METMA V International Workshop on Spatio-Temporal Modelling
Santiago de Compostela
30 Jun - 2 Jul 2010

Spatial threshold exceedances analysis through marked point processes

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Abstract. *Indicators of recurrence, persistence, and in general, distribution patterns of extremal events defined by random field threshold exceedances provide relevant information on critical phenomena for risk assessment. Such indicators are directly related to geometrical properties describing the structure of the corresponding excursion sets. Given the intrinsic nature of the latter, marked point processes provide a natural approach to analyze distribution patterns of such extremal events in relation to specific characteristics of interest. In this paper, based on simulations from a flexible model separating long range dependence and fractality effects, we analyze the structure of threshold exceedances in terms of various second-order characeristics. In particular, we focus on the variations in size and distance heterogeneities in the components of excursion sets, as well as in clustering/inhibition patterns, depending on both the underlying model parameters and the threshold specifications.*

Keywords. *Geometrical characteristics; Heterogeneities; Marked point processes; Spatial random fields; Threshold exceedances.*

1 Introduction and motivation

Analysis of environmental phenomena for risk assessment usually involves the construction of indicators related to structural characteristics of extremal events defined by exceedances over critical thresholds. Recurrence and persistence, among others, are examples of such characteristics, which provide information about the distribution patterns of extremal events. Formally, these concepts are intimately related to the geometrical characteristics of the excursion sets defined by threshold exceedances over a given (bounded) domain. In particular, useful mathematical descriptions can be given in terms of the Lipschitz-Killing curvatures or, equivalently, the intrinsic volumes of such sets (see Adler and Taylor, 2007). In

Angulo and Madrid (2010), the effect of blurring and deformation transformations on the structure of a random field is studied in terms of the modifications implied on the Euler characteristic and hypervolume of threshold exceedance sets, depending on fractality and long range dependence parameters, as well as on the threshold considered. Note that, under suitable conditions, the expected value of the Euler characteristic approximates, for high thresholds, the probability of exceedance in at least one point of the domain considered (Adler and Taylor, 2007), whilst the hypervolume provides an estimate of the probability of having an exceedance in a generic point of such domain.

Given the fragmented nature of threshold exceedance sets, depending on the variation properties inherited by sample paths from the probabilistic structure of a random field and the threshold considered, marked point processes provide a powerful framework for the analysis of their structural properties. In fact, this approach can be exploited to help establishing the bridge between the construction and interpretation of risk indicators and the properties of the underlying random field generating critical events. More specifically, connected components of a threshold exceedance set can be treated as single, isolated events, with some geometrical properties as size, contour length, relative intensity of dominant orientation, etc., being considered as possible marks of interest for complementary analysis of diverse forms of heterogeneity and anisotropy. Hence, a variety of marked point process characteristics can be used to describe some features of interest, in particular for risk assessment purposes.

2 Methodology and simulations

A point process is a stochastic model governing the locations of events $\{x_i\}$ in some set X, where X is considered a bounded region in $\Re^2$ (Stoyan *et al.*, 1995). If the locations contain associated measurements or marks, the point process is referred to as a marked point process.

In this paper we consider the Cauchy class, defined by the homogeneous and isotropic covariance function $C(h) = \sigma^2(1+h^\alpha)^{-\beta/\alpha}$, $\alpha \in (0,2]$ and $\beta > 0$. This class has an interesting property consisting of allowing a separately characterization of local variability and dependence ranges. Specifically, parameter α determines the fractal dimension of realizations, $D = n+1-\alpha/2$ (for a random field on $\Re^n$), and, independently, β specifies the Hurst coefficient, $H = 1-\beta/2$ (see Gneiting and Schlather, 2004). Several realizations are shown in Figure 1. When a threshold is fixed, we can build a spatial point pattern through the centroids of the connected components defining the corresponding excursion sets. We can then associate the size as a mark, defining thus the marked point pattern. Figure 2 shows the point patterns coming from Figure 1 when fixing the threshold to 0.9.

A marked point process is *stationary* if its distribution is invariant under translations. One way to investigate the stationarity condition is by analyzing the intensity (expected number of points per unit area) of the point process. The intensity may be constant (the process is said 'homogeneous') or may vary from location to location (the process is said 'inhomogeneous'). The intensity function or intensity measure can be estimated by nonparametric techniques such as quadrat counting and kernel smoothing. In classical literature, the homogeneous Poisson process (CSR) is usually taken as the appropriate 'null' model for a point pattern. Our basic task in analysing a point pattern is to find evidence against CSR by using quadrat counts. A widely used second-order characteristic is the K-function: for a stationary point process, $\lambda K(r)$ defines the expected number of other points of the process within a distance r of a typical point of the process. A commonly used transformation of K is the L-function, which transforms the Poisson K-function to the straight line $L_{pois}(r) = r$, making visual assessment of the graph much easier. The square root transformation also approximately stabilises the variance of the estimator. Both functions

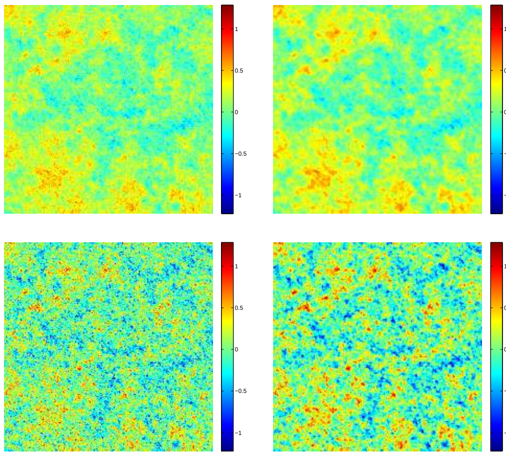

Figure 1: Simulated realizations of Cauchy class with $\alpha = 0.5, 2$ (from left to right) and $\beta = 0.1, 0.9$ (from top to bottom).

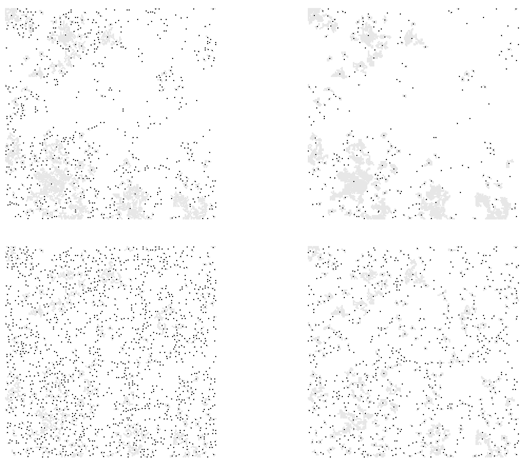

Figure 2: Excursion sets for threshold corresponding to percentile 0.9, based on realizations of Figure 1. Black dots represent centroids of connected components.

can be used to detect departures from random structures (see Mateu, 2000). There is a modification of the K and L functions that applies to inhomogeneous processes. Although the L-function is intended primarily for exploratory purposes, it is also possible to use it as a basis for statistical inference. We can use the language of hypothesis testing. Our null hypothesis is that the data point pattern is a realization of complete spatial randomness. The alternative hypothesis is that the data pattern is a realization of another, unspecified point process. Then, a Monte Carlo test can be run based on simulations from the null hypothesis and generate envelopes from the simulations; see Figure 3. We shall also make use of the mark correlation function, which is the natural extension of the pair correlation to the marked case. This function detects spatial dependencies between the marks (sizes) of the connected components.

3 Conclusions

We have focused on the analysis of structural changes in marked point processes based on excursion sets corresponding to different thresholds, depending on the fractality and long range dependence properties of the generating random field. Specifically, for thresholds corresponding to various high percentiles

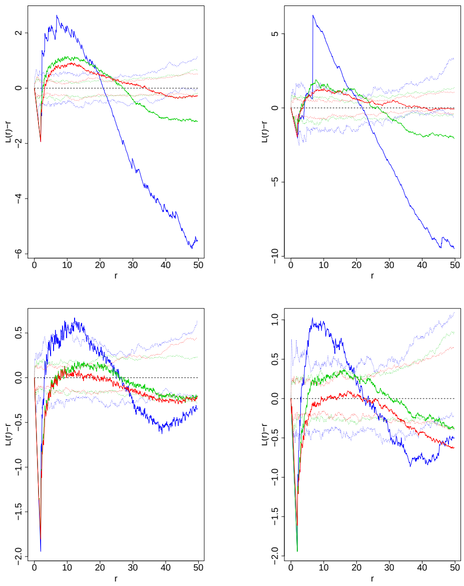

Figure 3: $L(r) - r$ function for point processes defined by centroids of connected components, for those excursion sets corresponding to percentiles 0.8 (red), 0.9 (green) and 0.97 (blue), based on realizations of Figure 1.

in the empirical distribution of sample-path values, we analyze size heterogeneities of isolated events defined by connected components, as well as distance ranges where the spatial distribution of the centroids representing such components display clustering/inhibition patterns. The results show significant differences, depending on the scenario determined by the model parameters, which have interesting interpretations related to the underlying random field probabilistic structure as well as in terms of risk indicators. It is shown that both an increase in the threshold and a decrease in the dependence range make the pattern inhomogeneous with more variability in the sizes and distances among components. In addition, the degree of clustering or inhibition is notoriously increased.

We aim at considering marks describing orientations to analyze anisotropic characteristics. Also evolution in time could be considered.

Acknowledgments. This paper has been partially supported by projects MTM2009-13250 and MTM2007-62923 of the SGPI, and P08-FQM-03834 of the Andalusian CICE, Spain.

References

[1] Adler, R.J. and Taylor, J.E. (2007). *Random Fields and Geometry*. Springer: New York.

[2] Angulo, J.M. and Madrid, A.E. (2010). Structural analysis of spatio-temporal threshold exceedances. *Environmetrics* **DOI:**10.1002/env.1018.

[3] Gneiting, T. and Schlather, M. (2004). Stochastic models that separate fractal dimension and the Hurst effect. *SIAM Review* **46**, 269-282.

[4] Mateu, J. (2000). Second-order characteristics of spatial marked processes with applications. *Nonlinear Analysis: Real World Applications* **1**, 145-162.

[5] Stoyan, D., Kendall, W.S. and Mecke, J. (1995). *Stochastic Geometry and its Applications*. Wiley, New York.