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Wavelet-based estimation of spatiotemporal long-range dependence

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Abstract. *The local singular behavior displayed by the spectral density of long-range correlated data motivates us to consider the wavelet transform, in the spectral domain, for parameter estimation of spatiotemporal long-range dependence. Two spectral scalogram-based functional estimation algorithms are proposed. A simulation study is developed to investigate the bias and efficiency of the computational methods proposed for functional parameter estimation of strong-dependent processes.*

Keywords. *Separable Riesz kernel; Spatiotemporal long-range dependence; Wavelet periodogram*

1 Introduction

Different approaches have been introduced in the statistical analysis of long-range correlated systems. Fractional time series [2] and regression [9] models provide a flexible framework to represent strong-dependence in data. The theory of self-similar and fractal fields also provides a suitable context for the introduction of spatiotemporal long-range dependence models (see [10]).

Estimation methods for long-range dependence parameters have been widely studied within the theory of fractional time series. Semiparametric procedures based on the periodogram have been formulated, for example, in [2], and [12]. Spectral-based approaches have been also proposed in the spatial and spatiotemporal contexts in [5],[8],[4],[6],[7] and [11]. In all the cited spectral-based approaches, the main difficulty relies on the lost of information caused by the truncation of the covariance (heavy) tails, when the discrete Fourier transform of the data is performed. This difficulty is overcome when measurements are produced in the spectral domain, e.g., measurements from spectral devices (spectrometers and spectroradiometers). In this paper, we refer to a special class of multispectral imaging systems, generated from fractal spectral Gaussian processes directly connected with strong-dependence spatiotemporal random fields. Specifically, the following spectral image sequence model is considered:

$$\hat{X}(d\omega, d\boldsymbol{\lambda}) = \hat{r}(\omega, \boldsymbol{\lambda})[f_Y(\omega, \boldsymbol{\lambda})]^{1/2}\varepsilon(\omega, \boldsymbol{\lambda})d\omega d\boldsymbol{\lambda}, \tag{1}$$

where $\varepsilon(\omega, \boldsymbol{\lambda})$ denotes spectral white noise on $\mathbb{R}^{d+1}$,

$$\hat{r}(\omega, \boldsymbol{\lambda}) = C_1|\omega|^{-2\nu}\Pi_{i=1}^{d}|\lambda_i|^{-2\beta_i}, \ \nu \in (0, 1/2), \ \beta_i \in (0, 1/2), \ i = 1, \dots, d,$$

and $f_Y(\omega, \boldsymbol{\lambda})$ satisfies the following conditions:

Condition 1. $|f_Y(\omega, \boldsymbol{\lambda})| \to C_1$, when $\omega \to 0$ and $\lambda_i \to 0$, for $i = 1, \dots, d$, with C_1 being a positive constant.

Condition 2. $\frac{|f_Y(\omega,\boldsymbol{\lambda})|}{(1+|(\omega,\boldsymbol{\lambda})|^2)^{-\widetilde{\nu}-\Sigma_{i=1}^{d}\widetilde{\beta}_i}} \to C_2$, when $\omega \to \infty$ and $\lambda_i \to \infty$, for $i = 1, \dots, d$, where C_2 is a positive constant, and $(\widetilde{\nu}, \widetilde{\beta}_1, \dots, \widetilde{\beta}_d) \in (1/2, \infty)^{d+1}$, and $(\nu, \beta_1, \dots, \beta_d) \in (0, 1/2)^{d+1}$.

2 Results and Methodology

The continuous discrete wavelet transform of the spectral random field $\hat{r}(\omega, \boldsymbol{\lambda})[f_Y(\omega, \boldsymbol{\lambda})]^{1/2}\varepsilon(\omega, \boldsymbol{\lambda})$ is performed, obtaining the wavelet-spectral random field

$$\widehat{WX}_j(\mathbf{b}) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \boldsymbol{\Psi}_{j,\mathbf{b}}(\omega, \boldsymbol{\lambda})\hat{r}(\omega, \boldsymbol{\lambda})[f_Y(\omega, \boldsymbol{\lambda})]^{1/2}\varepsilon(\omega, \boldsymbol{\lambda})d\omega d\boldsymbol{\lambda},$$

where we have considered a compactly supported wavelet basis $\{\boldsymbol{\Psi}_{j,\mathbf{b}}, \ \mathbf{b} \in L_j, \ j \in \mathbb{Z}\}$ with support $D = [-\varepsilon, \varepsilon]^{d+1}$, $\varepsilon <<, \ \varepsilon \sim 0$.

Since, under Conditions 1 and 2, the local behavior of the spectral random field $\hat{X}$ is given by

$$\hat{X}(d\omega, d\boldsymbol{\lambda}) \sim \hat{r}(\omega, \boldsymbol{\lambda})\varepsilon(\omega, \boldsymbol{\lambda})d\omega d\boldsymbol{\lambda}, \quad \|(\omega, \boldsymbol{\lambda})\| \to 0,$$

the weak-sense second-order moments of the square of the wavelet-spectral random field will display the asymptotic behavior given in the proposition below.

Proposition 1 *Let $\widehat{WX}_j(\cdot)$ be the wavelet-spectral random field defined above. For every absolutely integrable function g, the following limit holds:*

$$E\left[[\widehat{WX}]_j^2(g) - \mu_{[\widehat{WX}]_j^2(g)}\right]^2 \to 0, \quad j \to \infty,$$

where $[\widehat{WX}]_j^2(g)$ denotes the generalized random field associated with $[\widehat{WX}]_j^2$ applied to function g, and $\mu_{[\widehat{WX}]_j^2(g)}$ denotes its mean applied to the same test function g.

The proof of this result follows from the local self-similar behavior of the higher order spectra of spatiotemporal random field X, associated with spectral process $\hat{X}$, which is collected by the wavelet transform at high resolution levels. This result provides the weak-consistency of the functional estimators proposed.

2.1 Methodology

From equation (1), when $|\widetilde{\omega}| \to 0$,

$$\int_{\varepsilon(\widetilde{\omega})} C_1 |\omega|^{-2\nu} \prod_{i=1}^{d} |\lambda_i^0|^{-2\beta_i} \psi_{j:k}(\omega) d\omega \sim 2^{-j(-2\nu+1)} C(\psi, \boldsymbol{\lambda}^0).$$

Here, $C(\psi, \boldsymbol{\lambda}^0)$ represents a constant depending on the wavelet basis chosen and on the fixed spatial frequency value $\boldsymbol{\lambda}^0$. A similar behavior is displayed for $|\widetilde{\lambda}_i| \to 0$, $i = 1, \dots, d$ ($\sim 2^{-j(-2\beta_i+1)}$ $C(\psi, \omega^0, \dots, \lambda_{i-1}^0, \lambda_{i+1}^0, \dots, \lambda_d^0)$, for $C(\psi, \omega^0, \dots, \lambda_{i-1}^0, \lambda_{i+1}^0, \dots, \lambda_d^0)$ being a constant depending on the wavelet basis chosen, and the fixed frequency values $\omega^0, \dots, \lambda_{i-1}^0$, $\lambda_{i+1}^0, \dots$, λ_d^0 in a neighborhood of zero frequency). The long-range dependence parameter estimates are then computed by applying linear regression, from the square of the directional log-wavelet transform (at high resolution levels) of process $\hat{X}$ evaluated at different fixed spectral marginal values in a neighborhood of zero-frequency. Specifically, two functional estimation algorithms are designed, which respectively correspond to averaging and non-averaging the square of directional spectral curves in a zero-frequency neighborhood sequence (before applying the one-dimensional wavelet transform to such spectral curves, see, [3]).

3 Simulations

Spatiotemporal process X is defined as a Gaussian stationary process with spectral density given by

$$\begin{aligned} f_{X_1}(\omega, \lambda_1, \lambda_2) &= \left[\frac{1}{(1+|\omega|^2)^{\frac{\alpha_1}{2}}}\right] \left[\frac{1}{(1+|\lambda_1|^2)^{\frac{\alpha_2}{2}}}\right] \left[\frac{1}{(1+|\lambda_2|^2)^{\frac{\alpha_3}{2}}}\right] \\ &\times \ |\omega|^{-2\nu} |\lambda_1|^{-2\beta_1} |\lambda_2|^{-2\beta_2}, \end{aligned} \tag{2}$$

with $\alpha_i \in (0,1)$, $i = 1,2,3$. Functional spectral data are constructed from $256 \times 256 \times 256$ frequency points belonging to the interval $(\omega, \lambda_1, \lambda_2) \in [-127.5 * 10^{-8}, 127.5 * 10^{-8}]^3$, with discretization step size 10^{-8}. The simulation study is developed considering the following two structural parameter scenarios

Case I:	$\nu = 0.3$,	$\beta_1 = 0.375$,	$\beta_2 = 0.45$,	$\alpha_1 = 0.2$,	$\alpha_2 = 0.3$,	$\alpha_3 = 0.4$,
Case II:	$\nu = 0.15$,	$\beta_1 = 0.2$,	$\beta_2 = 0.25$,	$\alpha_1 = 0.2$,	$\alpha_2 = 0.3$,	$\alpha_3 = 0.4$.

The results obtained after implementation of the above-referred two functional estimation algorithms are displayed in Figures 1 and 2, where the three (temporal and spatial) long-range dependence parameter estimate sequences are represented. The spectral curve sample sizes considered are $n = 1, 4, 9, 16, 25, 36, 49, 64, 81, 100$, at temporal and spatial directions. Specifically, Figures 1 and 2 show $\hat{\nu}$, $\hat{\beta_1}$, and $\hat{\beta_2}$ values on top and standard deviations on bottom (blue, red and black line, respectively). Parameter values for ν, β_1, β_2 are displayed with dotted blue, red and black line, respectively.

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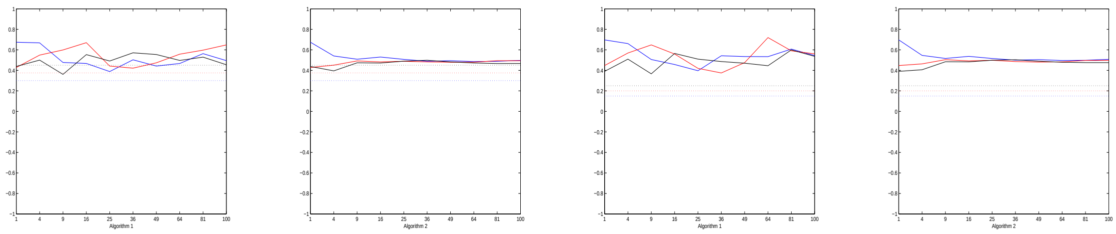

Figure 1: Functional estimates and real parameter values for case I (left) and case II (right).

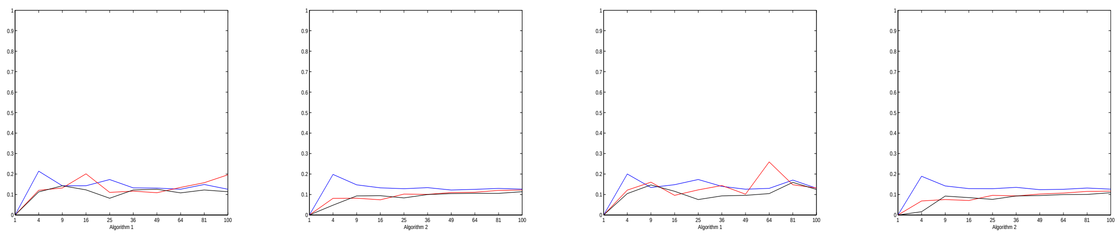

Figure 2: Standard deviations of $\hat{\nu}$, $\hat{\beta_1}$, and $\hat{\beta_2}$ estimators for case I and II.

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