) Moreover, as regards the \( Y; T \) and \( S \) variables, we consider:
i) \( a\{, (j;x) \cdot a_H\{, (j;x) \cdot b_H\{, (j;x) \); for all \( x \leq 1 \): (Compare with Woodroofe's results (Woodroofe, 1985) about identifiability of \( F \) for truncated data without covariables).

ii) The variable \( Y \) moves in an interval \([a; b]\) such that \( \inf\{ \frac{F}{1} \} (1; H(bj; x)) L(a_j x) < 2 \cdot 1 \). \( \mu > 0 \) (note that, if \( a\{, (j;x) < y < b\{, (j;x) \) then \( C(y j x) = \frac{1}{1} (1; H(y j x)) L(y j x) > 0 \), therefore condition ii) says that \( C(y j x) \mu > 0 \) in \([a; b] \cdot 1 \).

(H3) \( a < a\{, (j;x) \), for all \( x \leq 1 \):

(H4) The corresponding ( improper) densities of the distribution (subdistribution) functions \( L(y), H(y) \) and \( H_1(y) \) are bounded away from 0 in \([a; b]\):

\(^2\) The smoothness hypotheses:

(H5) The \(.rst\) derivatives with respect to \( x \) of functions \( m(x) \) and \( \theta(x) \) exist and are continuous in \( x \leq 1 \) and the \(.rst\) derivatives with respect to \( x \) of functions \( L(y j x); H(y j x) \) and \( H_1(y j x) \) exist and are continuous and bounded in \((y; x) 2 [0; 1] \cdot 1 \).

(H6) The second derivatives with respect to \( x \) of functions \( m(x) \) and \( \theta(x) \) exist and are continuous in \( x \leq 1 \) and the second derivatives with respect to \( x \) of functions \( L(y j x); H(y j x) \) and \( H_1(y j x) \) exist and are continuous and bounded in \((y; x) 2 [0; 1] \cdot 1 \).

(H7) The \(.rst\) derivatives with respect to \( y \) of functions \( L(y j x); H(y j x) \) and \( H_1(y j x) \) exist and are continuous in \((y; x) 2 [a; b] \cdot 1 \).

(H8) The second derivatives with respect to \( y \) of functions \( L(y j x); H(y j x) \) and \( H_1(y j x) \) exist and are continuous in \((y; x) 2 [a; b] \cdot 1 \).

(H9) The \(.rst\) derivatives with respect to \( x \) and second with respect to \( y \) of functions \( L(y j x), H(y j x) \) and \( H_1(y j x) \) exist and are continuous in \((y; x) 2 [a; b] \cdot 1 \).

Remark 1. Hypotheses H7 and H8 and relations:

\[ R_y \frac{dF(y j x)}{d(1; H(y j x))} = R_y \frac{d^2 H(y j x)}{d(1; H(y j x))} \]

\[ H_1(y j x) = \frac{\theta(x)}{H_1(y j x)} L(y j x) \]

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(see page 217 in Iglesias-Pérez and González-Manteiga (1999)) imply that the \(.rst\) and the second derivatives with respect to \( y \) of \( F(y j x) \) exist and are continuous in \((y; x) 2 [a; b] \cdot 1 \). We denote these derivatives as \( F^0(y j x) \) and \( F^0(y j x) \), respectively.

\(^2\) The kernel function assumptions:

(H10) The kernel function, \( K \), is a symmetrical density vanishing outside \((i; 1; 1)\) and the total variation of \( K \) is less than some \( \lambda < +1 \):\n
\(^2\) The bandwidth parameter hypothesis:

(H11) The bandwidth parameter \( h = (h_n) \) verifies: \( h > 0; \ln n \approx (nh) > 0 \) and \( nh^2 \approx \ln n \approx O(1) \):

Finally, we work with nonnegative variables as usual in survival analysis.
3. Some preliminary results.

In order to derive an almost sure representation of the GPLQE we establish four lemmas, which in themselves are of independent interest. The two rst lemmas are about the estimator \( \hat{F}_h \); dened by (1.1): Lemma 3.1 states the uniform strong consistency of \( \hat{F}_h \) and Lemma 3.2 studies the behavior of its modulus of continuity. The other two lemmas show the point and the uniform strong consistency of the conditional quantile estimator \( \hat{F}_h^{1-}. \)

Lemma 3.1. Suppose that assumptions H1-H11 hold. Then, for \( x \geq 1 \) and \( y \geq 1 \) \( [a; b] \); it follows that:

\[
\sup_{[a,b] \not \in I} \bar{T}_h(y_j \mid x) \quad \text{for} \quad F(y_j \mid x) = O \left( \frac{\ln n}{nh} \right) \quad \text{as:}
\]

Proof. The hypotheses in this lemma allows for the application of the a.s. representation of \( \hat{F}_h(y_j \mid x) \) which has been established in Theorem 2c) of Iglesias-Pérez and González-Manteiga (1999). This representation allows us to write, for \( x \geq 1 \) and \( y \geq 1 \) \( [a; b] \), that

\[
\hat{F}_h(y_j \mid x) = F(y_j \mid x) = (1 \in F(y_j \mid x)) \quad B_{hi}(x) \times (Z_i; T_i; \pm; y; x) + R_n(y_j \mid x) \quad (3.1)
\]

where \( \times (Z_i; T_i; \pm; y; x) \) is dened by (4.3) in Theorem 4.1, and

\[
\sup_{[a,b] \not \in I} \bar{R}_n(y_j \mid x) = O \left( \frac{\ln n}{nh} \right) \quad \text{as:} \quad (3.2)
\]

It is easy to see that \( P \left[ \frac{n}{i=1} B_{hi}(x) \times (Z_i; T_i; \pm; y; x) \right] \) is equal to

\[
\frac{Z}{n} \frac{dH_i^\#(s_j \mid x)}{C(s_j \mid x)} \quad \text{and} \quad \frac{Z}{n} \frac{C_i(s_j \mid x)}{C^2(s_j \mid x)} \quad \text{where} \quad H_i^\#(y_j \mid x) = P \left[ \frac{n}{i=1} B_{hi}(x) \right] \text{and} \quad C_i(y_j \mid x) = P \left[ \frac{n}{i=1} B_{hi}(x) \right] \text{are} \quad \text{the nonparametric estimators with Nadaraya-Watson weights of} \quad H_i^\#(s_j \mid x) \quad \text{and} \quad C(y_j \mid x), \quad \text{respectively.}
\]

Moreover, the rst term in the latter expression can be written as

\[
\frac{Z}{n} \frac{dH_i^\#(y_j \mid x)}{C(y_j \mid x)} \quad \text{and} \quad \frac{Z}{n} \frac{C_i(y_j \mid x)}{C^2(y_j \mid x)} = (3.3)
\]

So, the rates of uniform strong consistency of the estimators \( \hat{H}_i^\#(y_j \mid x) \) and \( \hat{C}_i(y_j \mid x) \) given by Theorem 1 of Iglesias-Pérez and González-Manteiga (1999) and assumption H2 lead to

\[
\sup_{[a,b] \not \in I} \bar{R}_n(y_j \mid x) = O \left( \frac{\ln n}{nh} \right) \quad \text{as:} \quad (3.5)
\]

\[
\text{Lemma 3.2}
\]
Finally, this result and (3.2) prove the lemma.

Lemma 3.2. Suppose that assumptions H1-H11 hold. Then, for \( x \geq 1 \); it follows that:

\[
\sup_{f_s,t_2[a:b]; s_1, t_1} \left| \int_{\mathbb{R}} \frac{F_h(s_j x)}{F_h(t_j x) + F(s_j x)} \right| = O \left( \frac{\ln n}{n^\beta} \right)
\]

as:

Proof. By using the a.s. representation (3.1) of \( F_h(y j x) \), we can write:

\[
\begin{align*}
& \sup_{f_s,t_2[a:b]; s_1, t_1} \left| \int_{\mathbb{R}} \frac{F_h(s_j x)}{F_h(t_j x) + F(s_j x)} \right| \\
& \quad = \sup_{f_s,t_2[a:b]; s_1, t_1} \left| \int_{\mathbb{R}} \frac{B_h(x) \mathcal{N}(Z_i; T_i; \pm; s_1, t_1)}{B_h(x) \mathcal{N}(Z_i; T_i; \pm; t_1)} \right| \\
& \quad = \sup_{f_s,t_2[a:b]; s_1, t_1} \left| \int_{\mathbb{R}} \frac{B_h(x) \mathcal{N}(Z_i; T_i; \pm; s_1, t_1)}{B_h(x) \mathcal{N}(Z_i; T_i; \pm; t_1)} \right| \\
& \quad + O \left( \frac{\ln n}{n^\beta} \right)
\end{align*}
\]

where the \( \beta \)st term on the right hand side in the above inequality is upper-bounded by:

\[
\begin{align*}
& \sup_{f_s,t_2[a:b]; s_1, t_1} \left| \int_{\mathbb{R}} \frac{B_h(x) \mathcal{N}(Z_i; T_i; \pm; s_1, t_1)}{B_h(x) \mathcal{N}(Z_i; T_i; \pm; t_1)} \right| \\
& \quad = 4 \left( \int_{\mathbb{R}} \frac{B_h(x) \mathcal{N}(Z_i; T_i; \pm; s_1, t_1)}{B_h(x) \mathcal{N}(Z_i; T_i; \pm; t_1)} \right)
\end{align*}
\]

As regards (3.7), we have (see (3.3) and (3.4)):

\[
\begin{align*}
& X_i \quad B_h(x) \mathcal{N}(Z_i; T_i; \pm; s_1, t_1) \\
& = 4 \left( \int_{\mathbb{R}} \frac{B_h(x) \mathcal{N}(Z_i; T_i; \pm; s_1, t_1)}{B_h(x) \mathcal{N}(Z_i; T_i; \pm; t_1)} \right)
\end{align*}
\]

now we have to prove that uniformly in \( s_1, t_1, \ldots, c(\ln n(nh))^{1/2} \) the three terms on the right side of above equality are \( O \left( \frac{\ln n}{n^\beta} \right) \) as: We denote them by S1, S2 and S3, respectively.

As regards S1 it is easy to see that:

\[
\begin{align*}
& \frac{\mathbb{Z}^3}{4} H^2_{1n} \mathcal{H}^2_{1n} (y j x) \quad \frac{\mathbb{Z}^3}{4} H^2_{1n} (y j x) \\
& - \frac{\mathbb{Z}^3}{4} H^2_{1n} (y j x) \quad \frac{\mathbb{Z}^3}{4} H^2_{1n} (y j x) \\
& - \frac{\mathbb{Z}^3}{4} H^2_{1n} (y j x) \quad \frac{\mathbb{Z}^3}{4} H^2_{1n} (y j x) \\
& - \frac{\mathbb{Z}^3}{4} H^2_{1n} (y j x) \quad \frac{\mathbb{Z}^3}{4} H^2_{1n} (y j x)
\end{align*}
\]
and we know that the two terms on the right hand side of the above inequality are, respectively, $O_{\text{I}}(\ln n \ln n)$ and $O_{\text{I}}(\ln n \ln n)$ due to Lemma 6 and Theorem 1 of Iglesias-Pérez and González-Manteiga (1999) jointly with hypothesis H2 and H7 and conditions about $j \leq t$. Secondly, we have that

$$jS2j \cdot \frac{1}{\mu_{y2[a;b]}} \sup_{i \in \Omega_{y2[a;b]}} H_{1}^{#}(y j x) \cdot H_{1}^{#}(y j x) \cdot c_{1} j \leq t$$

where $c_{1}$ is an upper bound of $C_{0}$ which exists from assumption H7. Thus, arguing as in the study of S1, we obtain that $\sup_{f_{s} \leq t} \sup_{j \leq t} c_{1} j \leq t = O_{\text{I}}(\ln n \ln n)$ due to Lemma 6 and (3.5) we obtain:

$$\sup_{f_{s} \leq t} \sup_{j \leq t} c_{1} j \leq t = O_{\text{I}}(\ln n \ln n)$$

Finally, in analogous way as in the calculation of the order of $\sup jS2j$, we have

$$jS3j \cdot \frac{1}{\mu_{y2[a;b]}} \sup_{i \in \Omega_{y2[a;b]}} \tilde{C}_{1}(y j x) \cdot C(y j x) \cdot c_{2} j \leq t$$

which implies that $\sup_{f_{s} \leq t} \sup_{j \leq t} c_{2} j \leq t = O_{\text{I}}(\ln n \ln n)$ due to Lemma 6 and (3.7) and (3.9) prove the lemma.

Now, we study term (3.8). By taking into account the following upper bound:

$$(3.8) \cdot jF^{0}(r) j \leq t \cdot C(y j x) \cdot B_{1}(x) = O_{\text{I}}(\ln n \ln n)$$

where $r$ is between $t$ and $s$, jointly with Remark 1, the fact of $j \leq t \cdot c_{1} j \leq t = O_{\text{I}}(\ln n \ln n)$ and the result (3.5) we obtain:

$$\sup_{f_{s} \leq t} \sup_{j \leq t} c_{2} j \leq t = O_{\text{I}}(\ln n \ln n)$$

This order, (3.6) and (3.9) prove the lemma.

Lemma 3.3. Assume that $F^{1}(p j x)$ is the unique solution of $F(y j x) = 0$ for $x \geq 1$ and $y \geq 1$ in $[a; b]$. Then, the strong consistency of $F^{1}_{1}(y j x)$ implies the strong consistency of $F^{1}_{1}(p j x)$, which is:

$$F^{1}_{1}(p j x) \cdot F^{1}_{1}(p j x) = O_{\text{I}}(\ln n \ln n)$$

Proof. This proof is similar to that of Theorem 2.3.1 of Ser‡ing (1980, p.75). It is only necessary to observe that the estimator $F^{1}_{1}(y j x)$ is a monotone increasing and right continuous function and that it is a strong consistent estimator of $F(y j x)$.
Lemma 3.4. Assume $H1-H11$. Let $0 < p_0 \cdot p_t < 1$ be such that $a < F_1^{-1}(p_0 j x) \cdot F_0(\#j x) < b$ and $\inf_{p_0, p_t} F_0 F_1(p_t j x) \cdot \xi \neq -\infty$ and $\xi > 0$, for $x \geq 1$. Then, for $p_2 \cdot [p_0, p_t]$:

$$
\sup_{p_0 \cdot p_t} \frac{F_0 F_1(p_t j x) \cdot \xi}{F_1^{-1}(p_0 j x) \cdot F_0(\#j x) \cdot \xi} = O \left( \frac{\ln n}{n} \right)^{1/2} \text{ as:}
$$

Proof. First, note that $F_1^{-1}(p_t j x) \cdot 2 [a; b]$ for all $n$ large enough, because Lemma 3.3 and the assumption $a < F_1^{-1}(p_t j x) < b$. This fact will be repeatedly used in this and the next proof.

Second, we need to study the difference between $F_0 F_1^{-1}(p_t j x) $ and $p$. In order to do so, we write:

$$
\sup_{a; b} \left[ F_0 F_1^{-1}(p_t j x) \cdot \xi \right] = \sup_{a; b} \left[ F_0 F_1^{-1}(p_t j x) \cdot \xi \right] = \sup_{a; b} \left[ F_0 F_1^{-1}(p_t j x) \cdot \xi \right] \cdot F_1^{-1}(p_0 j x) + F_0(\#j x) \cdot \xi
$$

where (see (3.5))

$$
\sup_{[a; b]} jR_P(n j x) = O \left( \frac{\ln n}{n} \right)^{1/2} \text{ as:}
$$

A Taylor expansion of $F(\xi j x)$ allows us to write (3.11) as

$$
F_0 F_1^{-1}(p_t j x) \cdot \xi = F_0 F_1^{-1}(p_t j x) \cdot \xi + F_0(\#j x) \cdot F_1^{-1}(p_0 j x) + R_P(n j x)
$$

for some $\#j$ between $F_1^{-1}(p_t j x)$ and $F_1^{-1}(p_0 j x)$. This latter equality is equivalent to

$$
F_0 F_1^{-1}(p_0 j x) \cdot \xi = F_0 F_1^{-1}(p_0 j x) \cdot \xi + F_0(\#j x) \cdot \xi R_P(n j x)
$$

Finally, (3.10), (3.12) and the assumption $\inf_{p_0, p_t} F_0 F_1(p_0 j x) \cdot \xi \neq -\infty$ and $\xi > 0$ prove the lemma.
4. Strong representation and asymptotic normality.

In this section we derive an almost sure representation of the conditional quantile estimator given by (1.3). As consequence of this representation we establish the asymptotic normality of the mentioned estimator. Theorem 4.1 and Theorem 4.2 provide these results.

Theorem 4.1. Assume H1-H11. Let \( 0 < p_0 \cdot p_1 < 1 \) be such that \( a < F^{-1}(p_0 \cdot x) \cdot F^{-1}(p_1 \cdot x) < b \) and \( \inf_{p_0 \cdot p_1} F^0(F^{-1}(p_0 \cdot x) \cdot x \cdot F^{-1}(p_1 \cdot x)) = \cdot > 0 \) for \( x \geq 1 \). Then, for \( p \geq 2 \) \( [p_0 ; p_1] \), it follows that:

\[
F_h^0(F^{-1}(p_0 \cdot x) \cdot x \cdot F^{-1}(p_1 \cdot x)) = F^0(F^{-1}(p_0 \cdot x) \cdot x \cdot F^{-1}(p_1 \cdot x)) + R_{n_1}(p \cdot x)
\]

(4.1)

\[
F^0(F^{-1}(p_0 \cdot x) \cdot x \cdot F^{-1}(p_1 \cdot x)) = \frac{1}{p_i} F^0(F^{-1}(p_0 \cdot x) \cdot x \cdot F^{-1}(p_1 \cdot x)) + R_{n_2}(p \cdot x)
\]

(4.2)

where

\[
\gamma(Z; T; \pm z; x) = 1_{Z \cdot \gamma} \frac{1}{C(Z \cdot x)} i \int \frac{Z}{C(s \cdot x)} dH_1(s \cdot x)
\]

(4.3)

Proof. We apply Lemma 3.2, Lemma 3.3 and a Taylor expansion to obtain:

\[
F^0(F^{-1}(p_0 \cdot x) \cdot x \cdot F^{-1}(p_1 \cdot x)) = \frac{1}{p_i} F^0(F^{-1}(p_0 \cdot x) \cdot x \cdot F^{-1}(p_1 \cdot x)) + R_{n_1}(p \cdot x)
\]

(4.4)
Theorem 4.2.

By using the representation (3.1) of

\[ F_h^{i-1}(p_j x) i F_i^{i-1}(p_j x) \]  
\[ + R_{n1}(p_j x) \]

where the term \( \sup_{p_0 \cdot p \cdot p_1} j(4.5) \) is \( O \left( \frac{3^{3+4}}{\max} \right) \) as (Remark 1 implies that \( F_0^{i}(\cdot x) \) is bounded).

We also have uniformly in \( p_0 \cdot p \cdot p_1 \) (from (3.10)) that the second term in (4.4) is, with probability one, equal to

\[ p_i F_h^{i-1}(p_j x) j x^{\xi} + \frac{\tilde{A}^{i} \mu_{\ln n}^{3+4}}{n} j x^{\xi} \]

So finally, we have obtained the representation given by (4.1), which is

\[ F_h^{i-1}(p_j x) i F_i^{i-1}(p_j x) = p_i F_h^{i-1}(p_j x) j x^{\xi} + R_{n1}(p_j x) \]

with \( \sup_{p_0 \cdot p \cdot p_1} jR_{n1}(p_j x) j = O \left( \frac{3^{3+4}}{\max} \right) \) as:

By using the representation (3.1) of \( F_h^{i}(y_j x) \), we can write (4.1) as

\[ F_h^{i-1}(p_j x) i F_i^{i-1}(p_j x) = p_i F_h^{i-1}(p_j x) j x^{\xi} + F_i^{i-1}(p_j x) j x^{\xi} i F_i^{i-1}(p_j x) j x^{\xi} + \frac{\tilde{A}^{i} \mu_{\ln n}^{3+4}}{n} j x^{\xi} + R_{n1}(p_j x) \]

\[ + R_{n1}(p_j x) \]

\[ = i (1 + p_i) F_h^{i-1}(p_j x) j x^{\xi} i F_i^{i-1}(p_j x) j x^{\xi} + R_{n2}(p_j x) \]

with \( \sup_{p_0 \cdot p \cdot p_1} jR_{n2}(p_j x) j = O \left( \frac{3^{3+4}}{\max} \right) \) as: This completes the proof of the theorem.

Theorem 4.2. Under the assumptions of Theorem 4.1 we have:

a) If \( nh^3 \leq 0 \) and \((\ln n)^3 = (nh) \) then

\[ (nh)^{1/2} F_h^{i-1}(p_j x) i F_i^{i-1}(p_j x) ! \]  
\[ d N \overset{\xi}{=} 0; s^2(p_j x) \]

where

\[ s^2(p_j x) = \frac{(1 + p_i)^2}{m^2(x)} \mu Z \tilde{A}^{i} i Z^{i} F_i^{i-1}(p_j x) j x^{\xi} dH^i(y_j x) \]

b) If \( h = Cn^{1/2} \) for some \( C > 0 \), then

\[ (nh)^{1/2} F_h^{i-1}(p_j x) i F_i^{i-1}(p_j x) ! \]  
\[ d N \overset{\xi}{=} b(p_j x); s^2(p_j x) \]

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where

\[ b(p_j x) = C^{5=2} \frac{i (1 \cdot i \cdot p)}{2m^2(x) F \sum(F^{i-1}(p_j x) j \cdot j)} \mu Z \int_{\mathbb{R}} u^2 K(u) du \left( \partial^{0}_{i}(p_j x) m^2(x) + 2\partial^{0}_{i}(p_j x) m^2(x) \right) \]

with

\[ \partial(p_j u) = E^{\frac{1}{i}x}(Z ; T) \partial F^{i-1}(p_j x) j \cdot T \cdot Z ; X = u \]

\[ = \frac{\partial H_{i}^{\flat}(y \cdot j) C(y \cdot j)}{C(y \cdot j)} \int_{0}^{1} C(y \cdot j) dH_{i}^{\flat}(y \cdot j) \]

\[ = \frac{1}{0} \int_{0}^{1} C(y \cdot j) dH_{i}^{\flat}(y \cdot j) \]

and

\[ \partial^{0}(p_j x) = \frac{1}{0} \int_{0}^{1} C(y \cdot j) dH_{i}^{\flat}(y \cdot j) \]

where \( \partial \) and \( \partial^{0} \) denotes the rst and second derivatives with respect to \( x \) in the \( x \) point, respectively.

Proof. a) The expression \( (4.1) \) in Theorem 4.1 and the condition \( \frac{(ln n)^3}{m^n}! \cdot 0 \), which implies that \( (nh)^{1-2} \frac{ln n}{m^n} \frac{4^n}{4^n} = \frac{ln n}{m^n} 1/n^4! \cdot 0 \), lead to

\[ (nh)^{1-2} F_{h}^{i-1}(p_j x) i \cdot F^{i-1}(p_j x) = \frac{(nh)^{1-2} F_{h}^{i-1}(p_j x) j \cdot x^{\xi} \cdot p}{F^{0}_{i}(p_j x) j \cdot x^{\xi}} + o(1) \quad as: \]

By applying Theorem 5.1. of Billingsley (1968, p.30), we only have to study the limiting distribution of \( (nh)^{1-2} F_{h}^{i-1}(p_j x) j \cdot x^{\xi} \cdot p \), which is shown in Corollary 3a) in Iglesias-Pérez and González-Manteiga (1999) under the present condition of \( nh^5! \cdot 0 \).

b) As consequence of that \( h = C n^{1-5} = (ln n)^3 = (nh)! \cdot 0 \), this proof parallels completely that of part a). We only emphasize that Corollary 3b) in Iglesias-Pérez and González-Manteiga (1999) provides the asymptotic distribution of \( (nh)^{1-2} F_{h}^{i-1}(p_j x) j \cdot x^{\xi} \cdot p \) under the condition \( h = C n^{1-5} \).

Remark 2. The assumption \( \inf_{p_j} p_j F^{0}_{i}(p_j x) j \cdot x^{\xi} > 0 \) in Theorem 4.2 can be replaced by \( F^{0}_{i}(p_j x) j \cdot x^{\xi} > 0 \) because it concerns a non uniform result.
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6. References