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The Banzhaf value in TU games with restricted cooperation

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Prefacio

Este trabajo es el resultado de mi iniciación a la investigación en Matemáticas, en concreto en el área de la Teoría de Juegos.

Antes de nada me gustaría expresar mi agradecimiento a Ignacio García Jurado, en primer lugar por despertar en mí el interés por este área con sus clases de Teoría de la Decisión y Teoría de Juegos, y en segundo lugar por toda la ayuda prestada para integrarme en el acogedor grupo de investigadores teóricos de juegos de Galicia. También quiero agradecer a todos los integrantes de SaGaTh y muy especialmente a José Mª Alonso Meijide y a Mª Gloria Fiestras Janeiro por haberme guiado en mis primeros años como investigador y por la gran ayuda prestada.

En este trabajo, se presentan mis primeras aportaciones a la Teoría del valor de juegos cooperativos con utilidad transferible, en concreto, se estudian modelos de juegos en los que la cooperación está restringida. En los capítulos 2, 3, 4 y 5 se presentan los distintos modelos estudiados, se repasan los principales resultados existentes y se proponen y caracterizan nuevos conceptos de solución . Por último, en el capítulo 6 se aplican los modelos estudiados a un caso práctico proveniente de la política.

El resto de trabajo ha sido escrito en inglés dado a que de esta forma podré utilizar partes de él para su futura publicación.

Contents

Introduction

The goal of Game Theory is the analysis of conflictive situations on which more than one player interact. In such a situation the agents or players have different preferences over the outcomes of the game. This research branch studies how rational individuals should behave when they have to face different kinds of conflictive situations.

Game Theory classifies such situations in two different groups: A situation is modeled as a non cooperative game when the players do not have mechanisms to make binding agreements before the game is played, in this first group each players 'best strategies' are obtained depending on each possible situation. A situation is modeled as a cooperative game when players have mechanisms to make binding agreements before the game itself is played. The class of cooperative games is divided into transferable and non transferable utility games. We assume that players obtain utility from each possible outcome of the game depending on their preferences. In the transferable utility games, TU games from now on, the utility that players get can be divided and transferred among other players without any loss. This work deals with TU games.

Chapter 1 starts introducing some basic concepts and notation dealing with TU games. Next, the existing solution concepts are briefly described, classifying them into the one point set and set solutions. The first Chapter ends recalling the main results concerning the Shapley and the Banzhaf values since they are the basis of the solution concepts that are studied in the rest of this report.

The remaining Chapters, except for Chapter 6, consider different extensions of the original model presented in Chapter 1. The extensions are built introducing external information to enrich the model. This external information is called externality.

Chapter 2 is devoted to the study of games with graph restricted communication model. Following the model proposed by Myerson (1977), the main results of the literature concerning the Myerson and the Banzhaf graph values are revised. In particular a characterization of the Banzhaf graph value proposed in Alonso-Meijide and Fiestras-Janeiro (2006) is presented in detail.

In Chapter 3 we study the model of games with incompatibilities. In this case the restrictions to the cooperation are given by means of a graph which describes the existing incompatibilities among players. We follow the approach proposed in Carreras (1991) and Bergantiños (1993). In this context, the existing literature deals with the extension of the Shapley value and nothing has been done using the Banzhaf value so far. Consequently, an extension of the Banzhaf value for games with incompatibilities is introduced and characterized.

Chapter 4 follows the first approach to restricted cooperation games proposed in Aumann and Drèze (1974). In this approach a system of a priori unions is given together with the TU game. The games with a priori unions, also called games with coalition structure, have been widely studied in the literature and different solution concepts have been proposed. Particularly the Owen value proposed in Owen (1977) has been successfully used to analyze the implications of the formation of coalitions in a parliament (Carreras and Owen

(1995)). In this Chapter three solution concepts (values on $U(N)$ under this scenario) are presented together with their parallel axiomatic characterizations by Alonso-Meijide and Fiestras-Janeiro (2002). These parallel characterizations of the values are very helpful in order to compare and sort out the differences among the solution concepts.

In Chapter 5 we consider the joint model of games with graph restricted communication and a priori unions as in Vázquez-Brage et al. (1996). In this way we are capable of building a model which adds more external information concerning the behavior of the players involved. In Vázquez-Brage et al. (1996) the Owen value is extended to this context and an axiomatic characterization of it provided. The remaining values on $U(N)$ presented in the previous Chapter are extended to this context. The two new values proposed are characterized and the proposed characterizations yield to a comparison among the three values.

Lastly, Chapter 6 provides two real world examples coming from the political field. The examples show the applicability of the studied models. The presented solution concepts are used in order to determine the power of each political party in a parliament. The relations among the different political parties and the emerging coalitions can constrain the way the cooperation occurs. The studied models try to include all this externalities into the original simple game.

1

Preliminaries

This Chapter is devoted to the basic concepts of the Cooperative Game Theory. In Section 1.1, the cooperative games with transferable utility are introduced and some important properties of them stated. Section 1.2 revises the different approaches that have been proposed in order to obtain solutions to this class of games. Section 1.3 concludes recalling with detail the main characterizations of the Shapley and the Banzhaf values.

1.1 Basic concepts of Game Theory

Definition 1.1.1. A *TU game* is a pair (N, v), where $N = \{1, \ldots, n\}$ is the (finite) set of players, and $v : 2^N \to \mathbb{R}$ is the characteristic function of the game, which satisfies $v(\emptyset) = 0$. In general, we interpret $v(S)$ as the benefit that S can obtain by its own, i.e., independent to the decisions of players in $N \setminus S$. We will denote by $G(N)$ the class of all TU games with set of players N.

To avoid cumbersome notation braces will be omitted whenever it does not lead to confusion, for example we will write $v(S \cup i)$ or $v(S \setminus i)$ instead of $v(S \cup \{i\})$ or $v(S \setminus \{i\})$. The cardinality of a finite set S, will be denoted by s, the corresponding lower case letter.

We can define the sum and the scalar product in the set $G(N)$ in the following way. Let $(N, v), (N, w) \in G(N)$ and $\lambda \in \mathbb{R}$,

- The sum game $(N, v+w) \in G(N)$ is defined by $(v+w)(S) = v(S) + w(S)$ for all $S \subseteq N$.

- The scalar product game $(N, \lambda v) \in G(N)$ is defined by $(\lambda v)(S) = \lambda v(S)$ for all $S \subseteq N$.

The set $G(N)$ together with the operations defined above has a vector space structure. The neutral element in this space is the null game $(N, v_0) \in G(N)$, defined by $v_0(S) = 0$ for all $S \subseteq N$. Shapley (1953) showed that this space has dimension $2^n - 1$ and that the family of unanimity games constitute a basis of it. Then, any TU game can be uniquely written as a linear combination of this type of games. In other words, given $(N, v) \in G(N)$, there exist a unique set of scalars $\{\lambda_S \in \mathbb{R}\}_{\emptyset \neq S \subseteq N}$, for which,

$$v = \sum_{\emptyset \neq S \subseteq N} \lambda_S u_S,$$

where $(N, u_S) \in G(N)$ denotes the unanimity game with carrier S defined by,

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{else} \end{cases}$$

for all $T \subseteq N$, and λ_S are the Harsanyi dividends (Harsanyi (1959, 1963)), defined by,

$$\lambda_S = \sum_{T \subseteq S} (-1)^{s-t} v(T).$$

Next we recall the definitions of some specially interesting games.

Definition 1.1.2. Let $(N, v) \in G(N)$ be a TU game.

- (N, v) is called *superadditive* if

$$v(S \cup T) \geq v(S) + v(T) \text{ for all } S, T \subseteq N, S \cap T = \emptyset.$$

- (N, v) is called *subadditive* if $(N, -v)$ is superadditive.
- (N, v) is called *additive* if

$$v(S \cup T) = v(S) + v(T) \text{ for all } S, T \subseteq N, S \cap T = \emptyset.$$

- (N, v) is called *monotone* if

$$v(S) \leq v(T) \text{ for all } S, T \subseteq N \text{ with } S \subseteq T.$$

- (N, v) is called *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \subseteq N.$$

- (N, v) is called a *zero sum* game if

$$v(S) + v(N \setminus S) = 0 \text{ for all } S \subseteq N.$$

- (N, v) is called a *simple game* if it is monotone, $v(N) = 1$, and

$$v(S) \in \{0, 1\} \text{ for all } S \subseteq N.$$

We will denote by $SG(N)$ the set of all simple games with player set N.

In general in superadditive or monotone games players have incentives to form the grand coalition, i.e., they benefit from the cooperation with the rest of players. It is easily seen that if a game is convex, then it is superadditive.

Some players have properties which will be specially interesting. We provide their definitions as follows,

Definition 1.1.3. Let $(N, v) \in G(N)$ be a TU game,

- Player $i \in N$ is called a *dummy player* if

$$v(S \cup i) = v(S) + v(i) \text{ for all } S \subseteq N \setminus i.$$

- Player $i \in N$ is called a *null player* if it is a dummy player and $v(i) = 0$.
- Players $i, j \in N$ are called *symmetric* if

$$v(S \cup i) = v(S \cup j) \text{ for all } S \subseteq N \setminus \{i, j\}$$

1.2 Solution concepts

The situations modeled by a TU game have a cooperative approach, therefore an implicit objective of TU games is the grand coalition, N, to be formed, and the generated benefits to be shared among the players. Hence, one of the goals of the Cooperative Game Theory is to distribute the worth of the grand coalition, $v(N)$, among the players involved. An allocation is simply a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, where each coordinate represents the amount allotted to each player. Hence, the aim is to provide rules which give allocations for any particular game. These rules will be called solution concepts.

The aim is to provide an allocation which is "admissible" for the players. But what does admissible mean? This issue has generated a big discussion and many different approaches have been developed in the last years in order to obtain an answer to such a question. Two of the most accepted principles are individual rationality and efficiency. Let us take a TU game $(N, v) \in G(N)$. We say that an allocation $x \in \mathbb{R}^n$ satisfies individual rationality if it allocates each player with at least what he can obtain on his own, i.e., if $v(i) \leq x_i$. An allocation $x \in \mathbb{R}^n$ is efficient if it completely shares the worth of the grand coalition, i.e., if $x_1 + \cdots + x_n = v(N)$. The allocations which satisfy these two properties are called imputations. Hence, most of the solution concepts existing in the literature provide allocations that lie within this set.

In general, solution concepts can be classified in two big groups. In the first one we have the set-valued solutions. They are mainly based on stability, i.e, these solutions try to provide a set of allocations on which players will possibly agree. In other words, this approach discards those allocations which are not acceptable for a group of players. It depends on the game, but these kind of solutions can be unique, can be a set of different vectors or can even be empty. The most well known such a solution concept is the core, which was introduced by Gillies (1953). The idea behind the core follows a coalitional rationality principle, which states that no coalition should have incentives to brake the grand coalition. Other set solution concepts are the Stable set (von Neumann and Morgenstern (1944)), the Bargaining set (Aumann and Maschler (1965)), the Kernel (Davis and Maschler (1965)), the Harsanyi set (Hammer et al. (1977), Vasil'ev (1980)), and the Weber set (Weber (1988)) among others. The second group are the so called one-point solutions. From now on we will focus our attention on them and we will also refer to them as values. These kind of rules provide an allocation which is fair in some sense. In other words, first some desirable properties need to be defined. And then, a value is built in such a way that is the only one satisfying the set of properties. The characterizations help us to sort out all the basic properties that each rule satisfies. The most popular one-point solution is the Shapley value (Shapley (1953)). There is a vast literature concerning this value and many different characterizations have been provided. The Shapley value does not verify individual rationality in general, however, for the class of superadditive games the Shapley value is an imputation. For a good survey on this issue see, for instance Winter (2002). Another value is the Banzhaf value (Banzhaf (1965)). As we will later see in the explicit expression the Banzhaf value is very similar to the Shapley value. Other one-point solution are the Nucleolus (Schmeidler (1969)), the τ-value (Tijs (1981)), and the Core center (González-Díaz and Sánchez-Rodríguez (2007)) among others. For the class of simple games $SG(N)$ the one-point solutions are usually called power indices.

1.3 The Shapley and Banzhaf values

This section is devoted to the description and comparison of the Shapley and Banzhaf values since these solution concepts are the basis of the following Chapters. Although most of the values existing in the literature are introduced axiomatically, i.e., first a set of desirable properties which a value should satisfy are stated, and then it is proved that there exist only one value satisfying them. Here, we will first give the explicit analytical definitions of both the Shapley and the Banzhaf values, to come to the discussion on the properties later.

By a *value on G(N)* we will mean a map f that assigns a vector $f(N,v) \in \mathbb{R}^n$ to every game $(N,v) \in G(N)$.

Definition 1.3.1. Shapley (1953). Given a TU game (N,v), the *Shapley value*, φ, is a value on $G(N)$ defined for every $i \in N$ as follows:

$$\varphi_i(N,v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left[v(S \cup i) - v(S) \right].$$

Definition 1.3.2. Banzhaf (1965). The *Banzhaf value*, β, is a value on $G(N)$ defined for every $i \in N$ as follows:

$$\beta_i(N,v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus i} \left[v(S \cup i) - v(S) \right].$$

The Shapley and Banzhaf values have a simple probabilistic interpretation. For doing so we will base our explanation on the work by Weber (1988) on probabilistic values.

Fix a player $i \in N$, and let $\{p_S^i | S \subseteq N \setminus i\}$ be a probability distribution over the collection of coalitions not containing i. A value f on $G(N)$ is a probabilistic value if for every $(N, v) \in G(N)$ and every $i \in N$,

$$f_i(N, v) = \sum_{S \subseteq N \setminus i} p_S^i \left[v(S \cup i) - v(S) \right].$$

Let i view his participation in a game as consisting merely of joining some coalition $S \subseteq N \setminus i$, and then receiving as a reward his marginal contribution $v(S \cup i) - v(S)$ to the coalition. If, for each $S \subseteq N \setminus i$, p_S^i is the (subjective) probability that he joins coalition S, then $f_i(N, v)$ is simply i's expected payoff from the game.

As it can be seen in Definition 1.3.1 and Definition 1.3.2, both φ and β are instances of probabilistic values. The Banzhaf value arises from the subjective belief that each player is equally likely to join any coalition, that is, $p_S^i = \frac{1}{2^{n-1}}$ for all $S \subseteq N \setminus i$. On the other hand, the Shapley value arises from the belief that for every player, the coalition he joins is equally likely to be of any size $s \quad (0 \leq s \leq n-1)$ and that all coalitions of a given size are equally likely. That is,

$$p_S^i = \frac{1}{n} \binom{n-1}{s}^{-1} = \frac{s!(n-s-1)!}{n!}, \text{ for every } S \subseteq N \setminus i.$$

Before we get into the characterizations, it is worth to make a comment on the origin of the Shapley and Banzhaf values. The Shapley value was proposed by Shapley (1953) in an axiomatic way and for the whole class of n-person cooperative TU games. We recall the original characterization in Theorem 1.3.3. The value proposed by Shapley was an interesting idea and one year later Shapley and Shubik (1954) considered the value for the class of simple games. In this framework it is known as the Shapley-Shubik index. The values restricted to simple games are called power indices. This comes from the use of simple games to study the distribution of power in a voting body as we will see in Chapter 6. The Shapley value became early the most popular probabilistic value. On the other hand, the underlying idea of the Banzhaf value was first presented in Penrose (1946) for simple games, but it lay unnoticed for decades. For that reason the Banzhaf value for simple games is known as the Banzhaf-Penrose index. Banzhaf (1965) was the first to present the power index in detail even though he provided no characterization. The Banzhaf value in the expression presented above, was introduced in Owen (1975), and the first characterization was not proposed until Lehrer (1988). The Banzhaf value has never received as much attention as the Shapley value.

To end with this first Chapter we will state different characterizations of both the Shapley and the Banzhaf values. For doing this we will need some properties a value f on $G(N)$ could be asked to satisfy. Let f be a value on $G(N)$.

- *Efficiency.* For any $(N, v) \in G(N)$,

$$\sum_{i \in N} f_i(N, v) = v(N).$$

- *Dummy player property.* For every $(N, v) \in G(N)$ and each dummy player $i \in N$ in (N, v),

$$f_i(N, v) = v(i).$$

- *Null player property.* For any TU game $(N, v) \in G(N)$ and each null player $i \in N$ in (N, v),

$$f_i(N, v) = 0.$$

- *Symmetry.* For all $(N, v) \in G(N)$ and each pair of symmetric players $i, j \in N$ in (N, v),

$$f_i(N, v) = f_j(N, v).$$

- *Anonymity.* For all $(N, v) \in G(N)$ and all permutations σ of N,

$$f_{\sigma(i)}(N, v) = f_i(N, \sigma v),$$

where the game σv is defined by $\sigma v(S) = v(\sigma(S))$ for all $S \subseteq N$.

- *Additivity.* For any pair of TU games $(N, v), (N, w) \in G(N)$,

$$f(N, v + w) = f(N, v) + f(N, w).$$

- *Transfer property.* For all $(N, v), (N, w) \in G(N)$,

$$f(N, v) + f(N, w) = f(N, v \vee w) + f(N, v \wedge w),$$

where $(N, v \vee w), (N, v \wedge w) \in G(N)$ are defined for all $S \subseteq N$ as $(v \vee w)(S) = \max\{v(S), w(S)\}$ and $(v \wedge w)(S) = \min\{v(S), w(S)\}$.

- *2-Efficiency.* For all $(N, v) \in G(N)$ and each pair of players $i, j \in N$,

$$f_i(N, v) + f_j(N, v) = f_p(N^{ij}, v^{ij}),$$

where (N^{ij}, v^{ij}) is the game obtained from (N, v) when players i and j merge in a new player p, i.e., $N^{ij} = (N \setminus \{i, j\}) \cup \{p\}$ and $v^{ij}(S) = \begin{cases} v(S) & \text{if } p \notin S \\ v((S \setminus p) \cup i \cup j) & \text{if } p \in S \end{cases}$, for all $S \subseteq N^{ij}$.

- *2-Efficiency*.* For all $(N, v) \in G(N)$ and each pair of players $i, j \in N$,

$$f_i(N, v) + f_j(N, v) \leq f_p(N^{ij}, v^{ij}).$$

- *Total power property.* For all $(N, v) \in G(N)$,

$$\sum_{i \in N} f_i(N, v) = \frac{1}{2^{n-1}} \sum_{i \in N} \sum_{S \subseteq N \setminus i} [v(S \cup i) - v(S)].$$

- *Strong monotonicity.* For any pair $(N,v),(N,w) \in G(N)$ and each $i \in N$ such that $v(S \cup i) - v(S) \leq w(S \cup i) - w(S)$, for all $S \subseteq N \setminus i$, then

$$f_i(N,v) \leq f_i(N,w).$$

- *Marginal contributions.* For any pair $(N,v),(N,w) \in G(N)$ and each $i \in N$ such that $v(S \cup i) - v(S) = w(S \cup i) - w(S)$, for all $S \subseteq N \setminus i$, then

$$f_i(N,v) = f_i(N,w).$$

A value is Efficient if it completely shares the worth of the grand coalition, $v(N)$, among the players. The Efficiency property may appear very reasonable, but it is not always essential. It depends on the situation being modeled. For example if simple games are being used to analyze the distribution of the power in a voting body, the Efficiency property sates that the sum of the power indices of each agent adds unity. This however, may not be necessary when we want to compare the strength of two players. Besides, if in a voting body, the unanimity is needed to reach an agreement, the voting body itself would have less power than if we consider the majority rule, because it will be more difficult to make a decision.

The Dummy player property states that a player whose marginal contribution to any coalition is always the worth he can obtain on his own, $v(i)$, should be allotted with exactly that amount. The Null player property is the Dummy player property only required for null players.

An allocation rule is Symmetric when it allocates the same amount to players whose marginal contribution to every coalition are equal. The Anonymity property states that the amount that a player recibes does not depend on his relative position inside N. Anonymity implies Symmetry while the reverse does not hold in general.

The Additivity is a standard property in the literature, even if it has been criticized for the use of the sum game. It states that the payoff of the sum game equals the sum of the original games payoff. The Transfer property avoids the use of the sum game but is very similar to the Additivity property.

The 2-Efficiency property states that the allocation rule that satisfies it is immune against artificial merging or splitting of players. It was first introduced as inequality (2-Efficiency*) and it is very helpful for some characterizations. The 2-Efficiency property requieres a value to be inmune only against artificial splitting.

The Total power property establishes that the total payoff obtained for the players is the sum of all marginal contributions of every player normalized by 2^{n-1}. Depending on the particular game this amount may be greater, lesser or equal to $v(N)$.

The last two properties, Strong monotonicity and Marginal contributions are quite similar. They link the payoffs of two games with the differences between the marginal contributions of the games mentioned.

There is a vast literature concerning characterizations of the Shapley and Banzhaf values. In this work we present the main results in the following Theorems. In Shapley (1953) the Shapley value was introduced in an axiomatic way.

Theorem 1.3.3. Shapley (1953). *The Shapley value, φ, is the unique value on $G(N)$ satisfying Efficiency, Null player property, Symmetry, and Additivity.*

In Young (1985) the Shapley value was characterized without using the additivity axiom, which was the most criticized one of the characterization by Shapley.

Theorem 1.3.4. Young (1985). *The Shapley value, φ, is the unique value on $G(N)$ satisfying Efficiency, Strong monotonicity, and Symmetry.*

Feltkamp (1995) presented alternative characterizations of both the Shapley and Banzhaf values. In this way a comparison of the properties that each solution concept satisfies could be done.

Theorem 1.3.5. Feltkamp (1995). *The Shapley value, φ, is the unique value on $G(N)$ satisfying Efficiency, Anonymity, Null player property, and Transfer property.*

The first characterization of the Banzhaf value which made use of the Symmetry, the Additivity and the Dummy player property was stated in Lehrer (1988). In this Theorem the Shapley's Efficiency axiom was substituted by 2-Efficiency*.

Theorem 1.3.6. Lehrer (1988). *The Banzhaf value, β, is the unique value on $G(N)$ satisfying 2-Efficiency*, Symmetry, Additivity, and the Null player property.*

Theorem 1.3.7. Feltkamp (1995). *The Banzhaf value, β, is the unique value on $G(N)$ satisfying Total power, Transfer property, Anonymity, and the Null player property.*

In Nowak (1997) it was shown that the Banzhaf value actually satisfy the 2-Efficiency property, which extend the property 2-Efficiency* used in Theorem 1.3.6.

Theorem 1.3.8. Nowak (1997). *The Banzhaf value, β, is the unique value on $G(N)$ satisfying 2-Efficiency, Symmetry, Dummy player property, and Marginal contributions.*

A recent work which provides a new characterization of the Banzhaf value is Lorenzo-Freire et al. (2007).

Theorem 1.3.9. Lorenzo-Freire et al. (2007). *The Banzhaf value, β, is the unique value on $G(N)$ satisfying Total power, Symmetry, and Strong monotonicity.*

To conclude this first Chapter, we present in Table 1.1 a summary of the properties that each of the values on $G(N)$ satisfies. A ✓ indicates that the value on $G(N)$ satisfies the corresponding property.

	φ	β
Efficiency	✓	
Dummy player property	✓	✓
Null player property	✓	✓
Symmetry	✓	✓
Anonymity	✓	✓
Additivity	✓	✓
Transfer property	✓	✓
2-Efficiency		✓
2-Efficiency*		✓
Total power		✓
Strong monotonicity	✓	✓
Marginal contributions	✓	✓

Table 1.1: Properties and values on $G(N)$

As it can be seen, there are few but important differences between φ and β. The Shapley value is efficient while the Banzhaf value is not. As we argue before, the efficiency property may not always be reasonable, for that reason, the decision whether to use φ or β depends on the situation being modeled. The Banzhaf value divides the amount indicated by the Total power property. The last property that makes the difference between φ and β is the 2-Efficiency. The Banzhaf value satisfies it. In addition, it is a very useful property since it leads to a characterization of β.

2 Games with graph restricted communication

In the basic TU game model there is no restriction to the cooperation, which means that any group of agents can reach agreements. In many real situations however, there is a priori information about the behavior of the players and only partial cooperation occurs.

The TU game with graph restricted cooperation model was first introduced in Myerson (1977). Few years later (Myerson (1980)) the question was extended to games without side payments (NTU games) where the communication is restricted by means of an hypergraph. An interesting survey in the field of games with graph restricted communication can be found in Borm et al. (1991), where several research lines are described.

In this Chapter the basic model introduced in Myerson (1977) will be described. In such a situation we have a graph together with the TU game which depicts the way cooperation may occur. Each link of the graph indicates that direct communication, and hence cooperation, is possible between agents located at each end. Also, communication between agents joined via a path in a given coalition is possible.

In Section 2.1 we will introduce the model and in Section 2.2 the solution concepts for such a class of games will be defined and characterized.

2.1 The model

An *undirected graph without loops* on N is a set B of unordered pairs of distinct elements of N. Each pair $(i:j) \in B$ is a *link*. Given $i, j \in S \subseteq N$, we say that i and j are *connected in S by B* if $i = j$ or if there is a path in S connecting them, i.e., there is some $k \geq 1$ and a subset $\{i_0, i_1, \ldots, i_k\} \subseteq S$ such that $i_0 = i$, $i_k = j$ and $(i_{h-1} : i_h) \in B$, for every $h = 1, 2, \ldots, k$. We denote by S/B the set of maximal connected components of S determined by B, i.e., the set of maximal subsets of elements connected in S by B. S/B is a partition of S. We denote by $g(N)$ the set of all undirected graphs without loops on N. Given $B \in g(N)$, the *dual or complementary graph* of B, $B^c \in g(N)$ is given by,

$B^c = \{(i : j) \in g(N) | (i : j) \notin B\}$. We will denote by $\emptyset^c$ the *complete graph* on N defined by $\emptyset^c = \{(i : j) | i, j \in N, i \neq j\}$.

Given $B \in g(N)$ we say that agent $i \in N$ is an isolated agent with respect to the graph B if there is no $j \in N \setminus i$ such that $(i : j) \in B$, that is, $\{i\} \in N/B$. Given a link $(i : j) \in B$, the graph $B^{-ij} \in g(N)$ is defined as the resulting graph after the elimination of the link $(i : j)$, that is

$$B^{-ij} = \{(h : k) \in B | (h : k) \neq (i : j)\}.$$

For any $i \in N$, we denote by B^{-i} the element of $g(N)$ obtained from B by breaking the links where agent i is involved, i.e.,

$$B^{-i} = \{(j : h) \in B | j \neq i \text{ and } h \neq i\}.$$

Definition 2.1.1. A *TU game with graph restricted communication* is a triplet (N, v, B) where $(N, v) \in G(N)$ and $B \in g(N)$. We denote by $C(N)$ the set of all such games. If $(N, v, B) \in C(N)$ the *communication game* $(N, v^B) \in G(N)$ is defined by

$$v^B(S) = \sum_{T \in S/B} v(T).$$

Notice that when $B = \emptyset^c$, we have $v^B = v$ and when $B = \emptyset$, $(N, v^B) \in G(N)$ is an additive game.

The definition of the communication game can be understood as follows. Consider a coalition of players $S \subseteq N$. If coalition S is internally connected, i.e., if all players in S can communicate with one another (directly or indirectly) without the help of players in $N \setminus S$, then they can fully coordinate their actions and obtain the worth $v(S)$. Nevertheless, if coalition S is not internally connected, then not all players in S can communicate with each other without the help of outsiders. Coalition S will then be split into communication components according to the partition S/B. The best that players in S can accomplish under these conditions is to coordinate their actions within each of these components. Players in different components cannot coordinate their actions and hence, the components will operate independently.

2.2 Myerson and Banzhaf graph values

After having described how several authors have approached the integration of restrictions on communication into TU games, in this Section we will focus on values which give us an assessment of the benefits for each player from participating in a graph restricted TU game.

By a *value on C(N)* we will mean a map f that assigns a vector $f(N, v, B) \in \mathbb{R}^n$ to every game with graph restricted communication $(N, v, B) \in C(N)$. In order to define such values on $C(N)$ we can use the communication game v^B defined above and the Shapley and Banzhaf values, φ and β. In doing this we come to the definitions of the well known Myerson and Banzhaf graph values.

Definition 2.2.1. Myerson (1977). The *Myerson value*, φ^c, is the value on $C(N)$ defined by

$$\varphi^c(N, v, B) = \varphi(N, v^B).$$

Definition 2.2.2. Owen (1986). The *Banzhaf graph value*, β^c, is the value on $C(N)$ defined by

$$\beta^c(N, v, B) = \beta(N, v^B).$$

If two players are in different communication components of a graph restricted game $(N, v, B) \in C(N)$, then they do not interact with each other at all. Consequently, it seems reasonable to expect that the values on $C(N)$ of coalitions that include players that are not connected to player $i \in N$ as well as links involving such players do not influence the payoff of player i. This requirement is satisfied by both the Myerson and the Banzhaf graph values, which are Component decomposable as it was shown in van den Nouweland (1993) and Alonso-Meijide and Fiestras-Janeiro (2006). Formally, let f be a value on $C(N)$.

- *Component decomposability.* For every $(N, v, B) \in C(N)$ and every player $i \in N$, it holds that

$$f_i(N, v, B) = f_i(S, v_{|S}, B_{|S}),$$

  where $S \in N/B$ such that $i \in S$, and $(S, v_{|S}, B_{|S}) \in C(S)$ is the graph restricted game obtained from $(N, v, B) \in C(N)$ when the players set is restricted to S.

Myerson (1977) found the Myerson value when he was looking for an allocation rule that satisfy the two properties Component efficiency and Fairness, which we describe below. Let f be a value on $C(N)$.

- *Component efficiency.* For every $(N, v, B) \in C(N)$ and every $S \in N/B$, it holds that

$$\sum_{i \in S} f_i(N, v, B) = v(S).$$

- *Fairness.* For every $(N, v, B) \in C(N)$ and any $i, j \in N$ such that $(i : j) \in B$, it holds that

$$f_i(N, v, B) - f_i(N, v, B^{-ij}) = f_j(N, v, B) - f_j(N, v, B^{-ij}).$$

An allocation rule is Component efficient if the payoffs of the players in a maximal connected component add up to the worth of that component. Using a Component efficient value on $C(N)$, the players distribute the worth of this component among themselves. Fairness reflects the equal gains equity principle that two players should gain or loose equally from their bilateral agreement.

Under this scenario, other properties that a value f, on $C(N)$ should satisfy have been proposed in the literature. Next we define four more properties which will be needed to characterize the Myerson and Banzhaf graph values.

- *Balanced contributions.* For every $(N, v, B) \in C(N)$ and any $i, j \in N$, it holds that

$$f_i(N, v, B) - f_i(N, v, B^{-j}) = f_j(N, v, B) - f_j(N, v, B^{-i}).$$

- *Isolation.* For every $(N, v, B) \in C(N)$ and any $i \in N$ isolated agent, it holds that

$$f_i(N, v, B) = v(i).$$

- *Component total power.* For every $(N, v, B) \in C(N)$ and any $S \in N/B$, it holds that

$$\sum_{i \in S} f_i(N, v, B) = \frac{1}{2^{s-1}} \sum_{i \in S} \sum_{T \subseteq S \setminus i} \left[v^B(T \cup i) - v^B(T) \right].$$

- *2-Efficiency.* For all $(N, v, B) \in C(N)$ and each pair of players $i, j \in N$ such that $(i : j) \in B$, it holds that

$$f_i(N, v, B) + f_j(N, v, B) = f_p(N^{ij}, v^{ij}, B^{ij}),$$

  where (N^{ij}, v^{ij}, B^{ij}) is the game obtained from (N, v, B) when players i and j merge in a new player p, i.e., (N^{ij}, v^{ij}) is defined as in Chapter 1 and given $l, k \in N^{ij}$,

$$(l : k) \in B^{ij} \text{ if and only if } \begin{cases} (l : k) \in B \text{ with } l, k \in N \setminus \{i, j\} \\ (l : i) \in B \text{ or } (l : j) \in B \text{ and } k = p \\ (i : k) \in B \text{ or } (j : k) \in B \text{ and } l = p \end{cases}$$

The Balanced contributions property generalizes the Fairness property. It states that a player's isolation from the graph benefits or damages other player in the same amount than if it happened the other way round. The Isolation is very similar to the Dummy player property, in fact it is just the Dummy player property applied to the communication game v^B, it states that a player who cannot communicate to any other player should be given what he can obtain on his own. The Component total power property indicates the amount that a connected component will receive. The 2-Efficiency property is just the generalization of the 2-Efficiency property defined in Chapter 1 for a pair of agents that are directly connected by the graph..

To conclude with the Chapter the main characterizations of the Myerson and the Banzhaf graph values will be presented. The first two characterizations by Myerson have been widely used in the literature.

Theorem 2.2.3. Myerson (1977). *The Myerson value, φ^c, is the unique graph value on $C(N)$ satisfying Component efficiency and Fairness.*

Theorem 2.2.4. Myerson (1980). *The Myerson value, φ^c, is the unique graph value on $C(N)$ satisfying Component efficiency and Balanced contributions.*

More recently, the Banzhaf graph value has been characterized using a similar set of properties. The difference between the Shapley and the Banzhaf value described in Chapter 1, is transferred to this context. The Banzhaf graph value characterizations change the Component efficiency property of φ^c, by the Total power property of β^c or the Isolation and the 2-Efficiency properties.

Theorem 2.2.5. Alonso-Meijide and Fiestras-Janeiro (2006).

- *The Banzhaf graph value, β^c, is the unique value on $C(N)$ satisfying Component total power and Fairness.*
- *The Banzhaf graph value, β^c, is the unique value on $C(N)$ satisfying Component total power and Balanced contributions*

Theorem 2.2.6. Alonso-Meijide and Fiestras-Janeiro (2006).

- *The Banzhaf graph value, β^c, is the unique value on $C(N)$satisfying Isolation, 2-Efficiency, and Fairness.*
- *The Banzhaf graph value, β^c, is the unique value on $C(N)$ satisfying Isolation, 2-Efficiency, and Balanced contributions.*

Proof. Next we recall the proof of the characterization by means of Isolation, 2-Efficiency, and Fairness.

(1) Existence. It is clear that β^c satisfies Isolation since $v^B(S \cup i) = v^B(S) + v(i)$ for any $\{i\} \in N/B$ and all $S \subseteq N \setminus i$.

Let $i, j \in N$ such that $(i:j) \in B$, then by definition

$$\beta_i^c(N,v,B) - \beta_j^c(N,v,B) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i,j\}} \left[2v^B(S \cup i) - 2v^B(S \cup j)\right]. \quad (2.1)$$

In a similar manner,

$$\beta_i^c(N,v,B^{-ij}) - \beta_j^c(N,v,B^{-ij}) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i,j\}} \left[2v^{B^{-ij}}(S \cup i) - 2v^{B^{-ij}}(S \cup j)\right]. \quad (2.2)$$

Since for all $S \subseteq N$ such that $\{i,j\} \nsubseteq S$, $v^B(S) = v^{B^{-ij}}(S)$, we have the equality of 2.1 and 2.2. Hence, it is proved that the Banzhaf graph value satisfies Fairness.

Using that the Banzhaf value satisfies 2-Efficiency,

$$\beta_i^c(N,v,B) + \beta_j^c(N,v,B) = \beta_i(N,v^B) + \beta_j(N,v^B) = \beta_p(N^{ij},(v^B)^{ij}).$$

In addition, for every $S \subseteq N^{ij}$,

$$(v^{ij})^{B^{ij}}(S) = \sum_{T \in S/B^{ij}} v^{ij}(T) = (v^B)^{ij}(S),$$

due to definition of $(v^B)^{ij}$ and the relationship between the component sets S/B and S/B^{ij}. Then,

$$\beta_p(N^{ij},(v^B)^{ij}) = \beta_p(N^{ij},(v^{ij})^{B^{ij}}) = \beta_p^c(N^{ij},v^{ij},B^{ij}).$$

We prove that β^c satisfies 2-Efficiency.

(2) Uniqueness. We prove it by induction on the number of links of B. Let f be a value on $C(N)$ that satisfies Isolation, 2-Efficiency, and Fairness. We claim that if $B = \emptyset$ by Isolation we have $f_i(N, v, B) = v(i) = \beta_i^c(N, v, B)$ for every $i \in N$. Then let $k \in \mathbb{N}$ and assume that $f(N, v, B) = \beta^c(N, v, B)$ for every graph B with less links than k. Let $(N, v, B) \in C(N)$ such that $|B| = k$ and $i \in N$. If agent i is isolated, then $f_i(N, v, B) = v(i) = \beta_i^c(N, v, B)$ by the Isolation property. Then there is some $j \in N$ such that $(i : j) \in B$, using that f and β^c satisfy Fairness and the induction hypothesis,

$$\begin{aligned} f_i(N, v, B) - f_j(N, v, B) &= f_i(N, v, B^{-ij}) - f_j(N, v, B^{-ij}) \\ &= \beta_i^c(N, v, B^{-ij}) - \beta_j^c(N, v, B^{-ij}) = \beta_i^c(N, v, B) - \beta_j^c(N, v, B) \end{aligned}$$

or equivalently,

$$f_i(N, v, B) - \beta_i^c(N, v, B) = f_j(N, v, B) - \beta_j^c(N, v, B) \tag{2.3}$$

By 2-Efficiency and the induction hypothesis,

$$\begin{aligned} f_i(N, v, B) + f_j(N, v, B) &= f_p(N^{ij}, v^{ij}, B^{ij}) \\ &= \beta_p^c(N^{ij}, v^{ij}, B^{ij}) = \beta_i^c(N, v, B) + \beta_j^c(N, v, B) \end{aligned}$$

where (N^{ij}, v^{ij}, B^{ij}) is the communication situation resulting as a consequence of the amalgamation of i and j. Then,

$$f_i(N, v, B) - \beta_i^c(N, v, B) = f_j(N, v, B) - \beta_j^c(N, v, B) \tag{2.4}$$

And combining 2.3 and 2.4, we get that,

$$f_i(N, v, B) = \beta_i^c(N, v, B)$$

which concludes the proof. □

To end with this second Chapter, we summarize in Table 2.1 the properties that each value on $C(N)$ satisfies. A ✓ indicates that the value satisfies the corresponding property.

	φ^c	β^c
Component decomposability	✓	✓
Component efficiency	✓	
Fairness	✓	✓
Balanced contributions	✓	✓
Isolation	✓	✓
Component total power		✓
2-Efficiency		✓

Table 2.1: Properties and values on $C(N)$

The Table shows that the differences between the Shapley and the Banzhaf values described in Chapter 1 are transferred to the differences between the Myerson and the Banzhaf graph values.

3

Games with incompatible players

In this Chapter we will study situations in which there are incompatible players.

The first work concerning cooperative games where there are players who cannot be together in a coalition can be found in Carreras (1991). In this paper a joint model which takes into account the affinities, described by a communication graph, and incompatibilities among the players was proposed for simple games. In Carreras and Owen (1996) a political application is provided taking into account the existence of incompatible players.

In Bergantiños (1993) and Bergantiños et al. (1993) both models are extended to transferable utility games. The existence of incompatible players is much more restrictive than the restrictions to the cooperation arriving from the affinities among the players, since players which are not connected by the affinities graph can still cooperate if there is a path connecting them while if two players are incompatible they could never be in the same coalition.

The joint model of TU games with affinities and incompatibilities was considered in Amer and Carreras (1995a). In this work the authors defined the cooperation index, which is a map $p : 2^N \to [0,1]$ that describes quantitative restrictions to the cooperation. The cooperation index is capable of modeling situations in which the affinities among players have different intensities, the only requirement is that $p(\{i\}) = 1$ for all $i \in N$ (non schizophrenic players). If $p(S) = 0$, it means that there are incompatible players on S, while $p(S) > 0$ means that players in S can communicate, and hence, cooperate. In Amer and Carreras (1995a) the Shapley value is extended and characterized for games with cooperation indices. The model studied in Chapter 2 is included in this new model if we consider the cooperation index p_B, given by $p_B(S) = 1$ if $S \subseteq N$ is connected by B and $p_B(S) = 0$ otherwise. In a similar way, the model that will be described in this Chapter is included in the approach of Amer and Carreras (1995a) as we will soon see.

The outline of the rest of the Chapter is as follows. In Section 3.1, we will present the TU games with incompatibilities. We recall the main results in this setting in Section 3.2. Finally, in Section 3.3 a new value for this class of games is proposed and characterized.

3.1 The model

First of all, the TU game with incompatibilities model is introduced formally in the next definition.

Definition 3.1.1. A *TU game with incompatibilities* is a triple (N, v, I) where $(N, v) \in G(N)$ is a TU game and $I \in g(N)$ is the incompatibility graph, i.e., $i, j \in N$ are incompatible if $(i : j) \in I$. We denote by $I(N)$ the set of all such games.

Given $(N, v, I) \in I(N)$ we will say that a coalition $S \subseteq N$ is *I-admissible* if there are not incompatible players contained on it. By $P(S, I)$ we will denote the set of all partitions of S whose elements are I-admissible coalitions.

Definition 3.1.2. Given a TU game with incompatibilities $(N, v, I) \in I(N)$, we denote by $(N, v^I) \in G(N)$ the *I-restricted game* whose characteristic function is given by,

$$v^I(S) = \max_{P \in P(S,I)} \sum_{T \in P} v(T) \text{ , for all } S \subseteq N.$$

The idea behind the I-restricted game is that players of a coalition S form I-admissible subcoalitions (which are the only feasible coalitions) and they choose them in such a way to maximize the sum of the worths of the subcoalitions of S.

As we mentioned before, this model is included in the games with cooperation index proposed by Amer and Carreras (1995a). We only need to define a cooperation index p_I given by $p_I(S) = 1$ if S is I-admissible, and $p_I(S) = 0$ otherwise.

In Bergantiños (1993) it is shown that the game with incompatibilities $(N, v, I) \in I(N)$ is not in general equal to the game with graph restricted communication $(N, v, I^c) \in C(N)$. This fact will be proved in Example 1. In Bergantiños (1993) it is also shown that the I-restricted game is always superadditive.

We end this Section, illustrating in Example 1, the way in which the I-restricted game is built and the difference between the I-restricted game and the communication game of the dual graph.

Example 1. Let $(N, v, I) \in I(N)$ be the game with incompatibilities where $N = \{1, 2, 3\}$, $I = \{(1 : 2)\}$, and the characteristic function v defined by:

$$v(i) = 0 \forall i \in N, \quad v(\{1, 2\}) = v(\{1, 3\}) = 1, \quad v(\{2, 3\}) = 2, \text{ and } v(N) = 10.$$

Let us compute the I-restricted game v^I following its definition.

$$v^I(i) = 0 \quad \forall i \in N, \quad v^I(\{1, 2\}) = v(1) + v(2) = 0,$$

$$v^I(\{1, 3\}) = v(\{1, 3\}) = 1, \quad v^I(\{2, 3\}) = v(\{2, 3\}) = 2,$$

$$v^I(N) = \max_{P \in P(N,I)} \sum_{T \in P} v(T) = v(\{2, 3\}) + v(1) = 2.$$

Next, let us consider the dual graph of I given by $I^c = \{(1:3),(2:3)\}$, the communication game $(N, v^{I^c}) \in G(N)$ is given by,

$$v^{I^c}(i) = 0 \quad \forall i \in N, \quad v^{I^c}(\{1,2\}) = v(1) + v(2) = 0,$$

$$v^{I^c}(\{1,3\}) = v(\{1,3\}) = 1, \quad v^{I^c}(\{2,3\}) = v(\{2,3\}) = 2,$$

$$v^{I^c}(N) = \sum_{S \in N/I^c} v(S) = v(\{1,2,3\}) = 10.$$

As it is seen, in the I-restricted game the grand coalition is not feasible since it has incompatible players contained. Nevertheless, the grand coalition N can cooperate jointly in the communication game v^{I^c} since players 1 and 2 can communicate through player 3. The incompatibility Shapley value and the Myerson value show this difference.

$$\varphi^I(N,v,I) = (1/6, 4/6, 7/6), \quad \varphi^c(N,v,I^c) = (17/6, 20/6, 23/6).$$

3.2 The incompatibility Shapley value

After having introduced the model of TU games with incompatibilities, we come now to the matter of how to allocate the benefits of the cooperation.

By a *value on I(N)* we will mean a map f that assigns a vector $f(N,v,I) \in \mathbb{R}^n$ to every game with incompatibilities $(N,v,I) \in I(N)$. In the literature, there is only a generalization of the Shapley value for this kind of games. Its formal definition is given next.

Definition 3.2.1. Bergantiños (1993). The *incompatibility Shapley value* φ^I, is a value on $I(N)$ defined as follows,

$$\varphi^I(N,v,I) = \varphi(N,v^I).$$

We will present the characterization of the incompatibility Shapley value by Bergantiños (1993), which is based on the following properties. Let $(N,v,I) \in I(N)$.

- *I-Efficiency.* An incompatibility value f is said to be I-Efficient if for all $S \in N/I^c$,

$$\sum_{i \in S} f_i(N,v,I) = \max_{P \in P(S,I)} \sum_{T \in P} v(T) = v^I(S).$$

- *I-Fairness.* An incompatibility value f is said to be I-Fair if for all $i,j \in N$ such that $(i:j) \notin I$,

$$f_i(N,v,I) - f_i(N,v,I \cup (i:j)) = f_j(N,v,I) - f_j(N,v,I \cup (i:j)).$$

- *I-Balanced contributions.* An incompatibility value f satisfies I-Balanced contributions if for all $i, j \in N$,

$$f_i(N, v, I) - f_i(N, v, I^{*j}) = f_j(N, v, I) - f_j(N, v, I^{*i}),$$

  where I^{*i} denotes the graph obtained from I when player i becomes incompatible with the rest of the players, i.e., $I^{*i} = I \cup \{(i : j) | j \in N \setminus i\}$.

The I-Efficiency property is similar to the efficiency proposed in Myerson (1977) and presented in the previous Chapter. The I-Fairness property has the same meaning as the Fairness property in Myerson (1977). It states that if two players are not incompatible anymore, both gain or loss the same amount.

In the next Theorem we present the first characterization of the incompatibility Shapley value.

Theorem 3.2.2. Bergantiños (1993). *The incompatibility Shapley value, φ^I, is the unique value on $I(N)$ satisfying I-Efficiency and I-Fairness.*

Recently Alonso-Meijide and Casas-Méndez (2007) presented an alternative characterization of φ^I, based on the property of I-Balanced contributions, we recall it here.

Theorem 3.2.3. Alonso-Meijide and Casas-Méndez (2007). *The incompatibility Shapley value, φ^I, is the unique value on $I(N)$ satisfying I-Efficiency and I-Balanced contributions.*

3.3 A new value on $I(N)$

In this setting we saw the lack of a value on $I(N)$ which generalizes the Banzhaf value. Therefore we propose a new incompatibility value using the Banzhaf value of the I-restricted game, which we introduce in the next definition.

Definition 3.3.1. The *incompatibility Banzhaf value*, β^I is a value on $I(N)$ defined as follows,

$$\beta^I(N, v, I) = \beta(N, v^I).$$

First of all we come to the discussion on the properties that this new value on $I(N)$ satisfies. To do so, we first need to define three more properties, which are natural modifications of the properties presented in the previous Chapter for this scenario. Let f be a value on $I(N)$.

- *I-Isolation.* For all $i \in N$ such that $(i : j) \in I$ for all $j \in N \setminus i$,

$$f_i(N, v, I) = v(i).$$

- *I-Total power.* For all $S \in N/I^c$,

$$\sum_{i \in S} f_i(N, v, I) = \frac{1}{2^{s-1}} \sum_{i \in S} \sum_{L \subseteq S \setminus i} \left[v^I(L \cup i) - v^I(L) \right].$$

These properties follow the same ideas as the analogous properties defined in the previous Chapter.

Lemma 3.3.2. *The incompatibility Banzhaf value satisfies properties I-Isolation, I-Fairness, I-Balanced contributions, and I-Total power.*

Proof.

I-Isolation It follows from the fact that for every $i \in N$ and $S \subseteq N \setminus i$,

$$v^I(S \cup i) = v^I(S) + v(i).$$

I-Fairness. Let $i, j \in N$ such that $(i:j) \notin I$, then

$$\begin{aligned}
&2^{n-1}\left[\beta_i^I(N,v,I) - \beta_i^I(N,v,I \cup (i:j))\right] \\
=&\sum_{S \subseteq N \setminus \{i,j\}} \left[v^I(S \cup i \cup j) - v^I(S \cup j) + v^I(S \cup i) - v^I(S)\right] \\
-&\sum_{S \subseteq N \setminus \{i,j\}} \left[v^{I\cup(i:j)}(S \cup i \cup j) - v^{I\cup(i:j)}(S \cup j) + v^{I\cup(i:j)}(S \cup i) - v^{I\cup(i:j)}(S)\right].
\end{aligned}$$

Since for all $S \subseteq N \setminus \{i,j\}$,

$v^I(S) = v^{I\cup(i:j)}(S)$, $v^I(S \cup i) = v^{I\cup(i:j)}(S \cup i)$, and $v^I(S \cup j) = v^{I\cup(i:j)}(S \cup j)$.

$$\begin{aligned}
&\beta_i^I(N,v,I) - \beta_i^I(N,v,I \cup (i:j)) \\
&\quad = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i,j\}} \left[v^I(S \cup i \cup j) - v^{I\cup(i:j)}(S \cup i \cup j)\right] \\
&\quad = \beta_j^I(N,v,I) - \beta_j^I(N,v,I \cup (i:j)).
\end{aligned}$$

I-Balanced contributions. Let $i, j \in N$, then

$$\begin{aligned}
&2^{n-1}\left[\beta_i^I(N,v,I) - \beta_i^I(N,v,I^{*j})\right] \\
=&\sum_{S \subseteq N \setminus \{i,j\}} \left[v^I(S \cup i \cup j) - v^I(S \cup j) + v^I(S \cup i) - v^I(S)\right] \\
-&\sum_{S \subseteq N \setminus \{i,j\}} \left[v^{I^{*j}}(S \cup i \cup j) - v^{I^{*j}}(S \cup j) + v^{I^{*j}}(S \cup i) - v^{I^{*j}}(S)\right].
\end{aligned}$$

Since for all $S \subseteq N \setminus j$,

$$v^I(S) = v^{I^{*j}}(S) \text{ and } v^{I^{*j}}(S \cup j) = v^I(S) + v(j).$$

$$\begin{aligned}
&\beta_i^I(N,v,I) - \beta_i^I(N,v,I^{*j}) \\
&\quad = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i,j\}} \left[v^I(S \cup i \cup j) - v^I(S \cup i) - v^I(S \cup j) + v^I(S)\right] \\
&\quad = \beta_j^I(N,v,I) - \beta_j^I(N,v,I^{*i}).
\end{aligned}$$

I-Total power Let $S \in N/I^c$ and take $(N, v^{I,S}) \in G(N)$ defined as follows,

$$v^{I,S}(T) = \max_{P \in P(T \cap S, I)} \sum_{L \in P} v(L) \text{ for all } T \in 2^N,$$

First of all we will see that $v^I = \sum_{S \in N/I^c} v^{I,S}$.

Let $P \in P(T, I)$ and $L \in P$. As L is I-admissible it follows that L is a connected component of I^c on N. Then, there exists $S' \in N/I^c$ such that $L \subseteq S'$. Hence, for any $S \in N/I^c$, each partition $P \in P(T, I)$ induces another partition $P \in P(T \cap S, I)$. Taking into account the definition of the I-restricted game we conclude that,

$$v^I(T) \leq \sum_{S \in N/I^c} v^{I,S}(T) \text{ for all } T \subseteq N.$$

On the other hand, let $T \subseteq N$, if $N/I^c = \{S_1, \ldots, S_m\}$ we may take $P_j \in P(T \cap S_j, I)$ for all $j = 1, \ldots, m$, and a partition P of T defined by those P_js. As $P \in P(T, I)$, we conclude that,

$$v^I(T) \geq \sum_{S \in N/I^c} v^{I,S}(T) \text{ for all } T \subseteq N.$$

Then the stated equality is proved. Given $S \in N/I^c$, and using the additivity of β, we have,

$$\sum_{j \in S} \beta_j(N, v^I) = \sum_{j \in S} \sum_{T \in N/I^c} \beta_j(N, v^{I,T}) = \sum_{T \in N/I^c} \sum_{j \in S} \beta_j(N, v^{I,T}).$$

For all $i \in N \setminus T$, i is a null player in $(N, v^{I,T}) \in G(N)$, hence,

$$\sum_{T \in N/I^c} \sum_{j \in S} \beta_j(N, v^{I,T}) = \sum_{j \in S} \beta_j(N, v^{I,S}) = \sum_{j \in S} \beta_j(S, v^{I,S}).$$

Lastly, using that the Banzhaf value satisfies the Total Power property we conclude that,

$$\sum_{j \in S} \beta_j(S, v^{I,S}) = \frac{1}{2^{s-1}} \sum_{j \in S} \sum_{L \subseteq S \setminus j} \left[v^I(L \cup j) - v^I(L) \right].$$

□

At this point we have the concepts and results we need to characterize the new proposed incompatibility value.

Theorem 3.3.3. *The incompatibility Banzhaf value, β^I, is the unique value on $I(N)$ satisfying I-Balanced contributions and I-Total power.*

Proof.

- *Existence*

  Shown in Lemma 3.3.2.

- *Uniqueness*

  Let f be an allocation rule satisfying the properties and $(N, v, \emptyset^c) \in I(N)$, then, $N/I^c = \{\{1\}, \{2\}, \ldots, \{n\}\}$. By I-Total power we have,

$$f(N, v, I) = (v(1), \ldots, v(n)),$$

and hence f is unique. Suppose that there are two different values f^1 and f^2 satisfying the properties. Then there exists $(N, v, I) \in I(N)$ such that $f^1(N, v, I) \neq f^2(N, v, I)$ and $I \neq \emptyset^c$, hence, we can take $I \in GR(N)$ with the maximum number of links for which the inequality holds. Let $i \in N$ such that $f_i^1(N, v, I) \neq f_i^2(N, v, I)$.

If for all $j \in N \setminus i$, $(i : j) \in I$. Then $\{i\} \in N/I^c$, applying the I-Total power property we come to contradiction.

If there is $j \in N \setminus i$ such that $(i : j) \notin I$. Then by I-Balanced contributions and the maximality of I,

$$\begin{aligned} f_i^1(N, v, I) - f_j^1(N, v, I) &= f_i^1(N, v, I^{*j}) - f_j^1(N, v, I^{*i}) = \\ &= f_i^2(N, v, I^{*j}) - f_j^2(N, v, I^{*i}) = f_i^2(N, v, I) - f_j^2(N, v, I) \end{aligned} \quad (3.1)$$

Moreover, let $S \in N/I^c$ such that $i \in S$, then $(i : j) \notin I$ for all $j \in S$ and there are $\{i_1, i_2, \ldots, i_k\} \subseteq S$ such that $i = i_1$, $j = i_k$, and $(i_l : i_{l+1}) \notin I$ for all $l = 1, \ldots, k-1$. Hence by (3.1) we have,

$$\begin{aligned} f_{i_1}^1(N, v, I) - f_{i_2}^1(N, v, I) &= f_{i_1}^2(N, v, I) - f_{i_2}^2(N, v, I) \\ &\vdots \\ f_{i_{k-1}}^1(N, v, I) - f_{i_k}^1(N, v, I) &= f_{i_{k-1}}^2(N, v, I) - f_{i_k}^2(N, v, I). \end{aligned}$$

Adding up both sides,

$$f_i^1(N, v, I) - f_j^1(N, v, I) = f_i^2(N, v, I) - f_j^2(N, v, I). \quad (3.2)$$

On the other hand, using the I-Total power property we have,

$$\sum_{i \in S} f_i^1(N, v, I) = \frac{1}{2^{s-1}} \sum_{i \in S} \sum_{T \subseteq S \setminus i} \left[v^I(T \cup i) - v^I(T)\right] = \sum_{i \in S} f_i^2(N, v, I). \quad (3.3)$$

By (3.2) and (3.3) it follows,,

$$s f_i^1(N, v, I) = s f_i^2(N, v, I).$$

Hence, we come to contradiction.

□

In a similar way, we can obtain a characterization of the value by means of the I-Fairness property instead of the I-Balanced contributions property. This result is presented in the next theorem without proof.

Theorem 3.3.4. *The incompatibility Banzhaf value, β^I, is the unique value on $I(N)$ satisfying I-Fairness and I-Total power.*

Proof.

- *Existence*

  Shown in Lemma 3.3.2.

- *Uniqueness* We can repeat the same argument as the proof of the Uniqueness in Theorem 3.3.3. Notice that we can use the I-Fairness property instead the I-Balanced contributions property.

□

The Chapter concludes summarizing in Table 3.1 the properties that each value on $I(N)$ satisfies. A ✓ indicates that the value on $I(N)$ satisfies the corresponding property.

	φ^I	β^I
I-Efficiency	✓	
I-Fairness	✓	✓
I-Balanced contributions	✓	✓
I-Isolation	✓	✓
I-Total power		✓

Table 3.1: Properties and values on $I(N)$

The difference between the incompatibility Shapley and the incompatibility Banzhaf value lies on the fact that the former satisfies I-Efficiency while the latter satisfies I-Total power.

4

Games with a priori unions

As mentioned in Chapter 2, in the basic TU game model there is no restriction to the cooperation, which means that any group of agents can reach agreements. In many real situations however, there is a priori information about the behavior of the players and only partial cooperation occurs.

Aumann and Drèze (1974) considered that restrictions in cooperation are given by a partition of the set of agents. This partition is capable of modeling the affinities among agents. The model including a TU game and such a partition is called a TU game with a priori unions. For this family of games, Owen (1977) proposed and characterized a modification of the Shapley value (Shapley (1953)) to allocate the total gains, the Owen value. This value initially splits the total amount among the unions, according to the Shapley value in the induced game played by the unions (quotient game). Then, once again using the Shapley value within each union, its total reward is allocated among its members (quotient game property), taking into account their possibilities of joining other unions. Owen (1982) defined a modification of the Banzhaf value following a similar procedure, known as the Banzhaf-Owen value. The first characterization of the Banzhaf-Owen value was proposed by Amer et al. (2002). Amer et al. also noted that the Banzhaf-Owen value does not satisfy two interesting properties: symmetry among unions and the quotient game property. Alonso-Meijide and Fiestras-Janeiro (2002) defined and characterized the symmetric coalitional Banzhaf value, a different modification of the Banzhaf value, that satisfies the two properties considered above. The symmetric coalitional Banzhaf value uses the Banzhaf value to allocate the payoff among the unions and the Shapley value to split this payoff within the members of each union. In Alonso-Meijide et al. (2007), a comparison among the three values on $U(N)$ considered is presented.

In Section 4.1 we recall the model of games with a priori unions. In Section 4.2 the main results concerning the Owen, the Banzhaf-Owen, and the symmetric coalitional Banzhaf values are presented.

4.1 The model

Let us consider a finite set of agents, say, $N = \{1, \dots, n\}$. We will denote the set of all partitions of N by $P(N)$. Each $P \in P(N)$, of the form $P = \{P_1, \dots, P_m\}$, is called a *system of a priori unions* on N. The so called *trivial coalition structures* are $P^n = \{\{1\}, \{2\}, \dots, \{n\}\}$, where each union is a singleton, and $P^N = \{N\}$, where the grand coalition forms. For $i \in P_k \in P$, P_{-i} will denote the partition obtained from P when player i leaves the union P_k and becomes a singleton, i.e.

$$P_{-i} = \{P_h \in P | h \neq k\} \cup \{P_k \setminus i, \{i\}\}.$$

Definition 4.1.1. A *TU game with a priori unions* is a triple (N, v, P) where $(N, v) \in G(N)$ and $P \in P(N)$. We denote by $U(N)$ the set of all such games. If $(N, v, P) \in U(N)$, with $P = \{P_k | k \in M = \{1, \dots, m\}\}$.

Given a TU game with a priori unions $(N, v, P) \in U(N)$, the associated *quotient game* $(M, v^P) \in G(M)$ is the TU game played by the unions and defined by

$$v^P(R) = v(P_R), \text{ for all } R \subseteq M,$$

where $P_R = \cup_{k \in R} P_k$. Note that if $P = P^n$, $v^P = v$.

4.2 Values on $U(N)$

In this Section we will recall the different allocation rules existing in the literature for this class of games. The values on $U(N)$ follow a two steps procedure. In the first step the worth of the grand coalition is shared among the unions and in the second step the amount allotted to each union is shared among the members of the union.

By a *value on U(N)* we will mean a map f that assigns a vector $f(N, v, P) \in \mathbb{R}^n$ to every game with a priori unions $(N, v, P) \in U(N)$. In this context we consider three possible extensions of the Shapley and Banzhaf values.

Definition 4.2.1. Owen (1977). The *Owen value*, ϕ, is the value on $U(N)$ defined for every $i \in N$ by

$$\phi_i(N, v, P) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{t!(p_k - t - 1)! r!(m - r - 1)!}{p_k! m!} \left[v(P_R \cup T \cup i) - v(P_R \cup T) \right],$$

where $i \in P_k \in P$.

Definition 4.2.2. Owen (1982). The *Banzhaf-Owen value*, ψ, is the value on $U(N)$ defined for every $i \in N$ by

$$\psi_i(N, v, P) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \left[v(P_R \cup T \cup i) - v(P_R \cup T) \right].$$

Definition 4.2.3. Alonso-Meijide and Fiestras-Janeiro (2002). The *symmetric coalitional Banzhaf value*, π, is the value on $U(N)$ defined for every $i \in N$ by

$$\pi_i(N, v, P) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{2^{m-1}} \frac{t!(p_k - t - 1)!}{p_k!} \left[v(P_R \cup T \cup i) - v(P_R \cup T) \right].$$

There exist a vast literature concerning values on $U(N)$ and their characterization, mostly in the case of the Owen value. However, for the purpose of this work we do not need to present all of them in detail.

The first characterization of the Owen value was proposed in Owen (1977), in the paper where the allocation rule was introduced. This first characterization was based on five properties, the Carrier property, two Anonymity properties, one for the unions and another for players, Additivity, and one last property which is the basis of the Quotient game property which will be present soon. The Carrier property together with the Null player property is equivalent to Efficiency. In Hart and Kurz (1983) three different characterizations of ϕ are proposed. The three of them are based on efficiency, symmetry, and additivity but they differ in the fourth axiom. In Winter (1992) the Owen value is characterized making use of a consistency property which states that the payoff of any player $i \in P_k \in N$ can be derived from a reduced game whose player set is a subset of P_k. Amer and Carreras (1995b) obtained a characterization of ϕ which is based on only three properties although two of them are quite demanding. Another characterization of the Owen value can be found in Vázquez-Brage et al. (1997). We will present it in this Section. More recently Hamiache (1999), Albizuri and Zarzuelo (2004), and Albizuri (2008) provide new characterizations of ϕ.

The first characterization of the Banzhaf-Owen value was proposed in Albizuri (2001), but only on the restricted domain of simple games. Amer et al. (2002) were the first to establish a characterization of ψ on the full domain of TU games. The authors use six properties, three well known properties in the literature (additivity, dummy player property, and symmetry), and three other properties which have never been used in other axiomatic systems, although they appear to be very interesting and easy to interpret. Two of these new properties are based on the delegation game, which is a game obtained from the original one, considering that a player delegates his role to another player, and a last property called many null players whose definition is quite cumbersome. As they said in Remark 3.3(b) their characterization is far from giving rise to an almost common axiomatization of both ϕ and ψ similar to Feltkamp's one for φ and β. With that target Alonso-Meijide et al. (2007) proposed a new characterization of the Banzhaf-Owen value, together with a survey on values on $U(N)$ which helps to understand the differences among the three allocation rules presented for games with a priori unions. The characterization will be presented in this Section.

The Symmetric coalitional Banzhaf value was characterized in Alonso-Meijide and Fiestras-Janeiro (2002) with two axiomatic systems. We will present one of them in detail next.

Some of the properties we need are just defined for games with a priori unions where the system of a priori unions is the trivial singleton coalition structure and other are defined for all the class of games with a priori unions. The first group of properties will be denoted by **b** while the second set of properties will be denoted by **a**. Let f be a value on $U(N)$.

b1 (*Efficiency*). For all $(N,v) \in G(N)$,

$$\sum_{i \in N} f_i(N,v,P^n) = v(N).$$

b2 (*2-Efficiency*). For all $(N,v) \in G(N)$ and any pair of distinct players $i,j \in N$,

$$f_i(N,v,P^n) + f_j(N,v,P^n) = f_p(N^{ij}, v^{ij}, P^{n-1}).$$

b3 (*Dummy player property*). For all $(N,v) \in G(N)$ and any $i \in N$ dummy player in (N,v),

$$f_i(N,v,P^n) = v(i).$$

b4 (*Symmetry*). For all $(N,v) \in G(N)$ and any pair of symmetric players $i,j \in N$ in (N,v),

$$f_i(N,v,P^n) = f_j(N,v,P^n).$$

b5 (*Equal marginal contributions*). For all $(N,v),(N,w) \in G(N)$ and all $i \in N$ such that $v(S \cup i) - v(S) = w(S \cup i) - w(S)$ for all $S \subseteq N \setminus i$, then

$$f_i(N,v,P^n) = f_i(N,w,P^n).$$

a1 (*Quotient game property*). For all $(N,v,P) \in U(N)$ and all $P_k \in P$,

$$\sum_{i \in P_k} f_i(N,v,P) = f_k(M,v^P,P^m).$$

a2 (*1-Quotient game property*). For all $(N,v,P) \in U(N)$ and every $i \in N$ such that there exists $k \in M$ such that $P_k = \{i\}$, then

$$f_i(N,v,P) = f_k(M,v^P,P^m).$$

a3 (*Balanced contributions within the unions*). For all $(N,v,P) \in U(N)$ and all $i,j \in P_k \in P$,

$$f_i(N,v,P) - f_i(N,v,P_{-j}) = f_j(N,v,P) - f_j(N,v,P_{-i}).$$

a4 (*Neutrality under individual desertion*). For all $(N,v,P) \in U(N)$ and all $i,j \in P_k \in P$,

$$f_i(N,v,P) = f_i(N,v,P_{-j}).$$

Finally we present the characterizations of the three values mentioned before.

Theorem 4.2.4. Vázquez-Brage et al. (1997). *The Owen value, ϕ, is the unique value on $U(N)$ satisfying* **b1**, **b3**, **b4**, **b5**, **a1**, *and* **a3**.

Theorem 4.2.5. Alonso-Meijide et al. (2007). *The Banzhaf-Owen value, ψ, is the unique value on $U(N)$ satisfying* **b2**, **b3**, **b4**, **b5**, **a2**, *and* **a4**.

Theorem 4.2.6. Alonso-Meijide and Fiestras-Janeiro (2002). *The Symmetric coalitional Banzhaf value, π, is the unique value on $U(N)$ satisfying* **b2**, **b3**, **b4**, **b5**, **a1**, *and* **a3**.

These three Theorems contribute to the understanding of the differences among the presented values on $U(N)$. As is seen, the only basic difference between ϕ and π lies on the fact that the former is the Shapley value for all $(N, v, P^n) \in U(N)$ whereas the later is the Banzhaf value for all games with a priori unions when we consider the trivial singleton coalition structure. Instead, the differences between ϕ and ψ arise in axioms **b1**-**b2**, **a1**-**a2**, and **a3**-**a4**. Finally, the differences between ψ and π are limited to **a1**-**a2** and **a3**-**a4**.

We conclude the Chapter presenting in Table 4.1, the properties that each value satisfies in short. A ✓ indicates that the property is satisfied by the corresponding value on $U(N)$.

	ϕ	ψ	π
b1 Efficiency	✓		
b2 2-Efficiency		✓	✓
b3 Dummy player property	✓	✓	✓
b4 Symmetry	✓	✓	✓
b5 Equal marginal contributions	✓	✓	✓
a1 Quotient game property	✓		✓
a2 1-Quotient game property	✓	✓	✓
a3 Balanced contributions within the unions	✓	✓	✓
a4 Neutrality under individual desertion		✓	

Table 4.1: Properties and values on $U(N)$

5

Games with graph restricted communication and a priori unions

The two extensions of TU games presented in Chapter 2 and Chapter 4 were first considered together by Vázquez-Brage et al. (1996). In this paper a value for games with graph restricted communication and a priori unions is introduced and characterized.

In this way, we can consider a more detailed approach of some situations, since we can enrich the model with more external information concerning the behavior of the players. The usefulness of this model will be clear when we come to the analysis of the Parliament of the Basque Country in Chapter 6.

Before we start formalizing the situation it is worth to mention the work by Amer and Carreras (1995b). The authors introduce a new model which considers the coalition structure together with a cooperation index as externalities of the TU game. A cooperation index is a map $p : 2^N \to [0,1]$ which can be used to describe the affinities and incompatibilities among players but also to describe the intensity of an affinity. In this work the Owen value is extended to games with cooperation indices and a priori unions and a characterization of it provided.

The rest of the Chapter is organized as follows. In Section 5.1 the model of games with graph restricted communication and a priori unions is introduced. In Section 5.2 the main results concerning the Owen graph value are revised and two new values for this kind of games are proposed. Finally, in Section 5.3 parallel axiomatizations of the considered values are proposed.

5.1 The model

In the next definition we introduce the TU games with graph restricted communications and a priori unions model formally.

Definition 5.1.1. A *graph restricted game with a priori unions* is a quadruple (N, v, B, P), where $(N, v) \in G(N)$, $B \in g(N)$, and $P \in P(N)$. We denote by CU the set of all such quadruples.

This class of games was studied with more detail in Vázquez-Brage (1998). In this PhD Thesis, the Owen value is extended to the class CU and two characterizations of the solution provided.

Associated to every TU game with graph restricted communication and a priori unions, we introduce a game which combines the ideas behind the quotient game, v^P, and the communication game, v^B.

Definition 5.1.2. Given $(N, v, B, P) \in CU$, the *communication quotient game* $(M, v^{BP}) \in G(M)$ is defined by

$$v^{BP}(R) = \sum_{L \in P_R/B} v(L), \text{ for all } R \subseteq M.$$

5.2 Values on CU

By a *value on CU* we will mean a map f that assigns a vector $f(N, v, B, P) \in \mathbb{R}^n$ to every graph restricted game with a priori unions $(N, v, B, P) \in CU$. As we will see in the following definitions, the values on $U(N)$ studied in Chapter 4 can be extended in a natural way to this new class of games using the communication game (N, v^B) associated to every game with graph restricted communication.

Definition 5.2.1. Vázquez-Brage et al. (1996). The *Owen graph value*, ϕ^c, is the value on CU defined by

$$\phi^c(N, v, B, P) = \phi(N, v^B, P).$$

Next we recall a characterization of this value based on the following properties. Let f be a value on CU.

A1 *Component efficiency.* For all $(N, v, B, P) \in CU$ and all $T \in N/B$,

$$\sum_{i \in T} f_i(N, v, B, P) = v(T).$$

A2 *Fairness in the quotient.* For all $(N, v, B, P) \in CU$ and all $P_k, P_s \in P$,

$$\begin{aligned}\sum_{i \in P_k} f_i(N, v, B, P) - \sum_{i \in P_k} f_i(N, v, B^{-(P_k,P_s)}, P) \\ = \sum_{i \in P_s} f_i(N, v, B, P) - \sum_{i \in P_s} f_i(N, v, B^{-(P_k,P_s)}, P),\end{aligned}$$

where $B^{-(P_k,P_s)} \in g(N)$ is the graph obtained from B deleting all links between members of P_k and P_s.

A3 *Balanced contributions within the unions.* For all $(N, v, B, P) \in CU$, all $P_k \in P$, and all $i, j \in P_k$,

$$f_i(N, v, B, P) - f_i(N, v, B, P_{-j}) = f_j(N, v, B, P) - f_j(N, v, B, P_{-i}).$$

Theorem 5.2.2. Vázquez-Brage et al. (1996). *There is a unique value on CU which satisfies* **A1**, **A2**, *and* **A3**. *It is the Owen graph value,* ϕ^c.

Proof. (1) Existence. Given $(N, v, B, P) \in CU$, consider, for every $S \in N/B$, the game u^S given by

$$u^S(T) = \sum_{L \in (T \cap S)/B} v(L), \text{ for all } T \subseteq N.$$

Clearly, S is a carrier for u^S, so as ϕ satisfies the carrier property, for every $S, T \in N/B$,

$$\sum_{i \in S} \phi_i(u^T, P) = \begin{cases} u^S(N) & \text{if } S = T \\ 0 & \text{if } S \neq T \end{cases}$$

Also, $v^B = \sum_{S \in N/B} u^S$, so the additivity of the Owen value implies that,

$$\sum_{i \in S} \phi_i(N, v^B, P) = \sum_{i \in S} \sum_{S \in N/B} \phi_i(N, u^S, P) = \sum_{S \in N/B} \sum_{i \in S} \phi_i(N, u^S, P)$$
$$= u^S(N) = \sum_{L \in S/B} v(L) = v(S).$$

Hence, ϕ^c satisfies **A1**.

To show that ϕ^c satisfies **A2**, take $P_k, P_s \in P$ and consider the game $z = v^B - v^{B^{-(P_k,P_s)}}$. For all $R \subseteq M \setminus \{k, s\}$ $z^P(R \cup k) = z^P(R \cup s) = 0$. Thus $\sum_{i \in P_k} \phi_i(N, z, P) = \sum_{i \in P_s} \phi_i(N, z, P)$, then as ϕ satisfies the additivity and the anonymity in the unions,

$$\sum_{i \in P_k} \phi_i(N, v^B, P) - \sum_{i \in P_k} \phi_i(N, v^{B^{-(P_k,P_s)}}, P) = \sum_{i \in P_s} \phi_i(N, v^B, P) - \sum_{i \in P_s} \phi_i(N, v^{B^{-(P_k,P_s)}}, P).$$

Finally, to show that the Owen graph value satisfies **A3**, take $(N, v, B, P) \in CU$ (with $P = \{P_1, \ldots, P_m\}$), $P_k \in P$, and $i, j \in P_k$. By Definition 4.2.1,

$$\phi_i(N, v, P) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus \{i,j\}} \frac{t!(p_k - t - 1)! r!(m - r - 1)!}{p_k! m!} \left[v(P_R \cup T \cup i) - v(P_R \cup T)\right]$$
$$+ \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus \{i,j\}} \frac{(t+1)!(p_k - t - 2)! r!(m - r - 1)!}{p_k! m!} \left[v(P_R \cup T \cup j \cup i) - v(P_R \cup T \cup j)\right].$$

P_{-j} can be expressed as

$$P_{-j} = \{P'_1, \ldots, P'_{m+1}\},$$

where $P'_l = P_l$ for all $l \in \{1, \ldots, k-1, k+1, \ldots, m\}$, $P'_k = P_k \setminus \{j\}$, and

$P'_{m+1} = \{j\}$. Writing M' for $\{1, \ldots, m+1\}$,

$$\phi_i(N, v, P_{-j}) = \sum_{R \subseteq M' \setminus k} \sum_{T \subseteq P'_k \setminus i} \frac{t!(p_k - t - 2)!r!(m-r)!}{(p_k - 1)!(m+1)!} \left[v(P_R \cup T \cup i) - v(P_R \cup T) \right]$$

$$= \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus \{i,j\}} \frac{t!(p_k - t - 2)!r!(m-r)!}{(p_k - 1)!(m+1)!} \left[v(P_R \cup T \cup i) - v(P_R \cup T) \right]$$

$$+ \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus \{i,j\}} \frac{t!(p_k - t - 2)!(r+1)!(m-r-1)!}{(p_k - 1)!(m+1)!} \left[v(P_R \cup T \cup j \cup i) - v(P_R \cup T \cup j) \right].$$

Hence,

$$\phi_i(N, v, P) - \phi_i(N, v, P_{-j}) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus \{i,j\}} A_1 \left[v(P_R \cup T \cup i) - v(P_R \cup T) \right]$$

$$+ \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus \{i,j\}} A_2 \left[v(P_R \cup T \cup j \cup i) - v(P_R \cup T \cup j) \right],$$

where

$$A_1 = \frac{t!(p_k - t - 1)!r!(m-r-1)!}{p_k!m!} - \frac{t!(p_k - t - 2)!r!(m-r)!}{(p_k - 1)!(m+1)!},$$

and

$$A_2 = \frac{(t+1)!(p_k - t - 2)!r!(m-r-1)!}{p_k!m!} - \frac{t!(p_k - t - 2)!(r+1)!(m-r-1)!}{(p_k - 1)!(m+1)!}.$$

Using some elementary algebra,

$$A_1 = -A_2 = \frac{t!(p_k - t - 2)!r!(m-r-1)!}{p_k!m!} \left(\frac{rp_k + P_k - mt - m - t - 1}{m+1} \right).$$

Thus,

$$\phi_i(N, v, P) - \phi_i(N, v, P_{-j}) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus \{i,j\}} A_1 \left[v(P_R \cup T \cup i) \right.$$
$$\left. - v(P_R \cup T) + v(P_R \cup T \cup j) - v(P_R \cup T \cup j \cup i) \right].$$

Since the right-hand side of the last equality depends on i in the same way as it depends on j,

$$\phi_i(N, v, P) - \phi_i(N, v, P_{-j}) = \phi_j(N, v, P) - \phi_j(N, v, P_{-i}),$$

and hence **A3** is satisfied.

(2) Uniqueness. If f^1 and f^2 are two different values on CU satisfying **A1**, **A2**, and **A3**, then there exist $(N, v) \in G(N)$, $P = \{P_1, \ldots, P_m\} \in P(N)$, and $B \in g(N)$ such that $f^1(N, v, B, P) \neq f^1(N, v, B, P)$. We may suppose that B

is the graph with the fewest links for which this inequality holds for the given (N, v). In a similar way, we may take P with the maximum number of unions for which the inequality holds for the given (N, v, B).

Let B^P be the graph induced by B on M,

$$B^P = \{(k:s) | \exists (i:j) \in B \text{ with } i \in P_k, j \in P_s\}.$$

Clearly, for every $R \in M/B^P$, $P_R = \cup_{r \in R} P_r$ can be expressed as a union of elements of N/B. Hence, as f^1 and f^2 both satisfy **A1**,

$$\sum_{i \in P_R} (f_i^1(N, v, B, P) - f_i^2(N, v, B, P)) = 0. \tag{5.1}$$

Also, as all the elements in R are connected in R by B^P, the minimality of B, and the fact that f^1 and f^2 both satisfy **A2** together imply that, for all $r \in R$,

$$\sum_{i \in P_r} (f_i^1(N, v, B, P) - f_i^2(N, v, B, P)) = c_R, \tag{5.2}$$

where c_R is a constant depending only on R. Equations 5.1 and 5.2 imply that, for all $P_k \in P$,

$$\sum_{i \in P_r} f_i^1(N, v, B, P) = \sum_{i \in P_r} f_i^2(N, v, B, P). \tag{5.3}$$

Now, select $P_k \in P$. If $P_k = \{i\}$, then from 5.3, $f_i^1(N, v, B, P) = f_i^2(N, v, B, P)$. If P_k has more that one member, take $i, j \in P_k$. Since f^1 and f^2 satisfy **A3**,

$$f_i^h(N, v, B, P) - f_i^h(N, v, B, P_{-j}) = f_j^h(N, v, B, P) - f_j^h(N, v, B, P_{-i}),$$

for all $h \in \{1, 2\}$. Hence, the maximality of P implies that

$$f_i^1(N, v, B, P) - f_i^2(N, v, B, P) = c_k \text{ for all } i \in P_k, \tag{5.4}$$

where c_k is a constant depending only on P_k. Together, 5.3 and 5.4 imply that $f_i^1(N, v, B, P) = f_i^2(N, v, B, P)$ for all $i \in P_k$, and the uniqueness follows. □

Note that as one might expect, this value generalizes the Shapley, Owen, and Myerson values. Table 5.1 depicts these generalizations for particular instances of the system of a priori unions and the communication graph.

graph \ unions	$P = P^n$	$P = P^N$	$P \in P(N)$
$B = \emptyset^c$	φ	φ	ϕ
$B \in g(N)$	φ^c	φ^c	ϕ^c

Table 5.1: The Owen graph value

We end by introducing two new values in this context based on the Banzhaf value as follows.

Definition 5.2.3. Alonso-Meijide et al. (to appear). The *Banzhaf-Owen graph value*, ψ^c, is the value on CU defined by

$$\psi^c(N, v, B, P) = \psi(N, v^B, P).$$

Definition 5.2.4. Alonso-Meijide et al. (to appear). The *symmetric coalitional Banzhaf graph value*, π^c, is the value on CU defined by

$$\pi^c(N, v, B, P) = \pi(N, v^B, P).$$

As one might expect these two values on CU generalize the values introduced in Definitions 4.2.1, 4.2.2, and 4.2.3 in the way shown in Table 5.2 and Table 5.3.

graph \ unions	$P = P^n$	$P = P^N$	$P \in P(N)$
$B = \emptyset^c$	β	β	ψ
$B \in g(N)$	β^c	β^c	ψ^c

Table 5.2: The Banzhaf-Owen graph value

graph \ unions	$P = P^n$	$P = P^N$	$P \in P(N)$
$B = \emptyset^c$	β	φ	π
$B \in g(N)$	β^c	φ^c	π^c

Table 5.3: The symmetric coalitional Banzhaf graph value

5.3 An axiomatic approach

Let us consider the following properties for a value f, on CU. We define some of the properties for games with the trivial system of a priori unions P^n. These properties will be denoted by **B** while the others will be denoted by **A** (as we did in Section 5.2).

B1 *Graph isolation.* For all $(N, v, B) \in C(N)$ and all $i \in N$ such that i is an isolated agent, i.e, $\{i\} \in N/B$, we have,

$$f_i(N, v, B, P^n) = v(i).$$

The idea behind this property is that an isolated agent with respect to the communication situation will only receive the utility he can obtain on his own because he will not be able to communicate with another agent in the game $(N, v, B, P^n) \in CU$. The graph isolation is based on the idea of the dummy player property which is standard in the literature.

B2 *Pairwise merging.* For all $(N, v, B) \in C(N)$ and all $i, j \in N$ such that $(i : j) \in B$, the following equality is satisfied,

$$f_i(N, v, B, P^n) + f_j(N, v, B, P^n) = f_p(N^{ij}, v^{ij}, B^{ij}, P^{n-1}),$$

where $(N^{ij}, v^{ij}, B^{ij}, P^{n-1})$ is the game such that player i and j have merged into the new player p.

Property **B2** states that a value is immune against artificial merging or splitting of two directly connected players in $(N, v, B, P^n) \in CU$. The idea behind this property was first introduced by Lehrer (1988) to characterize the Banzhaf value (Theorem 1.3.6) in a slightly different form (as an inequality), although it was soon discovered that the equality holds (see e.g. Carreras and Magaña (1994), Nowak (1997) (Theorem 1.3.8)). Recently, in Alonso-Meijide and Fiestras-Janeiro (2006) a similar property was used in the context of games with graph restricted cooperation to characterize the Banzhaf graph value (Theorem 2.2.6).

B3 *Fairness in the graph.* For all $(N, v, B) \in C(N)$ and all $i, j \in N$ such that $(i : j) \in B$, we have,

$$f_i(N, v, B, P^n) - f_i(N, v, B^{-ij}, P^n) = f_j(N, v, B, P^n) - f_j(N, v, B^{-ij}, P^n).$$

The Fairness in the graph property says that, given the trivial singleton coalition structure, if a player's payoff increases or decreases when breaking the link with another player, this other player should gain or lose the same amount, given the trivial singleton coalition structure, P^n. Property **B3** is based on the Fairness property introduced in Myerson (1977) (Theorem 2.2.3) and is also studied in Alonso-Meijide and Fiestras-Janeiro (2006) (Theorem 2.2.6). The property used in these papers was defined for games with graph restricted communication.

A4 *Neutrality under individual desertion.* For all $(N, v, B, P) \in CU$ and all $i, j \in N$ such that $\{i, j\} \subseteq P_k \in P$, we have,

$$f_i(N, v, B, P) = f_i(N, v, B, P_{-j}).$$

The Neutrality under individual desertion property states that the desertion of an agent from an a priori union does not affect the payoff of the remaining members of the union. Property **A4** is just a stronger version of **A3** which was first presented in Vázquez-Brage et al. (1996). The Neutrality under individual desertion was introduced in Alonso-Meijide et al. (2007) in the context of games with a priori unions to characterize the Banzhaf-Owen value (Theorem 4.2.5).

A5 *1-Quotient game property.* For all $(N, v, B, P) \in CU$ and all $i \in N$ such that $\{i\} = P_k \in P$,

$$f_i(N, v, B, P) = f_k(M, v^{BP}, B^M, P^m).$$

Property **A5** states that, using the value on CU in the original game, any isolated agent with respect to the system of a priori unions gets the same payoff as the union he forms in the communication quotient game with the trivial singleton coalition structure and the complete graph.

A6 *Quotient game property.* For all $(N, v, B, P) \in CU$ and all $P_k \in P$,

$$\sum_{i \in P_k} f_i(N, v, B, P) = f_k(M, v^{BP}, B^M, P^m).$$

Property **A6** states that the total payoff obtained by members of a union in the original game, is the amount obtained by the union itself in the communication quotient game with the trivial system of a priori unions and the complete graph. In the case where $P_k = \{i\}$ properties **A5** and **A6** are equal. The idea behind the Quotient game property was introduced by Owen (1977). A similar property was used in Vázquez-Brage et al. (1997) (Theorem 4.2.4) and Alonso-Meijide and Fiestras-Janeiro (2002) (Theorem 4.2.6) in the context of games with a priori unions.

At this point we have introduced the properties we need to give characterizations for each value on CU proposed in Definition 5.2.1, Definition 5.2.3, and Definition 5.2.4.

Theorem 5.3.1. Alonso-Meijide et al. (to appear). *There is a unique value on CU which satisfies* **B1**, **B2**, **B3**, **A4**, *and* **A5**. *It is the Banzhaf-Owen graph value,* ψ^c.

Proof.

First we will see that ψ^c satisfies the properties and next we will prove that it is the only value on CU satisfying them.

(1) Existence. As depicted in Table 5.2, $\psi^c(N, v, B, P^n) = \beta^c(N, v, B)$. Then, using the characterization of β^c in Theorem 2.2.6 it follows that ψ^c satisfies **B1**, **B2**, and **B3**. Take $(N, v, B, P) \in CU$ and let $i, j \in P_k$. Then we can write

$$\psi_i^c(N, v, B, P) = 2^{2-m-p_k} \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus \{i,j\}} \left[v^B(P_R \cup T \cup i \cup j) - \right.$$
$$\left. - v^B(P_R \cup T \cup j) + v^B(P_R \cup T \cup i) - v^B(P_R \cup T) \right].$$

Let $P_{-j} = \{P'_1, \ldots, P'_{m+1}\}$ where $P'_l = P_l$ for all $l \in \{1, \ldots, k-1, k+1, \ldots, m\}$, $P'_k = P_k \setminus j$ and $P'_{m+1} = \{j\}$. Let $M' = \{1, \ldots, m+1\}$. Then,

$$\psi_i^c(N, v, B, P_{-j}) = \sum_{R \subseteq M' \setminus k} \frac{1}{2^m} \sum_{T \subseteq P'_k \setminus i} \frac{1}{2^{p_k - 2}} (v^B(P_R \cup T \cup i) - v^B(P_R \cup T))$$
$$= \sum_{R \subseteq M \setminus k} \frac{1}{2^m} \sum_{T \subseteq P_k \setminus \{i,j\}} \frac{1}{2^{p_k-2}} (v^B(P_R \cup T \cup i) - v^B(P_R \cup T))$$
$$+ \sum_{R \subseteq M \setminus k} \frac{1}{2^m} \sum_{T \subseteq P_k \setminus \{i,j\}} \frac{1}{2^{p_k-2}} (v^B(P_R \cup T \cup j \cup i) - v^B(P_R \cup T \cup j)) = 2^{2-m-p_k}$$
$$\times \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus \{i,j\}} \left[v^B(P_R \cup T \cup i) - v^B(P_R \cup T) + v^B(P_R \cup T \cup i \cup j) - v^B(P_R \cup T \cup j) \right].$$

Hence, ψ^c satisfies **A4**.

Finally, we show that **A5** is satisfied. Take $i \in N$ and $k \in M$ such that $P_k = \{i\}$ and consider the communication quotient game $(M, v^{BP}, B^M, P^m) \in CU$. Then,

$$\psi_k^c(M, v^{BP}, B^M, P^m) = \sum_{R \subseteq M \setminus k} \frac{1}{2^{m-1}} ((v^{BP})^{B^M}(R \cup k) - (v^{BP})^{B^M}(R)) =$$
$$= \sum_{R \subseteq M \setminus k} \frac{1}{2^{m-1}} (v^{BP}(R \cup k) - v^{BP}(R)) = \psi_i^c(N, v, B, P).$$

(2) Uniqueness.

- Following a similar argument as in Theorem 2.2.6, we conclude that the first three properties characterize ψ^c for all $(N, v, B, P^n) \in CU$.
- Suppose that there are two different values on CU, ψ^{1c} and ψ^{2c} that satisfy the properties. Then, there exists $(N, v, B, P) \in CU$ such that $\psi^{1c}(N, v, B, P) \neq \psi^{2c}(N, v, B, P)$ and $P \neq P^n$. We may suppose that for the triple (N, v, B), P is a system of a priori unions with the maximum number of unions for which $\psi^{1c}(N, v, B, P) \neq \psi^{2c}(N, v, B, P)$ holds. Take $i \in N$ such that $\psi_i^{1c}(N, v, B, P) \neq \psi_i^{2c}(N, v, B, P)$. Two possible cases arise.
  - $\{i\} = P_k \in P$. By **A5** and the previous item we have,
  $$\psi_i^{1c}(N, v, B, P) = \psi_k^{1c}(M, v^{BP}, B^M, P^m)$$
  $$= \psi_k^{2c}(M, v^{BP}, B^M, P^m) = \psi_i^{2c}(N, v, B, P).$$
  - There is $j \neq i$ such that $i, j \in P_k$. Then, by **A4** and the maximality of P we have,
  $$\psi_i^{1c}(N, v, B, P) = \psi_i^{1c}(N, v, B, P_{-j})$$
  $$= \psi_i^{2c}(N, v, B, P_{-j}) = \psi_i^{2c}(N, v, B, P).$$

In both cases the inequality in the beginning is contradicted, and hence, the result is proved.

□

With a similar scheme, a characterization of the symmetric coalitional Banzhaf graph value is obtained, we just need to replace properties **A4** and **A5** by **A3** and **A6**.

Theorem 5.3.2. Alonso-Meijide et al. (to appear). *There is a unique value on CU which satisfies* **B1**, **B2**, **B3**, **A3**, *and* **A6**. *It is the symmetric coalitional Banzhaf graph value, π^c.*

Proof. Note that as π^c is based on β, for all $(N, v, B, P^n) \in CU$,

$$\pi_i^c(N, v, B, P^n) = \beta_i^c(N, v, B).$$

Then, following a similar argument as in Theorem 2.2.6 we have that π^c is characterized by properties **B1**, **B2**, and **B3** when the system of a priori unions is P^n. We just need to prove it for any $(N, v, B, P) \in CU$ where $P \neq P^n$.

(1) Existence. Using Theorem 2.2.6 it is straightforward to check that π^c satisfies **B1**, **B2**, and **B3**.

Let $P_k \in P$, from Definition 5.2.4,

$$\sum_{i \in P_k} \pi_i(N, v, P) = \sum_{R \subseteq M \setminus k} \frac{1}{2^{m-1}} \sum_{i \in P_k} \sum_{T \subseteq P_k \setminus i} \frac{t!(p_k - t - 1)!}{p_k!} \left[v(P_R \cup T \cup i) - v(P_R \cup T)\right], \tag{5.5}$$

For every $R \subseteq M$, we consider the TU game $(P_k, w_{P_R}) \in G(P_k)$, with characteristic function

$$w_{P_R}(T) = v(P_R \cup T) - v(P_R), \text{ for all } T \subseteq P_k.$$

Then the Shapley value of a player $i \in P_k$ is equal to:

$$\begin{aligned} \varphi_i(P_k, w_{P_R}) &= \sum_{T \subseteq P_k \setminus i} \frac{t!(p_k - t - 1)!}{p_k!} \left[w_{P_R}(T \cup i) - w_{P_R}(T)\right] \\ &= \sum_{T \subseteq P_k \setminus i} \frac{t!(p_k - t - 1)!}{p_k!} \left[v(P_R \cup T \cup i) - v(P_R \cup T)\right]. \end{aligned}$$

By the efficiency of the Shapley value, we obtain,

$$\sum_{i \in P_k} \varphi_i(P_k, w_{P_R}) = w_{P_R}(P_k) = v(P_R \cup P_k) - v(P_R).$$

Inserting this result in equation 5.5, we have:

$$\begin{aligned} \sum_{i \in P_k} \pi_i(N, v, P) &= \sum_{R \subseteq M \setminus k} \frac{1}{2^{m-1}} \left[v(P_R \cup P_k) - v(P_R)\right] \\ &= \frac{1}{2^{m-1}} \sum_{R \subseteq M \setminus k} \left[v^P(R \cup k) - v^P(R)\right] = \pi_k^c(M, v^P, B^M, P^m). \end{aligned}$$

Hence π^c satisfies **A6** since $(v^B)^P = v^{BP}$.

Let $P_k \in P$ and $i, j \in P_k$. By Definition 4.2.3,

$$\pi_i(N,v,P) = \sum_{R\subseteq M\setminus k}\sum_{T\subseteq N\setminus\{i,j\}} \frac{1}{2^{m-1}}\frac{1}{p_k}\frac{1}{\binom{p_k-1}{t}}\left[v(P_R\cup T\cup i) - v(P_R\cup T)\right]$$
$$+ \sum_{R\subseteq M\setminus k}\sum_{T\subseteq N\setminus\{i,j\}} \frac{1}{2^{m-1}}\frac{1}{p_k}\frac{1}{\binom{p_k-1}{t+1}}\left[v(P_R\cup T\cup j\cup i) - v(P_R\cup T\cup j)\right].$$

Let $P_{-j} = \{P'_1, \ldots, P'_{m+1}\}$ where $P'_l = P_l$ for all $l \in \{1, \ldots, k-1, k+1, \ldots, m\}$, $P'_k = P_k \setminus j$ and $P'_{m+1} = \{j\}$. Let $M' = \{1, \ldots, m+1\}$. Then,

$$\pi_i(N,v,B,P_{-j}) = \sum_{R\subseteq M'\setminus k}\sum_{T\subseteq P'_k\setminus i} \frac{1}{2^{m}}\frac{1}{p_k-1}\frac{1}{\binom{p_k-2}{t}}\left[v(P_R\cup T\cup i) - v(P_R\cup T)\right]$$
$$= \sum_{R\subseteq M\setminus k}\sum_{T\subseteq P_k\setminus\{i,j\}} \frac{1}{2^{m}}\frac{1}{p_k-1}\frac{1}{\binom{p_k-2}{t}}\left[v(P_R\cup T\cup i) - v(P_R\cup T)\right]$$
$$+ \sum_{R\subseteq M\setminus k}\sum_{T\subseteq P_k\setminus\{i,j\}} \frac{1}{2^{m}}\frac{1}{p_k-1}\frac{1}{\binom{p_k-2}{t}}\left[v(P_R\cup T\cup j\cup i) - v(P_R\cup T\cup j)\right].$$

Hence,

$$\begin{aligned}\pi_i(N,v,B,P) &- \pi_i(N,v,B,P_{-j}) \\ &= \sum_{R\subseteq M\setminus k}\sum_{T\subseteq P_k\setminus\{i,j\}} A_1\left[v(P_R\cup T\cup i) - v(P_R\cup T)\right] \\ &\quad + \sum_{R\subseteq M\setminus k}\sum_{T\subseteq P_k\setminus\{i,j\}} A_2\left[v(P_R\cup T\cup j\cup i) - v(P_R\cup T\cup j)\right],\end{aligned}$$

where

$$A_1 = \frac{1}{2^{m-1}}\frac{1}{p_k}\frac{1}{\binom{p_k-1}{t}} - \frac{1}{2^{m}}\frac{1}{p_k-1}\frac{1}{\binom{p_k-2}{t}},$$

and

$$A_2 = \frac{1}{2^{m-1}}\frac{1}{p_k}\frac{1}{\binom{p_k-1}{t+1}} - \frac{1}{2^{m}}\frac{1}{p_k-1}\frac{1}{\binom{p_k-2}{t}}.$$

And operating a bit we have, $A_1 + A_2 = 0$.

Then,

$$\begin{aligned}\pi_i(N,v,B,P) &- \pi_i(N,v,B,P_{-j}) \\ &= \sum_{R\subseteq M\setminus k}\sum_{T\subseteq P_k\setminus\{i,j\}} A_1\left[v(P_R\cup T\cup i) - v(P_R\cup T)\right. \\ &\qquad \left. - v(P_R\cup T\cup j\cup i) + v(P_R\cup T\cup j)\right].\end{aligned}$$

Since the last expression depends on i in the same way as it depends on j, we proved that **A3** is satisfied.

(2) Uniqueness. Suppose that there are two different values on CU π^{1c} and π^{2c} that satisfy the properties. Then, there exists $(N, v, B, P) \in CU$ such that $\pi^{1c}(N, v, B, P) \neq \pi^{2c}(N, v, B, P)$. Then $P \neq P^n$. We may suppose that for the triple (N, v, B), P is a system of a priori unions with the maximum number of unions for which $\pi^{1c} \neq \pi^{2c}$ holds. Then, there is $i \in N$ such that $\pi_i^{1c}(N, v, B, P) \neq \pi_i^{2c}(N, v, B, P)$. If $\{i\} = P_k \in P$, by **A6** and the uniqueness for the trivial singleton structure, we have,

$$\pi_i^{1c}(N, v, B, P) = \pi_k^{1c}(M, v^{BP}, \emptyset^c, P^m)$$
$$= \pi_k^{2c}(M, v^{BP}, \emptyset^c, P^m) = \pi_i^{2c}(N, v, B, P).$$

Assume that there is $j \in P_k \setminus i$. Then, by **A3** and the minimality of P,

$$\pi_i^{1c}(N, v, B, P) - \pi_j^{1c}(N, v, B, P) = \pi_i^{2c}(N, v, B, P) - \pi_j^{2c}(N, v, B, P). \tag{5.6}$$

Using **A6** and (5.6), we have,

$$p_k \pi_i^{1c}(N, v, B, P) = p_k \pi_i^{2c}(N, v, B, P).$$

We obtain a contradiction in both cases.

□

Lastly, we give a characterization of ϕ^c which will be useful to discuss the differences between the values studied in the paper.

Theorem 5.3.3. Alonso-Meijide et al. (to appear). *There is a unique value on CU which satisfies* **A1** *(for P^n),* **B3**, **A3**, *and* **A6**. *It is the Owen graph value, ϕ^c.*

Proof. Using that $\phi^c(N, v, B, P^n) = \varphi^c(N, v, B)$ for all $(N, v, B, P^n) \in CU$, we claim that ϕ^c satisfies **A1** (for P^n) and **B3** (Theorem 2.2.3). From Vázquez-Brage et al. (1997) we know that ϕ^c satisfies both **A3** and **A6**.

We know that the Myerson value is characterized by efficiency and fairness (Theorem 2.2.3). Hence, **A1** (for P^n) and **B3** uniquely determine ϕ^c for all $(N, v, B, P^n) \in CU$. It remains to prove the uniqueness for any $(N, v, B, P) \in CU$ with $P \neq P^n$. The proof follows the argument used in the proof of the uniqueness of Theorem 5.3.2.

□

Remark 5.3.4 (Independence of properties). In order to see that properties used in the previous results are independent we will consider the following values on CU.

For Theorem 5.3.1:

- The null value on CU, f^1, defined as $f^1(N, v, B, P) = 0$ satisfies properties **B2**, **B3**, **A4** and **A5**, but not **B1**.
- The value on CU f^2 defined as $f^2(N, v, B, P) = v(i)$ satisfies **B1**, **B3**, **A4** and **A5**, but not **B2**.

- The value on CU f^3 defined as

  i) If (N, v, B, P) with $N = \{i, j\}$, with $i \neq j$, $B = \emptyset^c$, $P = P^n$,

$$f_i^3(N, v, B, P) = \frac{3}{4}(v(N) - v(j)) + \frac{1}{4}v(i), \text{ and}$$

$$f_j^3(N, v, B, P) = \frac{1}{4}(v(N) - v(i)) + \frac{3}{4}v(j).$$

  ii) Otherwise, $f^3 = \psi^c$,

  satisfies **B1**, **B2**, **A4** and **A5**, but not **B3**.

- The value on CU π^c satisfies **B1**, **B2**, **B3**, and **A5**, but not **A4**.

- The value on CU f^4 defined as $f^4(N, v, B, P) = \beta^c(N, v, B)$ satisfies **B1**, **B2**, **B3**, and **A4**, but not **A5**.

For Theorem 5.3.2:

- The null value on CU, f^1, defined as $f^1(N, v, B, P) = 0$ satisfies properties **B2**, **B3**, **A3** and **A6**, but not **B1**.

- The value on CU ϕ^c satisfies properties **B1**, **B3**, **A3** and **A6**, but not **B2**.

- The value on CU f^5 defined as

  i) If (N, v, B, P) with $N = \{i, j\}$, with $i \neq j$, $B = \emptyset^c$, $P = P^n$,

$$f_i^5(N, v, B, P) = \frac{3}{4}(v(N) - v(j)) + \frac{1}{4}v(i), \text{ and}$$

$$f_j^5(N, v, B, P) = \frac{1}{4}(v(N) - v(i)) + \frac{3}{4}v(j).$$

  ii) Otherwise, $f^5 = \pi^c$,

  satisfies properties **B1**, **B2**, **A3**, and **A6** but not **B3**.

- The value on CU f^6, defined as $f_i^6(N, v, B, P) = \beta_k(M, v^{BP})/|P_k|$, for every $(N, v, B, P) \in CU$ with $i \in P_k$ and $k \in M$ satisfies B1, B2, B3, and A6, but not A3.

- The value on CU ψ^c, satisfies **B1**, **B2**, **B3**, and **A3**, but not **A6**.

For Theorem 5.3.3:

- The null value on CU, f^1, defined as $f^1(N, v, B, P) = 0$ satisfies properties **B3**, **A3**, and **A6**, but not **A1** (for P^n).

- The value on CU f^7, defined as

i) If (N, v, B, P) with $N = \{i, j\}$, with $i \neq j$, $B = \emptyset^c$, $P = P^n$,

$$f_i^7(N, v, B, P) = \frac{3}{4}(v(N) - v(j)) + \frac{1}{4}v(i), \text{ and}$$

$$f_j^7(N, v, B, P) = \frac{1}{4}(v(N) - v(i)) + \frac{3}{4}v(j).$$

ii) Otherwise, $f^7 = \phi^c$.

satisfies properties **A1** (for P^n), **A3**, and **A6**, but not **B3**.

- The value on CU f^8, defined as $f_i^8(N, v, B, P) = \varphi_k(M, v^{BP})/|P_k|$, for every $(N, v, B, P) \in CU$ with $i \in P_k$ and $k \in M$, satisfies **A1** (for P^n), **B3**, and **A6**, but not **A3**.
- The value on CU f^9, defined as $f^9(N, v, B, P) = \varphi^c(N, v, B)$, for every $(N, v, B, P) \in CU$ satisfies properties **A1** (for P^n), **B3**, and **A3**, but not **A6**.

We end by presenting in Table 5.4 the properties that are satisfied by each of the values studied above. A blank field indicates that the value does not satisfy the property while a ✓ means that it does satisfy the property described in the row.

	ϕ^c	ψ^c	π^c
A1 (component efficiency)	✓		
A2 (fairness in the quotient)	✓		✓
A3 (balanced contributions within the unions)	✓	✓	✓
B1 (graph isolation)	✓	✓	✓
B2 (pairwise merging)		✓	✓
B3 (fairness in the graph)	✓	✓	✓
A4 (neutrality under individual desertion)		✓	
A5 (1-quotient game property)	✓	✓	✓
A6 (quotient game property)	✓		✓

Table 5.4: Properties and values on CU

In conclusion, we would like to highlight that the differences between the Owen, the Banzhaf-Owen, and the Symmetric coalitional Banzhaf values are transferred to the corresponding values on CU. Therefore, the only difference between ϕ^c and π^c is that the former satisfies **A1**, while the latter satisfies **B2**. This difference comes from the fact that for every $(N, v, B) \in C(N)$, $\phi^c(N, v, B, P^n) = \varphi^c(N, v, B)$ and $\pi^c(N, v, B, P^n) = \beta^c(N, v, B)$, and the differences between the Myerson and the Banzhaf graph values depicted in Section 2.2 are maintained between ϕ^c and π^c. The Banzhaf-Owen graph value is the only one out of the three values that is not fair in the quotient (property **A2**) and does not satisfy the Quotient game property, however it is the only one satisfying the Neutrality under individual desertion property (**A4**). This three properties are the only fact in which ψ^c and π^c differ, while the difference between ϕ^c and ψ^c are quite bigger. ϕ^c satisfies **A1** while ψ^c satisfies **B2**, inheriting the distinction between φ^c and β^c, ψ^c is not fair in the quotient (**A2**), and finally the differences **A3**-**A5** and **A5**-**A6** described between ψ^c and π^c, are the same between ψ^c and ϕ^c.

To conclude, it is worth to emphasize the interest behind this kind of parallel axiomatic characterizations because they favor ease when comparing different options to be chosen as the preferred value. Anyway, there is no value able to cover all situations, and the idea is to analyze the situation in which the values are to be applied and all the additional information (not contained in the characteristic function, nor in the coalition structure, nor in the communication graph) in order to decide which one of the studied values should be used in each circumstance.

6

A political example. The Basque Parliament

In order to illustrate all the values studied in this work let us consider the Parliament of the Basque Country in two different periods of office. In the first example we analyze the distribution of the power in the Parliament between 1986 and 1990. We choose this term of office to illustrate the model presented in Chapter 3 because the used example has been studied in detail in Carreras and Owen (1995) and the proposed incompatibility graph is properly explained taking into account the situation and the relations between the agents at that time. This example has also been studied in Alonso-Meijide and Casas-Méndez (2007) using the TU games with incompatibilities model, and the proposed modification of the Public Good Index. In Carreras and Owen (1995) the decisiveness of each party was studied using an incompatibility graph and different a priori unions structures, but always with the Owen value ϕ, the incompatibility Shapley value φ^I, and a combination of both. In this case the incompatibility Banzhaf value β^I will be used and the obtained results compared with the ones in the cited papers. In the second example we analyze the distribution of the power among the different political parties using the models studied in Chapter 1, Chapter 2, Chapter 4, and Chapter 5. In this case we select the VIII term of office of the Parliament.

The Parliament of the Basque Country, one of Spain's seventeen regions, is constituted by 75 members. We model the situation by a simple game. The characteristic function of the game played by the parties with parliamentary representation is as follows, unity for any coalition summing 38 or more members, and zero for the rest, since most decisions are taken by majority.

6.1 The Parliament from 1986 to 1990

We will consider the situation in the Parliament after the elections in November 1986. The Parliament was composed by 19 members of the Spanish socialist party PSE, 17 members of the Basque nationalist conservative party PNV, 13 members of the Basque nationalist social democrat party EA, 13 members of the Basque nationalist left-wing party HB, 9 members of the Basque nationalist moderated left-wing party EE, 2 members of the Spanish conservative party CP, and 2 members of the Spanish centrist party CDS. In the papers mentioned above, taking into account the behavior of the parties and the declarations made by the representatives of the parties involved, it is assumed the incompatibility graph compound of the following links:

$$(PSE : HB), (PSE : CP), (HB : EE), (HB : CP), (HB : CDS).$$

For a more detailed description of each party and their political positions the reader is referenced to Carreras and Owen (1996).

In the original simple game there are twelve possible minimal winning coalitions, but when we consider the I-restricted game with the incompatibility graph only six minimal winning coalitions are feasible.

In Table 6.1, we present the Shapley and Banzhaf values, the incompatibility Shapley and incompatibility Banzhaf values, and the corresponding normalized values[1].

Party	Seats	φ	β	$\bar{\beta}$	φ^I	β^I	$\bar{\beta}^I$
PSE	19	.25238	.46875	.25424	.2333	.40625	.22414
PNV	17	.25238	.46875	.25424	.3167	.53125	.29310
EA	13	.15238	.28125	.15254	.2333	.40625	.22414
HB	13	.15238	.28125	.15254	.0333	.09375	.05172
EE	9	.15238	.28125	.15254	.1500	.28125	.15517
CP	2	.01905	.03125	.01695	0	0	0
CDS	2	.01905	.03125	.01695	.0333	.09375	.05172

Table 6.1: The Banzhaf and Shapley values, the corresponding values on $I(N)$, and the corresponding normalized values

The depicted results are similar to those presented in Carreras and Owen (1996) and Alonso-Meijide and Casas-Méndez (2007). PNV ranks first, even though PSE has more seats. PNV and EA are the only parties which increase their power significantly. CP becomes a null player because his rejection to PSE and HB.

6.2 The Parliament from 2005 to 2009

Since elections in 2005, the Parliament was composed by 22 members of the Basque nationalist conservative party EAJ/PNV, "A", 18 members of the Spanish socialist party PSE-EE/PSOE, "B", 15 members of the Spanish conservative party PP, "C", 9 members of the Basque nationalist left-wing party

[1] By $\bar{f}$ we denote the normalized f value

EHAK/PCTV, "D", 7 members of the Basque nationalist social democrat party EA, "E", 3 members of the Spanish left-wing party EB/IU, "F", and 1 member of the Basque nationalist moderated left-wing party Aralar, "J".

In order to build a model that takes into account the ideology of the political parties involved we propose a communication graph defined in Figure 6.2. The graph is based on the relations between the parties in such a way that we put a link between two agents whenever these parties have reached agreements in the past.

Finally, we propose a cooperation structure in terms of a priori system of unions. Since the government was formed by A, E, and F before the elections, we considered the following system of a priori unions:

$$P = \{P_1, P_2, P_3, P_4, P_5\}$$

where, $P_1 = \{A, E, F\}$, $P_2 = \{B\}$, $P_3 = \{C\}$, $P_4 = \{D\}$, and $P_5 = \{G\}$. The proposed coalition and communication structures are jointly depicted in Figure 6.2.

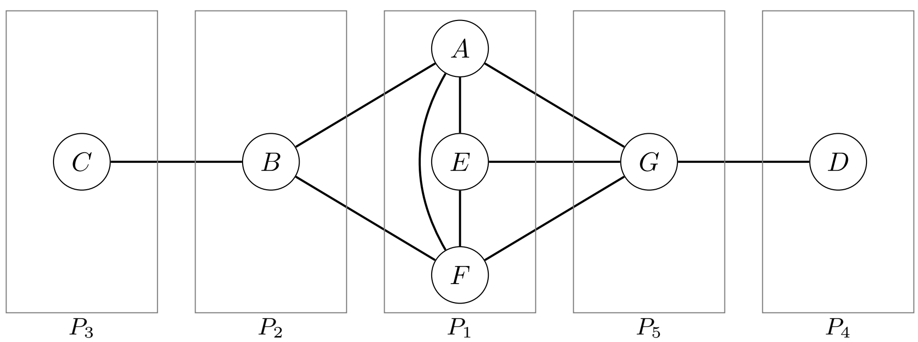

Figure 6.1: The communication graph and the a priori system of unions

Table 6.2 shows the values studied in Chapter 1 and Chapter 2.

Party	Label	Seats	φ	β	$\bar{\beta}$	φ^c	β^c	$\bar{\beta}^c$
EAJ/PNV	A	22	.3524	.5938	.3453	.3024	.4844	.248
PSE-EE/PSOE	B	18	.2524	.4062	.2364	.3690	.5313	.272
PP	C	15	.1857	.3438	.2000	.0357	.1250	.064
EHAK/PCTV	D	9	.0857	.1562	.0909	.0690	.1875	.096
EA	E	7	.0857	.1562	.0909	.0857	.250	.128
EB/IU	F	3	.0190	.0312	.0182	.0690	.1875	.096
Aralar	G	1	.0190	.0312	.0182	.0690	.1875	.096

Table 6.2: The Shapley, Banzhaf, Myerson, and Banzhaf graph values

Concerning the results presented in Table 6.2, the most remarkable effect of the communication graph restriction is that the most powerful player switches from A to B. In general those parties which have more links to other parties increase their payoffs at the expense of the parties located at the extremes of the graph.

Table 6.3 depicts the values studied in Chapter 4 and Chapter 5.

Party	ϕ	ψ	$\bar{\psi}$	π	$\bar{\pi}$	ϕ^c	ψ^c	$\bar{\psi}^c$	π^c	$\bar{\pi}^c$
A	.3833	.5938	.3878	.5833	.3889	.3806	.4688	.3571	.4479	.3525
B	.1667	.2500	.1633	.2500	.1667	.2500	.3750	.2857	.3750	.2951
C	.1667	.2500	.1633	.2500	.1667	0	0	0	0	0
D	.1667	.2500	.1633	.2500	.1667	.0833	.1250	.0952	.1250	.0984
E	.1000	.1563	.1020	.1458	.0972	.0722	.1250	.0952	.1042	.0821
F	.0167	.0313	.0204	.0208	.0139	.1306	.0938	.0714	.0938	.0738
G	0	0	0	0	0	.0833	.1250	.0952	.1250	.0984

Table 6.3: The Owen, Banzhaf-Owen, symmetric coalitional Banzhaf, and corresponding graph values

The left side of Table 6.3 shows the distribution of power based on the different values on $U(N)$ studied given the system of a priori unions P. This approach gives more power to A, which is the biggest party in the union P_1. On the other hand the next three parties, B, C, and D are each allotted with the same power. Finally, the smallest party becomes a null player.

When we consider the communication graph together with the system of a priori unions (right side of Table 6.3) the distribution of power changes significantly. The most remarkable change is that player C, the third most voted one, becomes irrelevant and that G, which has only one representative, is no longer null. This shows that the studied model is an accurate approach of the situation and different from the models presented in Section 2.2 and Section 2.3. Last but not least, if we focus on the values on CU, the difference between ϕ^c and the others lies on the power of parties E and F. The Owen graph value gives more weight to coalitions formed by many players (also by few players), while the other two (mostly ψ^c) are not so sensitive to the sizes of the coalitions where there are swings. This is the reason why parties E and F switch their order.

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