Exploring wind direction and SO_2 concentration by circular–linear density estimation

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- Investigation of the relation between pollutant concentrations from monitoring sites and the emission sources.

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- We focus on sulphur dioxide (SO_2) pollutants.

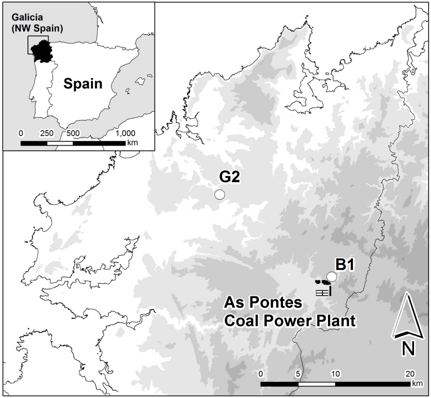
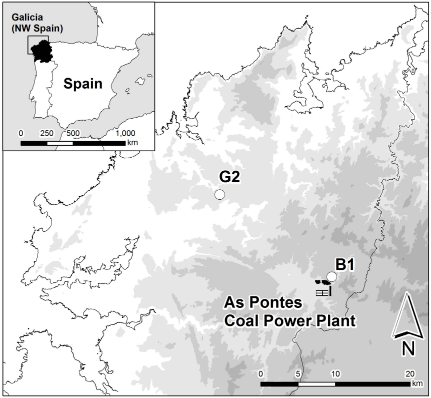


Figure: Locations of monitoring stations and power plant.



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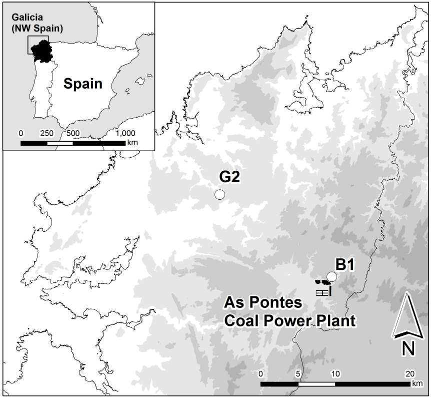
Distances to power plant

- B1: 0.9 km
- G2: 18.6 km



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Goal of the work

Study wind direction and SO_2 concentration relation.

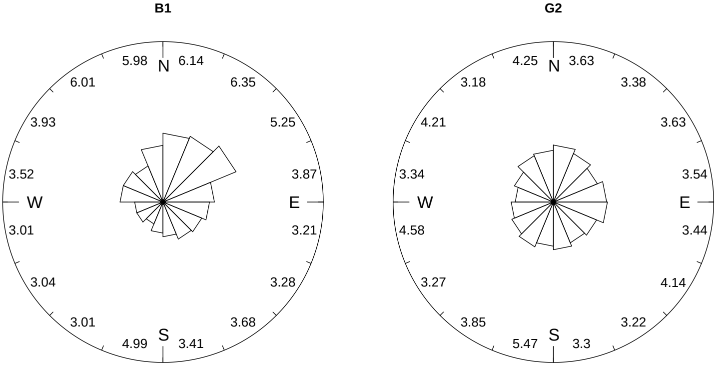

Figure: Rose diagrams for wind direction stations B1 and G2, with average SO_2 concentrations for August 2009.

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1. $f_\Theta(\theta) \geq 0,\ \forall \theta \in \mathbb{R}$.
2. $\int_r^{2\pi+r} f_\Theta(\theta) d\theta = 1,\ \forall r \in \mathbb{R}$.
3. $f_\Theta(\theta) = f_\Theta(\theta + 2\pi k),\ \forall \theta \in \mathbb{R},\ \forall k \in \mathbb{Z}$.

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Example (von Mises distribution $vM(\mu, \kappa)$)

The von Mises distribution has density

$$\varphi_{vM}(\theta; \mu, \kappa) = (2\pi I_0(\kappa))^{-1} \exp\left[\kappa \cos(\theta - \mu)\right],$$

where $\mu \in [0, 2\pi)$ is the circular mean, $\kappa \geq 0$ is the concentration in μ direction. Its distribution is denoted by Ψ_{vM}.

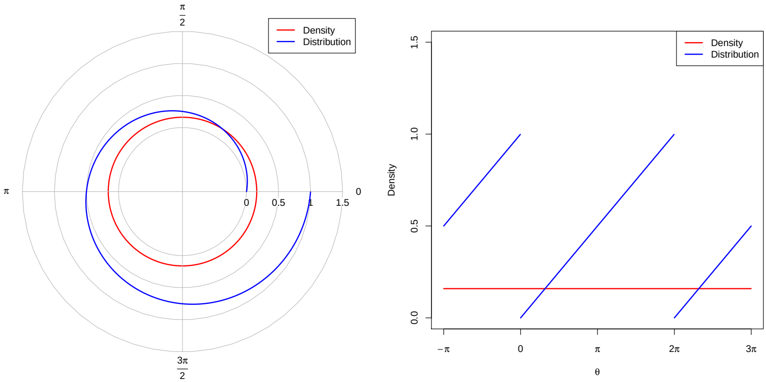


Figure: Circular and linear representations of the density and distribution of a von Mises with $\kappa = 0$ (**circular uniform distribution**).

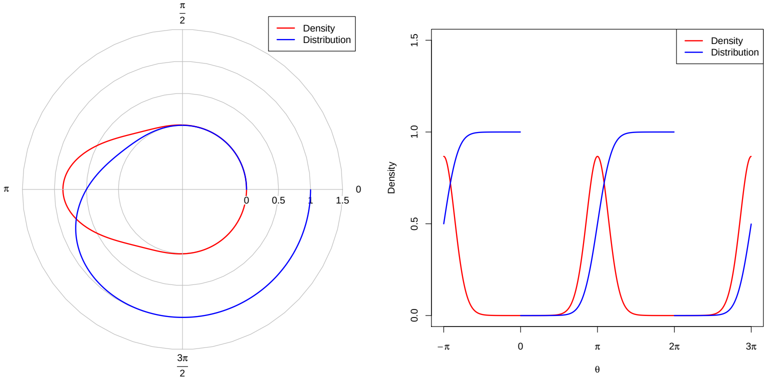
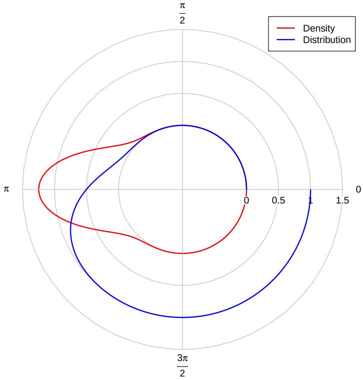


Figure: Circular and linear representations of the density and distribution of a von Mises with $\mu = \pi$ and $\kappa = 5$.



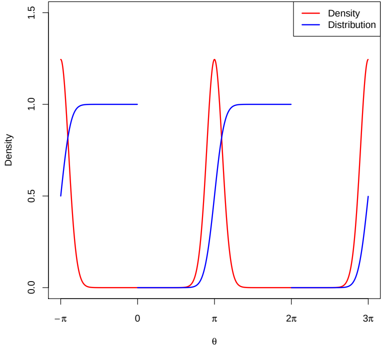


Figure: Circular and linear representations of the density and distribution of a von Mises with $\mu = \pi$ and $\kappa = 10$.

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Theorem (Johnson and Wehrly, 1978)

Let g be a circular density. Then

$$p(\theta, x) = 2\pi g\left[2\pi\left(\Psi(\theta) + F(x)\right)\right]\varphi(\theta)f(x)$$

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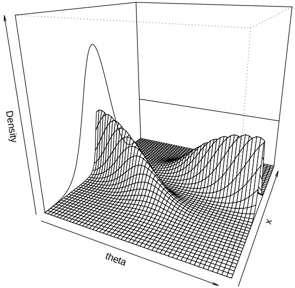
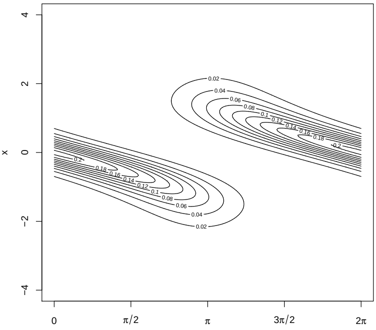
- It is a construction of p from φ and f, not a characterization.
- Θ and X independent $\Leftrightarrow g(\omega) = (2\pi)^{-1}$, $\forall\omega \in [0, 2\pi)$

Example (Circular uniform and Normal marginal densities)

Densities: $\varphi = (2\pi)^{-1}$, $f = \phi$ and $g = \varphi_{vM}(\mu, \kappa)$.

$$p_1(\theta, x) = (2\pi I_0(\kappa))^{-1} \exp\left[\kappa \cos(\theta - 2\pi\Phi(x) - \mu)\right] \phi(x)$$





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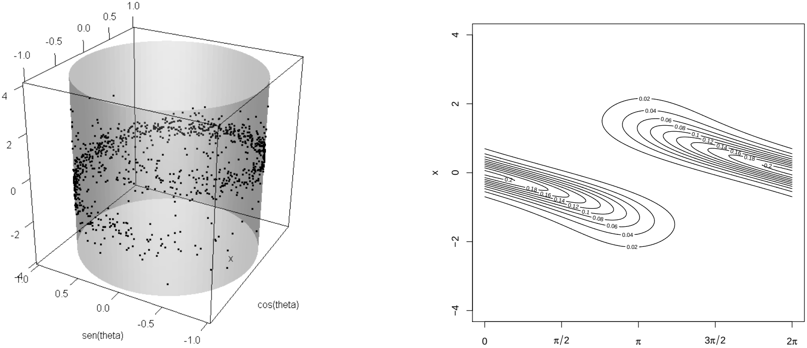

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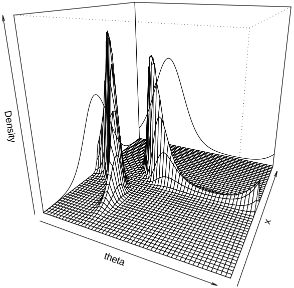
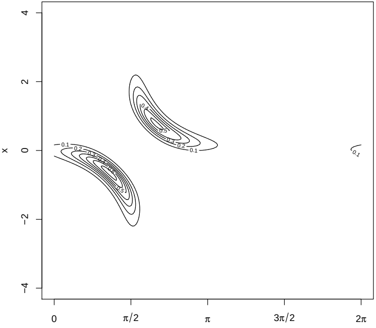
Example (von Mises and Normal marginal densities)

Densities: $\varphi = \varphi_{vM}(\mu_1, \kappa_1)$, $f = \phi$ and $g = \varphi_{vM}(\mu, \kappa)$.

$$p_2(\theta, x) = I_0(\kappa)^{-1} \exp\left[\kappa \cos\left(2\pi(\Psi_{vM}(\theta) - \Phi(x)) - \mu\right)\right] \varphi_{vM}(\theta)\phi(x)$$





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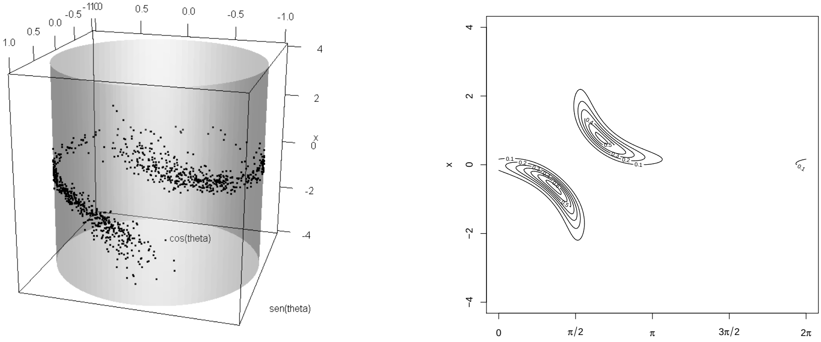

Figure: Joint density p_2 with $\mu_1 = \frac{\pi}{2}$, $\kappa_1 = 2$, $\mu = \pi$ and $\kappa = 5$.

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$$p(\theta, x) = 2\pi g\left[2\pi\left(\Psi(\theta) + F(x)\right)\right]\varphi(\theta) f(x)$$

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1. Obtain estimators for the marginal densities $\hat{\varphi}$, $\hat{f}$ and the corresponding marginal distributions $\hat{\Psi}$, $\hat{F}$.

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3. Obtain the circular–linear density estimator as

$$\hat{p}\left(\theta, x\right) = 2\pi\hat{g}\left[2\pi\left(\hat{\Psi}(\theta) + \hat{F}(x)\right)\right]\hat{\varphi}(\theta)\hat{f}(x).$$

Estimation approaches

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- **Nonparametric**. Estimate nonparametrically both marginals φ and f and the joining density g by kernel smoothing.

Kernel estimation

- Let f be a linear density and $X_1, \ldots, X_n$ a sample from $X \sim f$. The kernel estimator of f is

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right).$$

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- For a circular density φ and a sample $\Theta_1, \ldots, \Theta_n$, the kernel estimator defined by Hall, Watson and Cabrera (1987) is

$$\hat{\varphi}_\nu(\theta) = \frac{c_0(\nu)}{n} \sum_{i=1}^{n} L\left(\nu \cos(\theta - \Theta_i)\right).$$

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- Large values of ν undersmooth and small ones oversmooth (inverse behaviour of h).
- A possible choice of ν is by LSCV

$$\nu_{\mathsf{LSCV}} = \arg\min_{\kappa\geq 0} \int \hat{f}_\kappa(\omega)^2 d\omega - \frac{2}{n}\sum_{i=1}^n \hat{f}_\kappa^{-i}(\Theta_i).$$

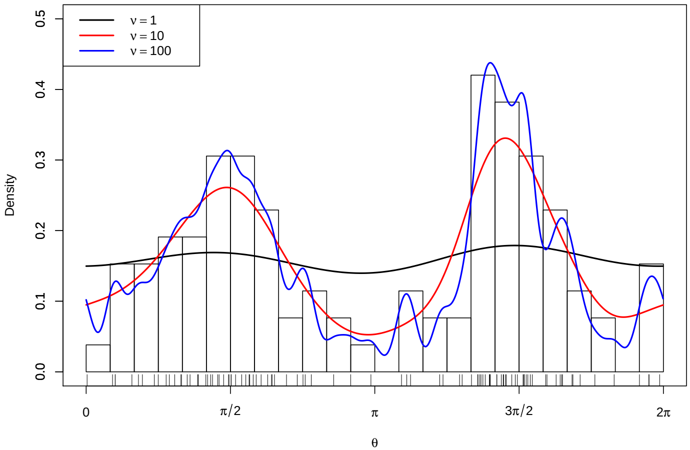


Figure: Effects of the circular bandwidth in the density estimator. Sample of size $n = 100$ from an equal mixture of $vM(\frac{\pi}{2}, 2)$ and $vM(\frac{3\pi}{2}, 5)$.

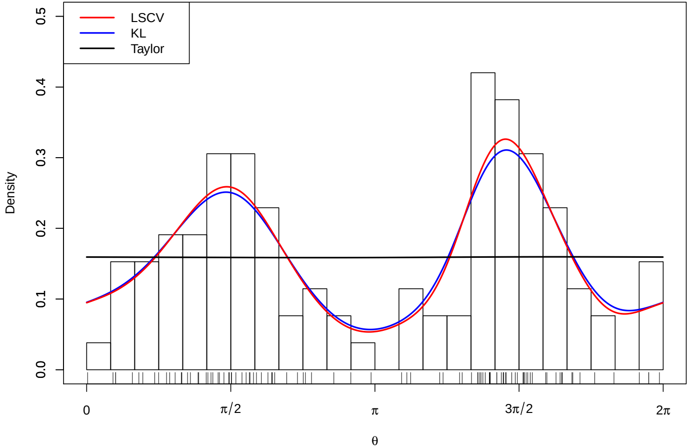
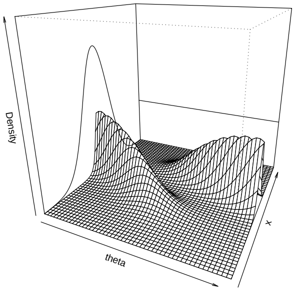


Figure: LSCV, KL and Taylor bandwidths. Sample of size $n = 100$ from an equal mixture of $vM(\frac{\pi}{2}, 2)$ and $vM(\frac{3\pi}{2}, 5)$.

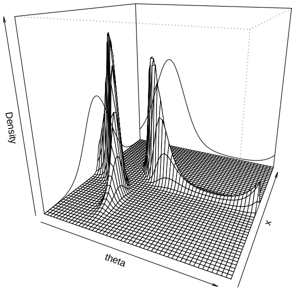
Example 1

- φ a circular uniform.
- $f \sim \mathcal{N}(0,1)$.
- $g \sim vM(\pi, 2)$.



Example 2

- $\varphi \sim vM\left(\frac{\pi}{2}, 2\right)$.
- $f \sim \mathcal{N}(0,1)$.
- $g \sim vM(\pi, 5)$.



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- Copula formulation helps for random simulation from p_1 and p_2 (our examples).

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Sklar's Theorem

Let X, Y two random variables with joint distribution F and marginals F_1 and F_2. There exists a copula C such that

$$F(x,y)=C(F_1(x),F_2(y)),\quad \forall x,y\in\mathbb{R}.$$

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Sklar's Theorem in terms of densities:

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- Simulation by copulas is easier due to the structure of p.
- The copula density is $c(u, v) = 2\pi g(2\pi(u + v))$.

Simulation setting

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- Other bandwidth selectors: Seather & Jones (linear); Kullblack–Leibler and Taylor (circular). Similar results.
- Sample sizes: $n = 50, 200, 500, 1000$. Samples generated using copula simulation.

Error criterion

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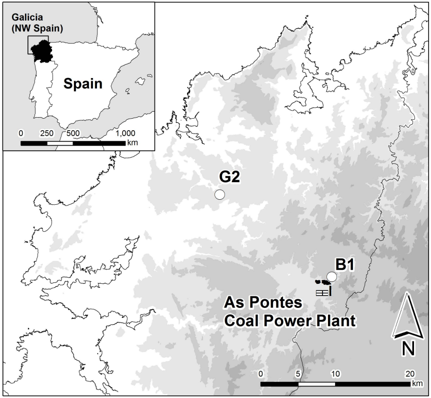
- Benchmark: Parametric model.
- Relative MISE efficiencies for Mixed and Nonparametric approaches.

	n	Estimation method: Param.	Estimation method: Mixed	Estimation method: Nonpar.	Relative efficiency: Mixed	Relative efficiency: Nonpar.
Example 1	50	0.0054	0.0095	0.0168	0.5674	0.3208
	200	0.0013	0.0029	0.0052	0.4802	0.2483
	500	0.0005	0.0012	0.0025	0.4267	0.2078
	1000	0.0003	0.0007	0.0014	0.3897	0.1840
Example 2	50	0.0402	0.0483	0.0977	0.8331	0.4115
	200	0.0104	0.0137	0.0363	0.7595	0.2862
	500	0.0043	0.0060	0.0185	0.7140	0.2296
	1000	0.0021	0.0032	0.0107	0.6783	0.2006

Table: MISE for estimating the circular–linear density in Example 1 and 2.



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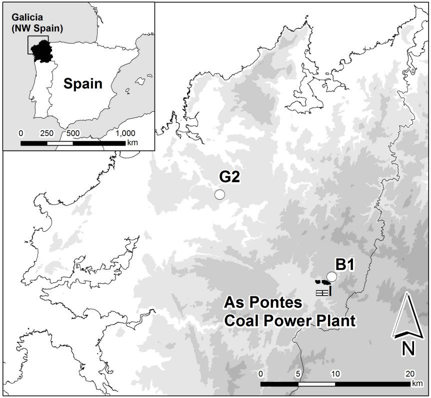

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Raw data

- SO_2 measured in $\mu g/m^3$. Detection limit: $> 3\mu g/m^3$.
- Wind direction.

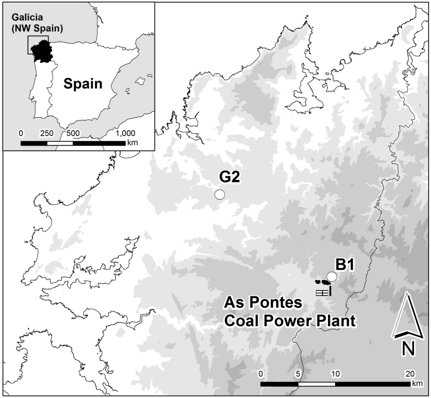

Figure: Locations of monitoring stations and power plant.

Raw data

- SO_2 measured in $\mu g/m^3$. Detection limit: $> 3\mu g/m^3$.
- Wind direction.

Our data

- Hourly averaged SO_2 and wind direction (circular mean).
- Perturbation to avoid repeated data.
- Box–Cox in SO_2.

How to proceed with repeated data?

- Linear case: Azzalini (1981) proposes a perturbation that allows a consistent estimation of the distribution:

$$\widetilde{X}_i = X_i + b\varepsilon_i,\ \varepsilon_i \sim \text{Epanech}\left(-\sqrt{5}, \sqrt{5}\right),$$

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with $d = n^{-\frac{1}{5}}$. Analogy with the bidimensional ($\mathbb{S}^1$) linear $b_{\text{AMISE}} = \mathcal{O}(n^{-\frac{1}{5}})$ (Liu and Yang, 2008).

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Circular–linear correlation coefficients (Mardia, 1976):

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Uniformity tests (Mardia and Jupp, 2000):

- Kuiper: Kolmogorov–type test.
- Watson: Cramer–von Mises test.
- Rayleigh: Alternative hypothesis is a unimodal distribution.
- Rao's Spacing test.

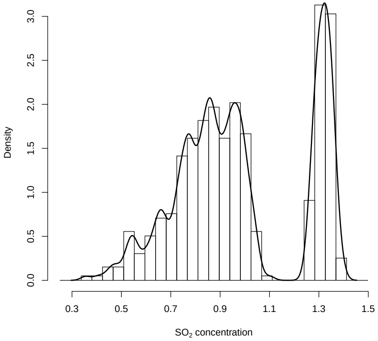


Figure: SO_2 concentration (Box–Cox) in B1.

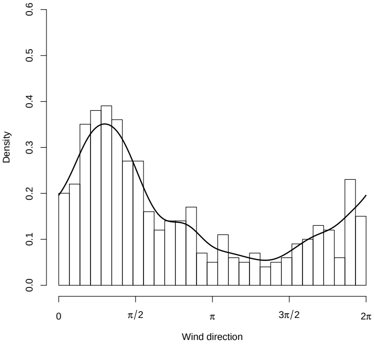


Figure: Wind direction in B1.

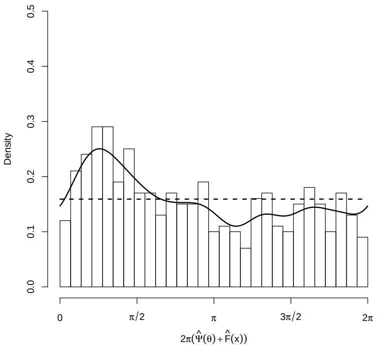


Figure: Estimation of g in B1.

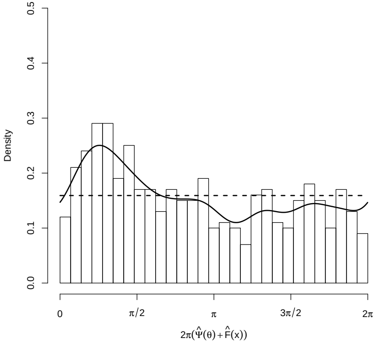
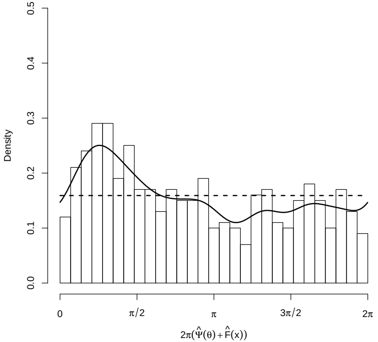


Figure: Estimation of g in B1.

Test	Statistic	p–value
Kuiper	2.8196	< 0.01
Watson	0.6425	< 0.01
Rayleigh	0.1552	< 0.01
Rao	140.8554	< 0.05

Table: Uniformity tests for g.



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Circular–linear correlation

- $\rho_{CL} = 0.1515$.
- $D_n = 0.1422$ with p–value=0.

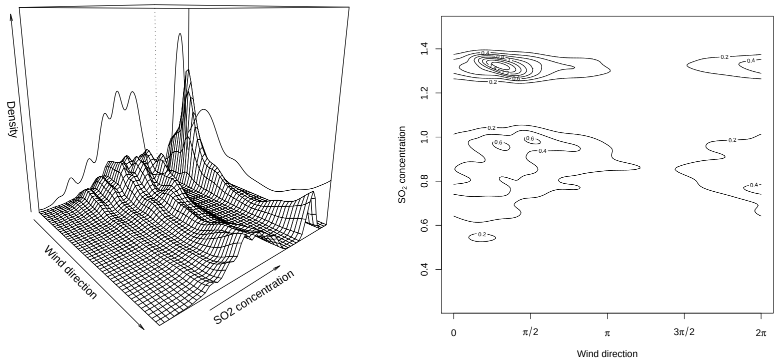

Figure: Surface and contourplot of the estimated circular–linear density in B1.

Another exploratory tool: circular regression.

- Consider a circular regression model:

$$Y = m(\Theta) + \varepsilon, \quad m(\theta) = \mathbb{E}\left(Y|\Theta = \theta\right)$$

with ε a zero–mean variable independent from Θ.

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- The circular Nadaraya–Watson, with von Mises kernel, is

$$\hat{m}(\theta; \nu) = \frac{\sum_{i=1}^{n} y_i \cdot \varphi_{vM}(\theta - \theta_i; 0, \nu)}{\sum_{i=1}^{n} \varphi_{vM}(\theta - \theta_i; 0, \nu)}$$

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- Possible selection of ν:

$$\nu_{\text{LSCV}} = \arg\min_{\kappa \geq 0} \sum_{i=1}^n \left(y_i - \hat{m}^{-i}(\theta_i;\kappa)\right)^2$$

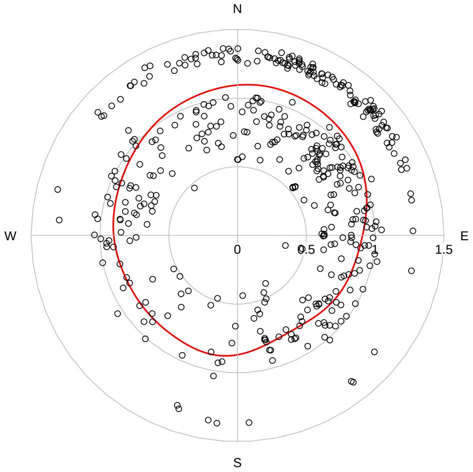

Figure: Circular regression of SO_2 (Box–Cox) in wind direction for B1.

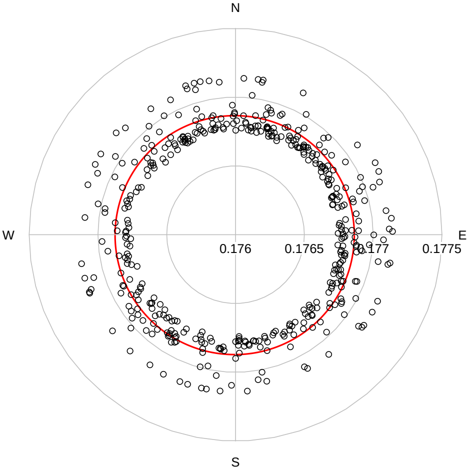

Figure: Circular regression of SO_2 (Box–Cox) in wind direction for G2.

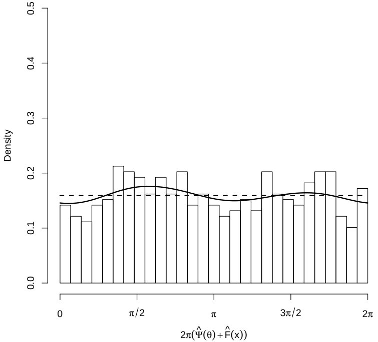

Figure: Estimation of g in G2.

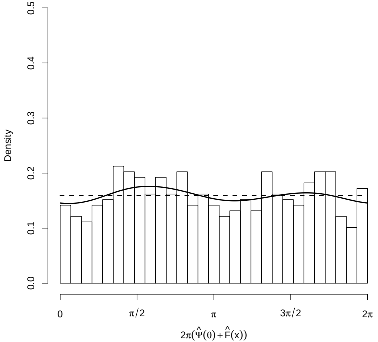

Figure: Estimation of g in G2.

Test	Statistic	p–value
Kuiper	1.2042	> 0.15
Watson	0.0748	> 0.10
Rayleigh	0.0259	0.737
Rao	130.7370	> 0.10

Table: Uniformity tests for g.

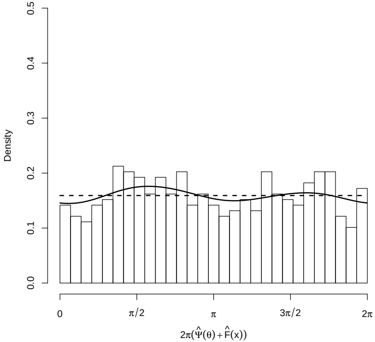

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Table: Uniformity tests for g.

Circular–linear correlation

- $\rho_{CL} = 0.0103$.
- $D_n = 0.0124$ with p–value$=0.0622$.

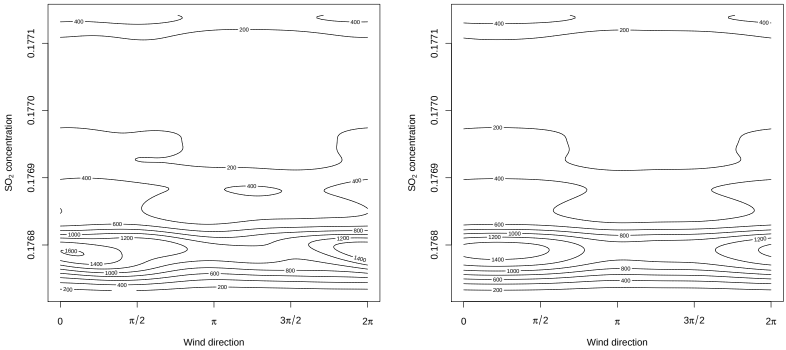

Figure: Right: contourplot of the estimated density in G2. Left: contourplot under independence.

Conclusions

- B1:
  - Moderate dependence between wind direction and SO_2.
  - Higher SO_2 concentrations linked to the NE and N wind, opposite direction to the power plant.
- G2: independence between wind direction and SO_2.

Open problems

1. Circular data perturbation.
2. Goodness–of–fit test for the Johnson and Wehrly family of circular–linear distributions.

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