Chapter 4

Ordinary Kriging

Variograms provide a lot of information about the parameter under study, but essentially they are tools for other geostatistical calculations. One of the possible (and perhaps the most important) use of variograms is in the estimation of parameter values at unsampled locations, and/or the estimation of the average of the parameter over a certain area. The simplest geostatistical procedure doing this is ordinary kriging. Ordinary kriging is the procedure which is most widely known (and often labeled by the single word kriging).

4.1 Point kriging

One of the most common interpolation (and extrapolation) problems is the estimation of a parameter at unsampled location u. In the framework of regionalized variables this can be done with the help of the procedure labeled point kriging.

A linear estimator, i.e. a linear combination of the values of the regionalized variable at known locations, is to be found. This means that the estimator is of the form:

$$Z^*(u) = \sum_{i=1}^{n} \lambda_i Z(u_i) \tag{4.1}$$

There are infinitely many possible choices for the weights λ_i. It is desirable to select them in order to have an unbiased estimator which also has the smallest possible estimation variance. Using the second order stationarity or the intrinsic hypothesis one has:

$$E[Z(u)] = m \text{ for all } u \in D \tag{4.2}$$

This means for the linear estimator

$$E[Z^*(u)] = \sum_{i=1}^{n} \lambda_i E[Z(u_i)] = m \tag{4.3}$$

so the weights have to fulfil:

$$\sum_{i=1}^{n} \lambda_i = 1 \tag{4.4}$$

This is the so called unbiasedness condition. Using the second order stationarity hypothesis the estimation variance can be calculated with the help of the covariance function $C(h)$ as:

$$\begin{aligned}
\sigma^2(u) = \mathrm{Var}[Z(u) - Z^*(u)] = E\left[(Z(u) - \sum_{i=1}^{n} \lambda_i Z(u_i))^2\right] = \\
= E\left[Z(u)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j Z(u_i) Z(u_j) - 2\sum_{i=1}^{n} \lambda_i Z(u_i) Z(u)\right] = \\
= C(0) + \sum_{j=1}^{n}\sum_{i=1}^{n} \lambda_j \lambda_i C(u_i - u_j) - 2\sum_{i=1}^{n} \lambda_i C(u_i - u)
\end{aligned} \tag{4.5}$$

The estimation variance is a quadratic function of the weights λ_i. The best linear unbiased estimator (BLUE) is the one which minimizes the estimation variance with respect to the unbiasedness condition. This constrained optimization problem can be solved with the help of a Lagrange multiplier μ. The function

$$\sigma^2(u) - 2\mu\left(\sum_{i=1}^{n} \lambda_i - 1\right) \tag{4.6}$$

is to be minimized. Using the partial derivatives with respect to the unknown parameters λ_i and μ one has to solve the linear equation system:

$$\sum_{j=1}^{n} \lambda_j C(u_i - u_j) - \mu \quad = \quad C(u_i - u) \quad i = 1, \ldots, n$$

$$\sum_{j=1}^{n} \lambda_j = 1 \tag{4.7}$$

Solving (4.7) yields the weights λ_i for the linear estimator. The equation system (4.7) is called kriging system in terms of covariances.

If the intrinsic hypothesis is used the estimation variance can be expressed with the help of the variogram:

$$\sigma^2(u) = Var[Z(u) - Z^*(u)] = -\sum_{j=1}^{n}\sum_{i=1}^{n} \lambda_j \lambda_i \gamma(u_i - u_j) + 2\sum_{i=1}^{n} \lambda_i \gamma(u_i - u) \tag{4.8}$$

The goal is to minimize $\sigma^2(u)$ under the unbiasedness conditions. This optimization problem can also be solved with the help of a linear equation system. Introducing the Lagrange multiplier μ the weights that minimize $\sigma^2(u)$ are the solution of:

$$\begin{aligned} \sum_{j=1}^{n} \lambda_j \gamma(u_i - u_j) + \mu &= \gamma(u_i - u) \quad i = 1, \ldots, n \\ \sum_{j=1}^{n} \lambda_j &= 1 \end{aligned} \tag{4.9}$$

The above equation system is called kriging system, the weights λ_i are the kriging weights. The minimal estimation variance can be obtained by substituting the kriging weights into (4.8). This variance is called kriging variance $\sigma_K^2(u)$. It can be proved that :

$$\sigma_K^2(u) = \sum_{i=1}^{n} \lambda_i \gamma(u_i - u) + \mu \tag{4.10}$$

This equation is of no theoretical interest, but it simplifies the calculation of the estimation variance.

EXAMPLE 4.1 :

Suppose that using two points on a straight line the value at a third point is to be estimated. The points are $u_1 = 1$ and $u_2 = -2$. The point for which the estimation is to be done is $u = 0$. Figure 4.1 shows the configuration. Let the measurement values be $Z(u_1) = 2$ and $Z(u_2) = 4$. Suppose the variogram is linear $\gamma(h) = h$.

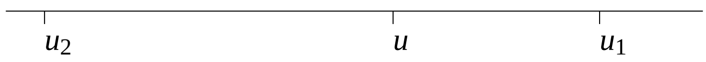

Figure 4.1: Data configuration for example 4.1

The kriging equations are:

$$
\begin{aligned}
0\lambda_1 + 3\lambda_2 + \mu &= 1 \\
3\lambda_1 + 0\lambda_2 + \mu &= 2 \\
\lambda_1 + \lambda_2 &= 1
\end{aligned}
\tag{4.11}
$$

From this one has $\lambda_1 = 0.6667$, $\lambda_2 = 0.3333$ and $\mu = 0$. Thus $\sigma^2 = 1.3333$ and $Z^*(u) = 2.6667$. It is clear that kriging yielded the same weights as linear interpolation or inverse distance method.

Suppose the configuration is changed and u_2 is moved to the other side of the origin: $u_2 = 2$. Figure 4.2 shows the modified configuration.

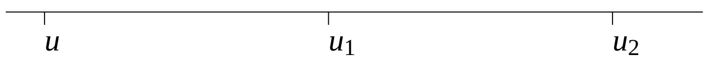

Figure 4.2: Modified data configuration for example 4.1

The kriging equations are:

$$0\lambda_1 + 1\lambda_2 + \mu = 1$$

$$\begin{aligned} 1\lambda_1 + 0\lambda_2 + \mu &= 2 \\ \lambda_1 + \lambda_2 &= 1 \end{aligned} \tag{4.12}$$

From this one has $\lambda_1 = 1.0$, $\lambda_2 = 0.0$ and $\mu = 1.0$. Thus $\sigma^2 = 2.0$ and $Z^*(u) = 2.0$. The result is different from the previous, but it would not be different in the case of the inverse distance method. This example demonstrates that the data configuration plays an important role in kriging. The increased estimation variance shows that the extrapolation in the second case is more uncertain than the interpolation in the first.

4.2 Block kriging

Quite often applications require average values of the parameter over certain areas, instead of point values. These averages could be calculated using point kriging for a great number of points in the area and taking their average. A simpler way of doing this is using block kriging.

Suppose the average of the parameter over a volume V (block) in the domain D is to be estimated.

$$Z(V) = \frac{1}{|V|} \int_{V} Z(u)\, du \tag{4.13}$$

Again a linear estimator of the form :

$$Z^*(V) = \sum_{i=1}^{n} \lambda_i Z(u_i) \tag{4.14}$$

is to be found. The unbiasedness condition leads again to:

$$\sum_{i=1}^{n} \lambda_i = 1 \tag{4.15}$$

The estimation variance in this case is:

$$\sigma^2(V) = Var[Z(V) - Z^*(V)] = -\bar{\gamma}(V,V) - \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_j \lambda_i \gamma(u_i - u_j) + 2 \sum_{i=1}^{n} \lambda_i \bar{\gamma}(u_i, V) \tag{4.16}$$

here $\bar{\gamma}$ is the average variogram value:

$$\bar{\gamma}(u_i, V) = \frac{1}{|V|} \int_{V} \gamma(u_i - u) \; du \tag{4.17}$$

$$\bar{\gamma}(V, V) = \frac{1}{|V|} \int_{V} \int_{V} \gamma(u - v) \, du \, dv \tag{4.18}$$

The minimization of $\sigma^2(V)$ under the unbiasedness condition leads to the linear equation system:

$$\begin{aligned} \sum_{j=1}^{n} \lambda_j \gamma(u_i - u_j) + \mu &= \bar{\gamma}(u_i, V) \;\; i = 1, \ldots, n \\ \sum_{j=1}^{n} \lambda_j &= 1 \end{aligned} \tag{4.19}$$

EXAMPLE 4.2 :

Suppose that for the same configuration as in the first part of example 4.1 instead of point $u = 0$ the average over the interval $[-0.5, 0.5]$ is to be found. Block kriging is applied for the estimation. The left hand side of the equation system is identical to the point kriging case. The right hand side is:

$$\bar{\gamma}(u_1, V) = \int_{-0.5}^{+0.5} |t - 1| \, dt = 1$$

$$\bar{\gamma}(u_2, V) = \int_{-0.5}^{+0.5} |t + 2| \, dt = 2$$

Thus the kriging equations are again:

$$\begin{aligned} 0\lambda_1 + 3\lambda_2 + \mu &= 1 \\ 3\lambda_1 + 0\lambda_2 + \mu &= 2 \\ \lambda_1 + \lambda_2 &= 1 \end{aligned} \tag{4.20}$$

From this one has $\lambda_1 = 0.6667$, $\lambda_2 = 0.3333$ and $\mu = 0$. To calculate the estimation variance one also needs the value of $\bar{\gamma}(V, V)$. This is:

$$\bar{\gamma}(V, V) = \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} |t - s| \, dt \, ds = 2 \int_{-0.5}^{+0.5} \int_{-0.5}^{s} s - t \, dt \, ds = \frac{1}{3}$$

Thus $\sigma^2 = 1.000$ and $Z^*(V) = 2.6667$. For this case block kriging yielded the same weights as point kriging, but the estimation variance is smaller using block kriging. (The weights calculated for the center of a block using point kriging are not necessarily equal to the weights corresponding to the block !)

4.3 Properties of ordinary kriging

The kriging estimator has several interesting partly advantageous and partly disadvantageous properties. First some general properties are listed, then the relationship between kriging and the variogram is investigated.

4.3.1 Kriging as an interpolator

Kriging is an interpolation (and extrapolation) technique. Important properties of the kriging interpolator are:

1. Kriging is an exact interpolator: for each observation point u_i $Z(u_i) = Z^*(u_i)$, and the corresponding estimation variance is zero. This is because taking $\lambda_i = 1$ and $\lambda_j = 0$ if $i \neq j$ the kriging equations are satisfied.

2. Kriging weights are calculated with the help of the variogram and the locations of the measurement points and the point to be estimated. Not only distances between measurement points and the point to be estimated are considered but also the relative position of the measurement points.

3. Kriging weights sum up to 1, but they can also be negative. Thus the usual hypothesis
$$\max\{Z(u_i)\} \leq Z^*(u) \leq \min\{Z(u_i)\}$$
is not true.

4. Kriging weights are not influenced by the measurement values. If the same configuration appears at two different locations the kriging weights will be

the same, independently from the measured values. The measured values influence the variogramm which is the basis for the calculation of the kriging weights.

5. Kriging weights show a screening effect, distant points receive lower weights if closer measurements are available. This effect is demonstrated in example 4.3.

EXAMPLE 4.3 :

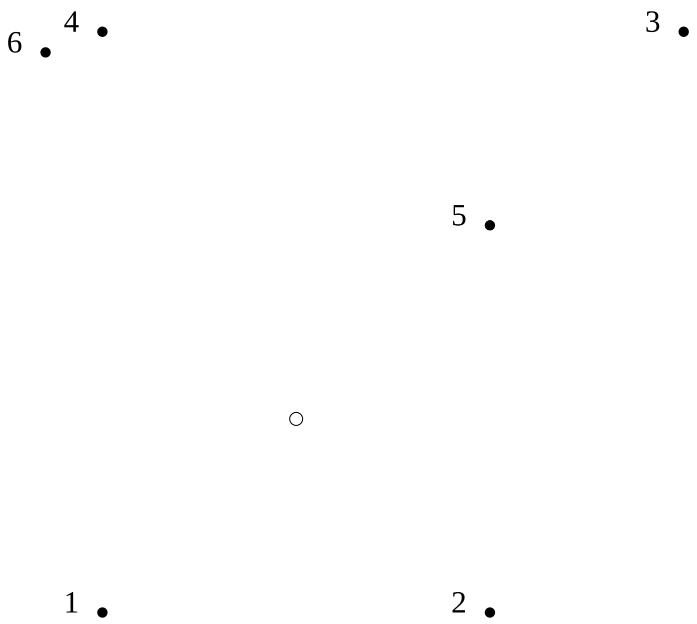

Figure 4.3: Data configuration for example 4.3

Suppose the value of the regionalized variable has to be estimated at the point (0,0) with the help of a subset of the points listed in table 4.1. The configuration is also displayed on figure 4.3. The variogram is known :

$$\gamma(h) = C_0 + C_1\gamma_S(h) \text{ for } h > 0 \tag{4.21}$$

where $\gamma_S(h)$ is a spherical model with a range $a = 10$. $C_0 = 0.05$ is the nugget effect and $C_1 = 0.20$.

Three different cases are considered:

1. kriging using points 1,2,3 and 4

2. kriging using points 1,2,3,4 and 5

No.	x	y
1	-1.00	-1.00
2	1.00	-1.00
3	2.00	2.00
4	-1.00	2.00
5	1.00	1.00
6	-1.10	1.90

Table 4.1: Different possible measurement locations

3. kriging using points 1,2,3,4 and 6.

Weights calculated for each case are shown in table 4.2.

Comparing case 1 and case 2 one can see that the weight corresponding to point 3 decreased substantially because of the inclusion of point 5. The other weights did not change drastically.

In case 3 part of the weight associated to point 4 was shifted to point 6, the other weights were much less influenced.

These two examples show that kriging filters out the useful information and assigns less weight to points which are close to other points or which are screened by other points.

4.3.2 Kriging and the variogram

As the estimation variance is calculated with the help of the variogram, and the kriging equations also contain variogram values it is obvious that the variogram plays a central role in kriging.

Using the variogram kriging delivers not only estimated values but also pro-

Point	Weights		
	Case 1	Case 2	Case 3
1	0.322	0.294	0.304
2	0.317	0.255	0.311
3	0.144	0.047	0.130
4	0.217	0.163	0.123
5	—	0.240	—
6	—	—	0.132

Table 4.2: Kriging weights for the three different cases

vides corresponding estimation variances. (Unfortunately these weights only depend on the data configuration and the variogram but not on the actual data values.) These estimation variances express the quality of the interpolation, high estimation variance means uncertain interpolation — low estimation variance shows good interpolation. Estimation variances are often used as normal error variances.

As mentioned previously the estimation variance is zero if the parameter is to be estimated at a measurement point location. In the neighbourhood the estimation variance is low (depending on the variogram) and as the distance from measurement points increases so does the estimation variance. Points (or blocks) with high estimation variances indicate areas where the estimation is uncertain.

Comparing estimation variances obtained using point and block kriging one can see that the latter are substantially smaller. This is because of the additional term $\bar{\gamma}(V,V)$ for the block variances. As $\bar{\gamma}(V,V)$ increases with the block dimensions the estimation variance decreases. This fact is in full agreement with the fact known from statistics, that a mean can be estimated with much higher accuracy than an individual value.

EXAMPLE 4.4 :

To show the role of the nugget effect consider the data of example 4.3. Three different variogram models were used to calculate the kriging weights.

$$\gamma(h) = C_0 + C_1 \gamma_S(h) \text{ for } h > 0 \tag{4.22}$$

where $\gamma_S(h)$ is a spherical model with a range $a = 10$. For γ_1 $C_0 = 0.05$ is the nugget effect and $C_1 = 0.20$. For γ_2 $C_0 = 0.20$ is the nugget effect and $C_1 = 0.05$. For γ_3 $C_0 = 0.0$ is the nugget effect and $C_1 = 0.25$.

Point	Weights		
	γ_1	γ_2	γ_3
1	0.322	0.265	0.341
2	0.317	0.262	0.352
3	0.144	0.230	0.098
4	0.217	0.243	0.210

Table 4.3: Kriging weights for the three different variograms

Kriging weights for the three different models are shown in table 4.3. Note that for γ_2, where the nugget value is increased, the weights are almost equal. The highest weight differences are for the case of γ_3, where there is no nugget effect. This example shows that a high nugget effect leads to estimators around the sample mean.

If the variogram $\gamma(h)$ is replaced by its constant multiple $c\gamma(h)$ then the kriging weights do not change. This is a consequence of (4.8), as the estimation variance is also multipled by the same constant, thus the minimum variance is realized using the same weights.

If $\gamma(h)$ is replaced by another variogram which is close to it, then the kriging weights do not change substantially. Unfortunately the possible changes depend both on the configuration of the data points and the actual data values.

4.4 Practice of kriging

4.4.1 Selection of the neighbourhood

As example 4.3 already demonstrated the screening property of kriging leads to small weights for distant samples. On the other hand the intrinsic hypothesis is supposed to hold locally within a certain distance. These two facts and the numerical efficiency of the solution imply that only the closest few samples should be used in kriging.

Usually the points used for the kriging of a point or block are selected within a certain distance (usually around the range) with taking into account the anisotropy. If there are still too many points in such a neighbourhood the closest n are taken, where n is a prescribed limit.

It is important to notice that the above procedure fails to work properly if the points are very irregularly spaced. In such a case different criteria have to be given. (for example directional search)

In three dimensions when the number of points is too high a regrouping of the points into blocks and then kriging from these blocks can reduce the computations.

4.4.2 Kriging with a "false" variogram

Kriging is sometimes used also without the calculation of an experimental variogram, but only assuming a theoretical model. As mentioned above the selection of the variogram parameters can influence the kriging results. Usually a complex model of two elements a nugget effect and a simple model (spherical, exponential, gaussian or linear) is assumed. As the multiplication of the variogram by a constant does not influence the kriging results, the most important factor in this case is the relative nugget effect (= sill divided by the nugget effect).

In any case an interpolator having the above mentioned properties is used. The estimation variances calculated without a proper variogram will be meaningless.

4.5 Cross validation

As previously mentioned the uniqueness of the realization makes the use of statistical test in geostatistics quite difficult. However, the subjective "by eye" fit of theoretical variograms should be checked somehow to reduce its effects. One possible way of doing this is the so called "cross validation". This procedure tests the variogram by a procedure where it is most often used, namely the kriging procedure.

For each measurement location u_i the values are estimated (using kriging) as if they were unknown. This estimator is now denoted by $Z^v(u_i)$ and the corresponding kriging standard deviation is $\sigma^v(u_i)$. Then the estimated values are compared with the true values $Z(u_i)$. If the kriging standard deviation can be interpreted as an estimation error with normal distribution then

$$S(u_i) = \frac{Z^v(u_i) - Z(u_i)}{\sigma^v(u_i)} \tag{4.23}$$

should be normally distributed with 0 mean and 1 as standard deviation ($N(0,1)$). The mean indicates whether the estimator is unbiased or not, the variance of S indicates the correctness of the kriging standard deviations.

The calculation of the $S(u_i)$ values with the fitted variogram is the first test of the appropriateness of the fit. If the distribution is different from $N(0,1)$ then variation of the coefficients can improve the fit.

Cross validation techniques can be used to detect outliers of the measurement values.

4.6 Kriging with uncertain data

It is quite often the case that the same parameter is measured or estimated with the help of different methods. If these methods yield different accuracies the corresponding measurement values should also be handled differently.

Suppose that for each point u_i there is an unknown error term $\varepsilon(u_i)$ having the following properties:

1. Unbiased :

$$E[\varepsilon(u_i)] = 0 \tag{4.24}$$

2. Uncorrelated :

$$E[\varepsilon(u_i)\varepsilon(u_j)] = 0 \text{ if } i \neq j \tag{4.25}$$

3. Uncorrelated with the parameter value:

$$E[\varepsilon(u_i)Z(u_i)] = 0 \tag{4.26}$$

For convenience the estimation for a block V is given here, but the same applies for point values, too. The linear estimator in this case is:

$$Z^*(V) = \sum_{i=1}^{n} \lambda_i \left(Z(u_i) + \varepsilon(u_i)\right) \tag{4.27}$$

The unbiasedness condition has to hold as in the case of ordinary kriging. So :

$$\sum_{i=1}^{n} \lambda_i = 1 \tag{4.28}$$

The estimation variance is:

$$Var[Z(V) - Z^*(V)] = -\bar{\gamma}(V,V) - \sum_{j=1}^{n}\sum_{i=1}^{n} \lambda_j \lambda_i \gamma(u_i - u_j) + 2\sum_{i=1}^{n} \lambda_i \bar{\gamma}(u_i, V) + \sum_{i=1}^{n} \lambda_i^2 E[\varepsilon(u_i)^2] \tag{4.29}$$

To minimize the estimation variance an equation system similar to the ordinary kriging system has to be solved. Namely:

$$\begin{aligned}\sum_{j=1}^{n}\lambda_j\gamma(u_i-u_j)+\lambda_i E[\varepsilon(u_i)^2]+\mu &= \bar{\gamma}(u_i,V)\;\; i=1,\ldots,n \\ \sum_{j=1}^{n}\lambda_j &= 1\end{aligned} \tag{4.30}$$

To illustrate the above methodology consider the following example:

EXAMPLE 4.5 :

Hydraulic conductivity is measured with different methods:

1. Direct measurements
2. Gravimetric measurements
3. Nuclear measurements

In the case of gravimetric and nuclear measurements the logarithm of the hydraulic conductivity is estimated from the measured water content and the dry density with the help of a nonlinear regression. The regression error for gravimetric measurements is $D[\varepsilon_G]=0.30997$, for nuclear measurements $D[\varepsilon_N]=0.32828$. The measurement data are listed in table 4.4. The average log K value of the square block V with opposite corner coordinates (0,0) and (3,3) is to be estimated. Figure 4.4 shows the data configuration.

The variogram of log K was estimated on the basis of other measurement data, and a theoretical model was fitted:

$$\gamma(h)=C_0+C_1\gamma_S(h) \text{ for } h>0 \tag{4.31}$$

where $\gamma_S(h)$ is a spherical model with a range $a=6\ m$. $C_0=0.05$ is the nugget

No.	x	y	log K	Measurement type
1	-1.00	-1.00	-7.07	Direct
2	4.00	1.50	-7.89	Direct
3	-1.00	1.50	-6.41	Gravimetric
4	4.00	-1.00	-6.84	Gravimetric
5	4.00	4.00	-7.69	Nuclear
6	1.50	-1.00	-7.94	Nuclear

Table 4.4: Different log K measurement data

effect and $C_1 = 0.15$. The equation system (4.30) for this case is:

$$
\begin{array}{ccccccccccccccc}
 & + & 0.199\lambda_2 & + & 0.138\lambda_3 & + & 0.194\lambda_4 & + & 0.200\lambda_5 & + & 0.138\lambda_6 & + & \mu & = & 0.167 \\
0.199\lambda_1 & + & & + & 0.194\lambda_3 & + & 0.138\lambda_4 & + & 0.138\lambda_5 & + & 0.167\lambda_6 & + & \mu & = & 0.141 \\
0.138\lambda_1 & + & 0.194\lambda_2 & - & 0.096\lambda_3 & + & 0.199\lambda_4 & + & 0.199\lambda_5 & + & 0.167\lambda_6 & + & \mu & = & 0.141 \\
0.194\lambda_1 & + & 0.138\lambda_2 & + & 0.199\lambda_3 & - & 0.096\lambda_4 & + & 0.194\lambda_5 & + & 0.138\lambda_6 & + & \mu & = & 0.167 \\
0.200\lambda_1 & + & 0.138\lambda_2 & + & 0.199\lambda_3 & + & 0.194\lambda_4 & - & 0.108\lambda_5 & + & 0.199\lambda_6 & + & \mu & = & 0.167 \\
0.138\lambda_1 & + & 0.167\lambda_2 & + & 0.167\lambda_3 & + & 0.138\lambda_4 & + & 0.199\lambda_5 & - & 0.108\lambda_6 & + & \mu & = & 0.141 \\
\lambda_1 & + & \lambda_2 & + & \lambda_3 & + & \lambda_4 & + & \lambda_5 & + & \lambda_6 & + & & = & 1
\end{array}
$$

The solution of the equation system is shown in table 4.5. The value of $\bar{\gamma}(V,V)$ is 0.1003, the estimation variance is 0.0778 and the estimated log K value is -7.36.

In the case of ordinary kriging without error terms the kriging equations would be the same except the main diagonal being zero. The solution in this case is can also be found in table 4.5.

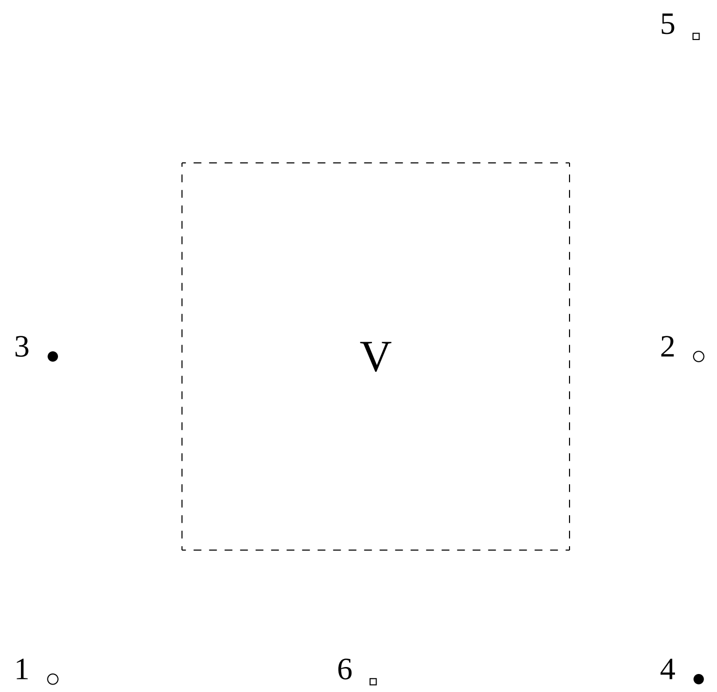

Figure 4.4: Data configuration for example 4.5

Note that observations 2,3, and 6 have similar weights as they are the closest observations to the block to be estimated. Weights for the direct measurements decreased, as all measurements are handled equally in this case.

Weights	Kriging with uncertainty	Point kriging
λ_1	0.147	0.042
λ_2	0.303	0.252
λ_3	0.210	0.294
λ_4	0.077	0.051
λ_5	0.108	0.126
λ_6	0.155	0.235
μ	0.020	0.009

Table 4.5: Weight calculated using uncertain and exact data

4.7 Simple Kriging

The Ordinary Kriging procedure is based on the assumption that the expected value of the underlying process is the same over the domain under study. The knowledge of this constant was not neccessary. Simple kriging is an alternative to OK supposing the mean $m(u)$ is known (not neccessarily constant) in the whole domain. In this case the estimator: Again a linear estimator of the form :

$$Z^*(u) = m(u) + \sum_{i=1}^{n} \lambda_i (Z(u_i) - m(u_i)) \tag{4.32}$$

is to be found. The unbiasedness condition means in this case:

$$E[Z^*(u) - Z(u)] = m(u) + \sum_{i=1}^{n} \lambda_i E[Z(u_i) - m(u_i)] - m(u) = 0 \tag{4.33}$$

This condition does not imply any additional constraints. The variance of the estimator is expressed using the covariance function C:

$$\begin{gathered} \mathrm{Var}[Z^*(u) - Z(u)] = E[Z^*(u)^2 + Z(u)^2 - 2Z^*(u)Z(u)] = \\ \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j C(u_i - u_j) + C(0) - 2\sum_{i=1}^{n} \lambda_i C(u_i - u) \end{gathered} \tag{4.34}$$

The estimation variance is minimal if:

$$\frac{\partial \mathrm{Var}[Z^*(u) - Z(u)]}{\partial \lambda_i} = 0 \tag{4.35}$$

This leads to the simple kriging equation system:

$$\sum_{j=1}^{n} \lambda_j C(u_i - u_j) = C(u_i - u) \tag{4.36}$$