International Workshop on Spatio-Temporal Modelling
Santiago de Compostela
30 Jun - 2 Jul 2010

Testing for anisotropy in spatial point processes

O. Nicolis[1,*], J. Mateu[2] and R. D'Ercole[2]

[1] *Department of Information Technology and Mathematical Methods, University of Bergamo, 24044 Dalmine, Italy; orietta.nicolis@unibg.it*
[2] *Department of Mathematics, Campus Riu Sec, E-12071 Castellon, Spain; mateu@mat.uji.es, dercole@mat.uji.es*
[*] *Corresponding author*

Abstract. *We propose a wavelet based hypothesis test for assessing isotropy in spatial point processes. We use the empirical logarithm of directional scalogram to build the test statistic. Under the null hypothesis of isotropy, a random point pattern should be expected to have the same value of the directional scalogram for all possible directions. Hence, Monte Carlo simulations of the logarithm of directional scalograms over all directions are used to approximate the test distribution and the critical values. Artificial and real data are then used to show the method.*

Keywords. *Anisotropy; Directional wavelets; Scalogram; Spatial point process.*

---

1 Introduction

Spatial point process models are useful tools for the analysis of scattered point patterns in many fields (biological, ecological, epidemiological, seismical, etc.). Most applications refer the detection of clusters, or particular patterns or the determination of weather the points are randomly distributed through the space. Relatively little attention has been focused on the analysis and testing of anisotropy. A spatial point process is said isotropic if its distribution is invariant under rotations about the origin. Otherwise is said anisotropic. In order to detect anisotropy or to assess isotropy it is necessary to perform a directional analysis of the point process. Some hypothesis tests of isotropy for spatial point processes have been proposed by Ohser and Stoyan (1981), Castelloe (1998), Rosenberg (2004), and Guan, Sherman, and Calvin (2006).

In this paper, we propose a wavelet based approach to test for isotropy in spatial point processes. The method can be considered an extension of Rosenberg (2004) by using the two dimensional directional wavelet transforms.

2 Directional wavelets

For any function $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^2$, the continuous directional wavelet transform for a scale a and orientation θ is given by

$$W_f(a, \mathbf{b}, \theta) = \langle f, \psi_{a,\mathbf{b},\theta} \rangle = \int_{\mathbb{R}^2} f(\mathbf{x}) \overline{\psi_{a,\mathbf{b}}(\mathbf{x}, \theta)} d\mathbf{x}, \quad (1)$$

where the overline denotes complex conjugate. In literature, many different directional wavelets $\psi_{a,\mathbf{b}}(\mathbf{x}, \theta)$ have been proposed. Neupauer and Powell (2005) introduced a flexible function called fully-anisotropic directional Morlet wavelet and given by:

$$\psi(\mathbf{x}, \theta) = e^{i\mathbf{k_0} \cdot \mathbf{Cx}} e^{1/2\mathbf{Cx} \cdot \mathbf{A}^T \mathbf{ACx}} \quad (2)$$

where $\mathbf{k_0} = (0, k_0)$ is a wave vector with $k_0 \geq 5.5$; $\mathbf{A} = diag(D, 1)$ denotes a diagonal matrix and D is the anisotropy ratio defined as the ratio of the length of the elliptical envelope in the y-direction to length of the elliptical envelope in the x-direction. The matrix $\mathbf{C}$ is a linear transformation given by

$$C = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$$

In order to identify the behavior of the process in different directions, Kumar (1995) introduced the function

$$\eta(a, \theta) = \int |W_f(a, \mathbf{b}, \theta)|^2 d\mathbf{b} \quad (3)$$

which characterizes the distribution of the energy at different scales and directions. The variance of wavelet coefficients for each scale and each direction, that is $E[|W_\psi f(\mathbf{b}, a, \theta)|^2]$, produces the so called "directional scalogram".

3 A Test of Significance

In this section, the Morlet wavelet transforms proposed by Neupauer and Powell (2005) are applied to the intensity function $\hat{\lambda}(\mathbf{x})$, estimated by the box counting method, where $\mathbf{x}$ denotes locations. Assume that we have m possible directions $\theta_i \in (0, \pi]$ with $i = 1, ..., m$, L possible scales a_j with $j = 1, ..., L$, and N points, b_k, with $k = 1, ..., N$. The directional wavelet transforms $W_{\hat{\lambda}}(a_j, b_k, \theta_i)$ are conducted for a range of scales and orientations at all positions in the domain of $\hat{\lambda}(\mathbf{x})$. Denote by S_{θ_i, a_j} the variance of the corresponding wavelet coefficients for a particular direction θ_i and scale a_j,

$$S_{\theta_i, a_j} = \frac{1}{N} \sum |W_{\hat{\lambda}}(a_j, b_k, \theta_i))|^2. \quad (4)$$

The significance of the directional wavelet coefficients is determined through a Monte Carlo simulation. We assume that the null hypothesis is isotropy. Hence, a isotropic random point pattern should be expected to have the same value of the directional scalogram for all possible directions. Then, we consider the logarithm of the variance of the wavelet coefficients over all scales as test statistic, given by

$$T(\theta_i) = \frac{1}{L} \sum_j \log S_{\theta_i, a_j}. \quad (5)$$

In order to estimate to approximate the distribution of the test statistic $T(\theta_i)$, we simulated 1000 data sets with the same number of points as in the observed data. Because we assume that the directional scalogram has the same value in all possible directions under the null hypothesis of isotropy, we evaluate the distribution of (5) for all simulations and over all directions θ_i. The critical value under the null hypothesis of isotropy is given by the maximum values of $T(\theta_j)$ in the empirical distribution.

3.1 Simulations

In order to show the ability of the method to assess the presence of anisotropic linear patterns, we performed a Monte Carlo simulation study. We considered different classes of spatial point processes, each describing a different type of directional pattern. For each class we simulated 100 data sets of 1000 points and we used the significance test of Section 3 for testing the null hypothesis of isotropy. In Figure 2 shows the directional wavelet analysis for two examples: the isotropic case (Fig. 1 a-c) and the anisotropic case with $\theta = 45$ as dominant direction (Fig. 1 d-f). The behavior of $T(\theta_i)$ for each direction θ_i (with $i = 1, \ldots, 180$), (in Fig. 1b) shows that no significant dominant directions are detected and the proposed statistical test accepts the null hypothesis of isotropy. Differently, in the anisotropic example of Figure 1 (d) the behavior of $T(\theta_i)$ outlines a dominant direction around 45 degree (peak). In this case the test rejects the null hypothesis of isotropy. The anisotropic component is also evident from the directional scalogram of Fig.1(f).

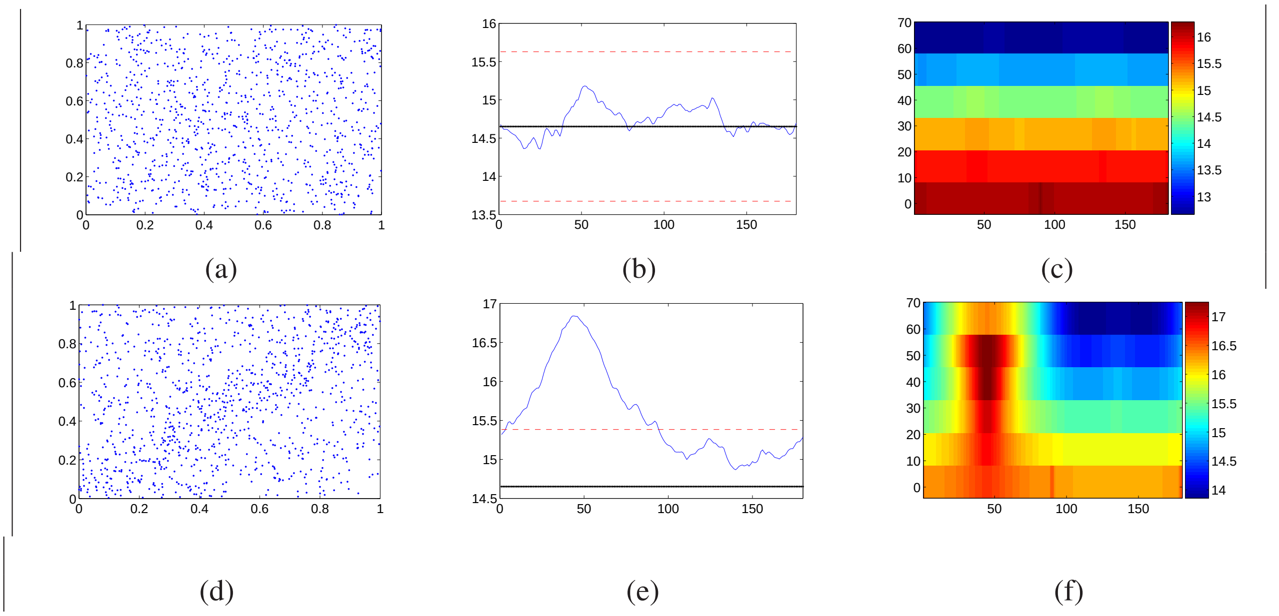
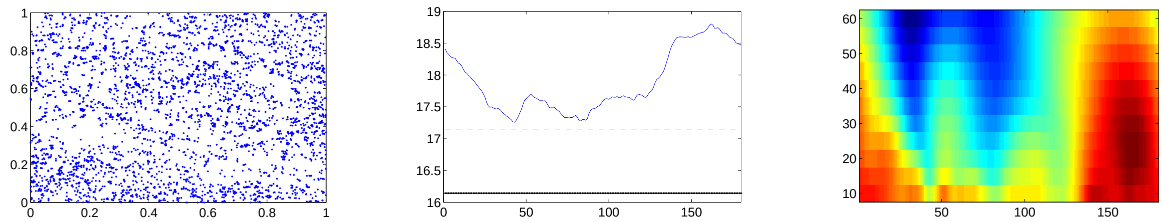


Figure 1: Directional wavelet analysis of $N = 1000$ simulated isotropic random points (upper) and anisotropic random points (bottom) with $\theta = 45$ degree. (b) and (c) show the behavior of the test statistic $T(\theta_i)$. Dotted and long dashed lines indicate the mean and the critical value of the empirical distribution, respectively. (c) and (f) show directional scalograms.

3.2 Application

We consider the data set *Ambrosia Dumosa* consisting of 4358 points (plants) in a 1-ha ($100m \times 100m$) covering the area of Joshua Tree National Park in California (see, Miriti et al. 1998 for a detailed description). Fig. 2 shows the wavelet analysis of the data set. The directional scalogram (Fig. 1d)



(a) (b) (c)

Figure 2: Directional wavelet analysis for the *Ambrosia Dumosa* data set. (a) Ambrosia Dumosa data; (b) Test statistic $T(\theta_i)$. Dotted and long dashed lines indicate the mean and the critical value of the empirical distribution, respectively. (c) Directional scalogram.

identify a dominant direction around 165 degree and a second direction around 43 degree. The statistical test (Fig. 1b) clearly rejects the null hypothesis of isotropy.

References

[1] Besag, J. and Diggle, P.J. (1977). Simple Monte Carlo tests for spatial patterns. *Applied Statistics*, **26**, 327-333.

[2] Castelloe, J.M. (1998). Issues in reversible jump Markov chain Monte Carlo and composite EM analysis, applied to spatial Poisson cluster processes. Ph.D. Thesis, University of Iowa, Iowa City, Iowa.

[3] Guan, Y., Sherman, M., and Calvin, J.A. (2004). A nonparametric test for spatial isotropy using subsampling. *Journal of the American Statistical Association, Theory and Methods*, **99**, 810-821.

[4] Guan, Y., Sherman, M. and Calvin, J.A. (2006). Assessing isotropy for spatial point processes. *Biometrics*, **62**, 119-125.

[5] Kumar, P. (1995). A wavelet based methodology for scale-space anisotropic analysis, *Geophys. Res. Lett.*, 22, **20**, 2777-2780.

[6] Miriti, M.N., Howe, H.F. and Wright, S.J. (1998). Spatial patterns of mortality in a Colorado desert plant community. *Plant Ecol.* **136**, 41-51.

[7] Ohser, J. and Stoyan, D. (1981). On the second-order and orientation analysis of planar stationary point processes. *Biometrical Journal*, **23**, 523-533.

[8] Rosenberg, M.S. (2004). Wavelet analysis for detecting anisotropy in point patterns. *Journal of Vegetation Science*, **15**, 277-284.

[9] Neupauer, R.M., and Powell, K. L. (2005). A fully-anisotropic Morlet wavelet to identify dominant orientations in a porous medium. *Computers and Geosciences*, **31**, 465-471.

[10] Zimmerman, D.L. (1993). Another look at anisotropy in geostatistics. *Mathematical Geology*, **25**, 453-470.