First-Price Winner-Takes-All Contests*

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Abstract

We introduce a unifying model for contests with perfect discrimination and show that it can be used to model many well known economic situations such as auctions, Bertrand competition, politically contestable rents and transfers, tax competition, litigation problems. Furthermore, we hope that the generality of our model can be used to study models of contests in settings in which they have not been applied yet. Our main result is a classification of the set of Nash equilibria of first-price winner-takes-all contests with complete information. Finally, we discuss the implications of our results in each one of the specific models.

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1 Introduction

Models of contests are pervasive in the economic literature. Among them, maybe the models of auctions are the most studied ones, probably because of the simplicity of the model and the richness of its applications. Also the models of Bertrand competition fall within the literature on contests; the consumers being the contestable good. But there are many more situations where different agents are engaged in some competition with the common objective of getting some prize: rent-seeking, political campaigns, patent races, R&D, political lobbying, war of attrition models, tax competition,...

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In this paper we present a model that encompasses many of the situations quoted above. Therefore, we provide a formal framework that might allow to improve the comprehension of the different models and the connections between them. Although most connections between the different models have already been established by the specialized literature, we go one step further and formally place the models under the same umbrella.[1] Hopefully, future applications of contests can also benefit from the generality of our unifying model and the results we present here.

The contests we unify have the following structure. There is a set of agents who want to get a prize and each of them has to choose the effort (investment) he wants to make in order to get the prize. Efforts are chosen independently. Then, for each player, depending on his productivity, his effort translates into a certain effect. In the end, the prize is awarded to the agent who has achieved the highest effect, provided that there are no ties. In case of a tie, some tie-breaking rule has to be specified to "share" the prize among the winners. So note that, once the game is over, we can split the players among losers and winners; *i.e.,* following Hilman and Riley (1989), we say that there is *perfect discrimination*.[2] Now, we present one more property that characterizes the models of contests we deal with in this paper. Essentially, the payoff of each agent at the end of the game only depends on two things: i) his own effort and ii) if he is a winner or a loser. These kind of contests are normally called *first-price winner-takes-all contests*, hereafter FP-WTA contests. It is important to note that with the previous property we have excluded, for instance, both war of attrition models and second-price auctions; in both of them, the payoff of the winners depends on the effort made by the losers and not on their own effort.

The main result of this paper provides, under mild assumptions, a closed form characterization of the whole set of Nash equilibria and of the equilibrium payoffs of FP-WTA contests with complete information. Thus, even though every FP-WTA contest is a game with discontinuous payoffs, the approach we take here is different from the one taken in the literature initiated in Dasgupta and Maskin (1986). They look for conditions that ensure the existence of a Nash equilibrium in a given game with discontinuous payoffs, not for characterizations of the set of equilibria.

A related research has been independently developed in Siegel (2006, 2009). In Siegel (2009) the author extends the standard characterizations of the equilibrium payoffs to a setting similar to ours, but allowing for contests with more than one prize. Further, in Siegel (2006), the author also gets an equilibrium characterization with the aid of some extra assumptions, which are neither weaker non stronger than the ones we present here.

Our framework allows to model a wide range of asymmetries among the agents and study their impact in the underlying economy. As already argued in Siegel (2009), by

[1]Moldovanu and Sela (2001) make a detailed review of these connections in the introduction to their paper.

[2]In Baye and Hoppe (2003) the authors do a similar exercise to the one we present here and formally establish the strategic equivalence of three models of contests that do not assume perfect discrimination. Also, refer to Skaperdas (1996) and Cornes and Hartley (2005) for two papers without perfect discrimination (on the set of winners).

assuming complete information, one can study the effect of purely economic asymmetries independently of that of informational ones. Moreover, we allow the nature of efforts to be different across players. For instance, in a litigation problem, the efforts of prosecutor and defense attorneys to win a trial might go in very different directions. All that matters for our model are the effects the agents achieve in return to their efforts. The player who wins is the one with the highest effect, which needs not coincide with the one with the highest effort.[3]

The paper is structured as follows. In Section 2 we enumerate various models of contests. In Section 3 we present our unifying model and formally define our general model for FP-WTA contests. In Sections 4 and 5 we characterize the set of Nash equilibria of FP-WTA contests and discuss the implications of our results within the specific models we generalize. Finally, in Section 6 we briefly discuss some directions in which our model can be used to analyze new economic situations.

2 Some Models of Contests

Here we present an enumeration of several models our unifying approach can account for. We refer the reader to González-Díaz (2010) for a more detailed revision of these models.

FPA: First price auctions.

APA: All-pay auctions (Baye et al., 1990, 1996). They have been applied to a wide variety of environments, the most representative being political campaigning and lobbying processes (Baye et al., 1993; Konrad, 2004; Sahuget and Persico, 2006; Che and Gale, 1998, 2006; Kaplan and Wettstein, 2006). Other applications are politically contestable rents (Hilman and Riley, 1989) and tournaments (Groh et al., 2003).

PCT: Politically contestable transfers (Hilman and Samet, 1987; Hilman and Riley, 1989).

BM: Standard price competition: Bertrand model (Tirole, 1988, Chapter 5).

MS: Varian's model of sales: price competition with loyal customers (Varian, 1980; Narasimham, 1988; Baye and de Vries, 1992; Baye et al., 1992; Deneckere et al., 1992).

LS: Litigation systems (Baye et al., 2005).

TC: Tax competition (Hatfield, 2006; Wang, 2004).

MM: Price competition between market makers (Dennert, 1993).

TG: (Silent) timing games (Hamers, 1993; González-Díaz et al., 2007).

[3]Indeed, Baye et al. (2005) already pointed out that it would be interesting to extend their results to situations where such asymmetries in the returns for the efforts are present (yet, they do not assume complete information, being this extension far more difficult in their setting).

3 The Unifying Model

Now we present a general model for which the ones enumerated above become specific cases. Despite of the extra generality of our model, we show that the equilibrium results of the papers quoted in Section 2 carry out.

There is a set of agents that want to get a prize. In order to do so, each of them has to make some effort (investment). These efforts are chosen simultaneously and independently. The effort of each player translates into a certain effect that is given by his productivity function. Finally, the prize is awarded to the agent whose effort leads to the highest effect. In case of a tie, the prize is shared according to some rule. The results we present in Sections 4 and 5 show that, with a lot of generality, the selected tie-breaking rule is not relevant for the equilibrium analysis.

The set of players N is assumed to have at least two players and is fixed throughout the paper. Let 2^N denote the set of all possible subsets of N and, for each $S \subseteq N$, $|S|$ denotes its cardinality. Informally, in a *FP-WTA contest* each player $i \in N$ chooses the *effort* he wants to make, that is, a number in $[m_i, M_i]$, where $m_i \in \mathbb{R}$, $M_i \in \mathbb{R} \cup \{+\infty\}$, and $M_i > m_i$. Then, each effort produces an *effect* in $[m, M]$, where $m \in \mathbb{R}$, $M \in \mathbb{R} \cup \{+\infty\}$, and $M > m$.[4] Finally, the prize is awarded to the player that has achieved the highest effect (ties are discussed below). Each player, depending on his effort and on whether he has achieved the prize or not, gets a certain payoff. Before formally defining a FP-WTA contest, we define a *FP-WTA form* as a 4-tuple $(\{r_i\}_{i\in N}, \{b_i\}_{i\in N}, \{p_i\}_{i\in N}, \{T_i\}_{i\in N})$ as follows:

- *Productivity functions.* For each $i \in N$, there is a one-to-one strictly increasing mapping $r_i : [m_i, M_i] \to [m, M]$ that returns, for each effort, the produced effect.[5] The literature on contests has essentially focused on models where the r_i functions coincide with the identity, *i.e.*, efforts and effects are the same thing.

- *Base payoff functions.* For each $i \in N$, there is a weakly decreasing and continuous function $b_i : [m_i, M_i] \to \mathbb{R}$. For each level of effort $e \in [m_i, M_i]$, $b_i(e)$ denotes the payoff (cost) to player i for an effort e.

- *Prize payoff functions.* For each $i \in N$, there is a weakly decreasing and continuous function $p_i : [m_i, M_i] \to \mathbb{R}$ with $p_i(m_i) > 0$. For each level of effort $e \in [m_i, M_i]$, $p_i(e)$ denotes the extra payoff of player i when he gets the prize with effort e.

- *Tie payoff functions.* For each $i \in N$, $T_i : [m_i, M_i] \times 2^N \backslash \{\emptyset\} \to \mathbb{R}$, determines i's "share of the prize" when there is a tie. The element in $[m_i, M_i]$ denotes the effort made by player i and the subset of N is the set of players with the highest effect. The T_i functions have the following properties:

  T1) $T_i(e, \{i\}) = p_i(e)$, *i.e.*, if i is the only winner he gets the prize.

[4] We are making the following abuse of notation. If $M = +\infty$, then $[m, M] := [m, +\infty)$, and the same for the $[m_i, M_i]$ intervals.

[5] Note that the definition of the r_i functions implies that they are continuous and, moreover, it also implies that either $M < \infty$ and, for each $i \in N, M_i < \infty$ or, for each $i \in N, M = M_i = \infty$.

T2) Let $S \subsetneq N$ be such that $i \notin S$. Then, $T_i(e, S) = 0$, *i.e.*, if i is not a winner he gets no extra payoff.

T3) Let, $\epsilon \in [m, M]$ and $S \in 2^N \backslash \{\emptyset\}$, with $|S| > 1$. For each $i \in S$, let $e_i := r_i^{-1}(\epsilon)$. Then,

i) for each $i \in S$, if $p_i(e_i) \geq 0$, then $0 \leq T_i(e_i, S) \leq p_i(e_i)$. Winners get at most their prize payoff at e_i, provided that it is non negative,

ii) for each $i \in S$, if $p_i(e_i) < 0$, then $T_i(e_i, S) < 0$. If i's valuation of the prize at e_i is negative, he must get something negative,[6]

iii) if $\sum_{i \in S} p_i(e_i) > 0$, then there is $j \in S$ such that $T_j(e_j, S) < p_j(e_j)$. If the sum of the prize payoffs of the tied players at e is positive, then at least one of them would be better off if he were the only winner.

Note that not only the productivity functions can be different for the different players. Also base and prize payoff functions are player dependent. Hence, situations where the valuation of the prize may be different across players are included. Note as well that we allow for constant prize functions, *i.e.*, the effort does not necessarily affect the extra payoff when getting the prize. We have defined tie functions in a very general way, indeed, not only natural shares of the prize but also unnatural ones can be defined within our family of tie functions. We show below that most of the results do not depend on the chosen tie functions as far as they satisfy properties T1-T3.

Remark 1. *Since the r_i mappings are one to one, all the players can achieve any effect in $[m, M]$. This implies that, for all the players, zero effort leads to zero effect. Since the r_i functions are intended to reflect asymmetries among the productivities of the players, it might seem awkward that they do not allow for the initial situations to be different. Similarly, it also seems natural to allow for situations in which some players cannot achieve effect M. Yet, there is a lot of freedom left in the model to (essentially) cover the above asymmetries. For instance, even though all the players can produce effect M, the associated cost might vary across players, that is, the functions b_i, p_i, and T_i can be used to model situations in which M is, in practice, unattainable for certain players.*

Remark 2. *Most of the literature on contests has restricted attention to linear functions. Here we do not impose any such restriction, neither in the productivity functions, nor on the base or prize payoff functions. Yet, it is worth to mention that there are already a few papers in contests in which non-linear functional forms have been considered; refer, for instance, to Golding and Slutsky (1999), Kaplan et al. (2003), and Siegel (2006, 2009).*

Let $f := (\{r_i\}_{i \in N}, \{b_i\}_{i \in N}, \{p_i\}_{i \in N}, \{T_i\}_{i \in N})$ be a FP-WTA form. Now, the associated *FP-WTA contest with pure strategies* for the players in N, denoted by C^f_{pure},

[6] Here we even allow for situations in which players "lose less" when they are tied than they would lose being alone. Note that this is the case, for instance, in first price auctions when players are tied at bids that exceed their valuations.

is defined as $C^f_{\text{pure}} := (\{E_i\}_{i\in N}, \{u_i\}_{i\in N})$, where, for each $i \in N$, $E_i := [m_i, M_i]$ and the payoff functions are defined as follows. For each effort profile $\sigma = (e_1, \ldots, e_n)$, let w^σ denote the set of winners, *i.e.*, $w^\sigma := \operatorname{argmax}_{i\in N}\{r_i(e_i)\}$. Now, for each $i \in N$ and each $\sigma = (e_1, \ldots, e_n) \in [m, M]^n$, $u_i(\sigma) := b_i(e_i) + T_i(e_i, w^\sigma)$. Note that, in the game C^f_{pure}, for each $i \in N$, $b_i(m_i)$ can be interpreted as the minimum right of player i, since, regardless of what the other players do, player i can always ensure himself $b_i(m_i)$ with the strategy m_i. Hereafter we restrict to FP-WTA contests with complete information.

From the above definitions, both the base and prize payoff functions are weakly decreasing. Nonetheless, it is natural to assume that either the b_i functions or the p_i functions are strictly decreasing so that we ensure that the payoff functions are sensitive to the invested efforts.

Assumption: ALL-PAY. For each $i \in N$, the function $b_i(\cdot)$ is strictly decreasing.

Assumption: WINNER-PAYS. For each $i \in N$, the function $p_i(\cdot)$ is strictly decreasing.

Depending on whether we assume ALL-PAY or WINNER-PAYS, a relevant part of the contest at hand is changed. Under ALL-PAY, the payoff of the players is strictly decreasing in e, regardless of whether they get the prize or not; models APA, PCT, MS, LS, MM, and TG fall within this category. On the other hand, under WINNER-PAYS, only the payoff achieved when getting the prize has to be strictly decreasing in e; this situation corresponds with the FPA, BM, and TC models.[7]

Now, for each $i \in N$, let $\bar{e}_i := \sup\{e \in [m_i, M_i] \ : \ b_i(m_i) \leq b_i(e) + p_i(e)\}$. The interpretation of the $\bar{e}_i$ is as follows. If $\bar{e}_i < M_i$, then $b_i(m_i) = b_i(\bar{e}_i) + p_i(\bar{e}_i)$. That is, $\bar{e}_i$ is an upper bound for the effort that i is willing to make because higher efforts are weakly dominated by m_i and, under ALL-PAY, they are strictly dominated. Hereafter we assume, without loss of generality, that players are ordered such that $i < j$ implies that $r_i(\bar{e}_i) \geq r_j(\bar{e}_j)$. That is, the players with lower indices are the ones willing to produce higher effects.

Assumption: *M*-BOUNDING. For each $i \in N$, $\bar{e}_i < M_i$.

Note that M-BOUNDING is satisfied by all the models with the exception of TC. Moreover, this requirement becomes very natural when translated into the different settings. Next result shows that, under ALL-PAY and M-BOUNDING, FP-WTA contests do not have Nash equilibria (in pure strategies).

Proposition 1. *If the FP-WTA contest C^f_{pure} satisfies* ALL-PAY *and* M-BOUNDING, *then it does not have any Nash equilibrium.*

Proof. Suppose $\sigma = (e_1, \ldots, e_n) \in \prod_{i\in N}[m_i, M_i]$ is a Nash equilibrium of C^f_{pure}. By ALL-PAY and M-BOUNDING, strategies above $\bar{e}_i$ are strictly dominated for player i.

[7] In BM we assume that the demand and cost functions satisfy that $(\bar{p} - e)D(\bar{p} - e) - c_i(D(\bar{p} - e))$ is strictly decreasing in e. We briefly discuss about the necessity of this assumption in Section 6.

Hence, for each $i \in N$, $e_i \leq \bar{e}_i < M_i$. By ALL-PAY, for each $i \in N$, the function $b_i(\cdot) + p_i(\cdot)$ is strictly decreasing. Hence, if $|w^\sigma| = 1$, the winner would be better off by doing less effort, but still ensuring himself to be the only winner (recall that the r_i functions are continuous). Hence, $|w^\sigma| > 1$, *i.e.*, there is a tie at some $\epsilon \in [m, M)$. We distinguish two cases:

$\boldsymbol{\sum_{i \in w^\sigma} p_i(e_i) > 0}$**:** By T3, there is $i \in w^\sigma$ such that $T_i(e_i, w^\sigma) < p_i(e_i)$. Now, since functions $b_i(\cdot)$ and $p_i(\cdot)$ are continuous, there is $\varepsilon > 0$ such that $u_i(\sigma) = b_i(e_i) + T_i(e_i, w^\sigma) < b_i(e_i + \varepsilon) + p_i(e_i + \varepsilon) = u_i(\sigma_{-i}, e_i + \varepsilon)$. Hence, player i would deviate.

$\boldsymbol{\sum_{i \in w^\sigma} p_i(e_i) \leq 0}$**:** Clearly, for each $i \in w^\sigma$, $e_i > m_i$ and there is $i \in w^\sigma$ such that $p_i(e_i) \leq 0$. By T3, $T_i(e_i, w^\sigma) \leq 0$. Hence, $u_i(\sigma) = b_i(e_i) + T_i(e_i, w^\sigma) \leq b_i(e_i)$. Since $b_i(\cdot)$ is strictly decreasing, player i would be better off by playing m_i and getting $b_i(m_i) > b_i(e_i)$. □

For an appropriate analysis of FP-WTA contests we need mixed strategies. First, we introduce some notation concerning distribution functions. Let $F : \mathbb{R} \to [0, 1]$ be such that i) F is non decreasing, ii) F is right-continuous, and iii) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$. For each $x \in \mathbb{R}$, $F(x^-) := \lim_{y \to x,\, y < x} F(y)$. The support of F is $S(F) := \{x \in \mathbb{R} : \text{for each } a, b \in \mathbb{R} \text{ with } a < x < b,\ F(b^-) > F(a)\}$; $S(F)$ is a closed set. The set of *jumps* (discontinuities) of F is $J(F) := \{x \in \mathbb{R} : F(x) > F(x^-)\}$.

Formally, a *mixed strategy* of player i in a FP-WTA contest is a distribution function G such that $S(G) \subseteq [m_i, M_i]$. Suppose that player i chooses pure strategy e_i and all other players choose mixed strategies $\{G_j\}_{j \neq i}$. For each $j \neq i$, let $e_j := r_j^{-1}(r_i(e_i))$, *i.e.*, the effort j needs to exert to produce the same effect as i does with e_i. Then the expected payoff for player i is

$$u_i(G_1, \ldots, G_{i-1}, e_i, G_{i+1}, \ldots, G_n) = A + B + C,$$

where

$$\begin{aligned} A &= \prod_{j \neq i} G_j(e_j^-)(b_i(e_i) + p_i(e_i)), \quad B = \big(1 - \prod_{j \neq i} G_j(e_j)\big) b_i(e_i), \text{ and} \\ C &= \sum_{i \in S, S \subseteq N} \Big(\prod_{j \notin S} G_j(e_j^-) \prod_{j \in S \setminus \{i\}} \big(G_j(e_j) - G_j(e_j^-)\big) \Big) T_i(e_i, S). \end{aligned}$$

That is, A is the probability that i wins alone multiplied by the corresponding payoff, B is the probability that i does not win multiplied by the corresponding payoff, and C is the payoff originated in the different ties in which i can be involved.

If player i also chooses a mixed strategy G_i, whereas all other players stick to mixed strategies $\{G_j\}_{j \neq i}$, then the expected payoff for player i can be computed by the use of the Lebesgue-Stieltjes integral:

$$u_i(G_1, \ldots, G_n) = \int u_i(G_1, \ldots, G_{i-1}, e_i, G_{i+1}, \ldots, G_n) dG_i(e_i). \tag{1}$$

Note that, with a slight abuse of notation, the functions u_i do not only denote payoffs to players when pure strategies are played, but also when mixed strategies are used.[8]

Hence, given the FP-WTA form $f := (\{r_i\}_{i\in N}, \{b_i\}_{i\in N}, \{p_i\}_{i\in N}, \{T_i\}_{i\in N})$, the associated FP-WTA contest for the players in N, denoted by C^f, is defined as the pair $C^f := (\{X_i\}_{i\in N}, \{u_i\}_{i\in N})$, where for each $i \in N$, X_i is i's set of mixed strategies and i's (expected) payoff function is defined by Eq. (1).

Finally, for notational convenience, for each strategy profile $G = (G_1, \ldots, G_n) \in \prod_{i\in N} X_i$ and each $e_i \in [m_i, M_i]$, we denote the payoff $u_i(G_1, \ldots, G_{i-1}, e_i, G_{i+1}, \ldots, G_n)$ by $u_i^G(e_i)$. That is, $u_i^G(e_i)$ is the expected payoff of player i when he chooses the pure strategy e_i and all the other players act in accordance with G. Let $\mathcal{C}$ denote the class of FP-WTA contests with mixed strategies.

3.1 Direct FP-WTA contests

Let $C^f \in \mathcal{C}$. Then, C^f is a *direct FP-WTA contest* if, for each $i \in N$, $m_i = m$, $M_i = M$, and, for each $e_i \in [m, M]$, $r_i(e_i) = e_i$. In direct FP-WTA contests there is no heterogeneity in the productivities of the players. The sets of strategies are now common for all the players and the player that makes the highest effort is the winner. Next proposition states that each FP-WTA contest is strategically equivalent to the direct FP-WTA contest in which the strategies of the players consist in directly choosing the effects they want to produce instead of choosing efforts.

Proposition 2. *Let $f := (\{r_i\}_{i\in N}, \{b_i\}_{i\in N}, \{p_i\}_{i\in N}, \{T_i\}_{i\in N})$ be a FP-WTA form and C^f the corresponding FP-WTA contest. Let*

$$f' := (\{\mathrm{id}(\cdot)\}_{i\in N}, \{b_i(r_i^{-1}(\cdot))\}_{i\in N}, \{p_i(r_i^{-1}(\cdot))\}_{i\in N}, \{T_i(r_i^{-1}(\cdot), \cdot)\}_{i\in N}).$$

Then, $C^{f'}$ is a direct FP-WTA contest and the games C^f and $C^{f'}$ are strategically equivalent. That is, for each strategy profile $(e_1, \ldots, e_n)$ in C^f, the strategy profile $(r_1(e_1), \ldots, r_n(e_n))$ in $C^{f'}$ leads to the same payoffs and, conversely, for each strategy profile $(\epsilon_1, \ldots, \epsilon_n)$ in $C^{f'}$, the strategy profile $(r_1^{-1}(\epsilon_1), \ldots, r_n^{-1}(\epsilon_n))$ in C^f leads to the same payoffs.

Proof. Straightforward from the definitions of the payoff functions of the FP-WTA contests. □

Proposition 2 implies that the productivity asymmetries, originally modeled via the r_i functions, can be incorporated in the base, prize, and tie payoff functions.

[8] We have to mention a subtle technical detail. To rigorously define the Lebesgue-Stieltjes integrals with respect to the distribution functions we should integrate over $\mathbb{R}$. Hence, we should also define payoff functions over $\mathbb{R}$. But note that, no matter the extension of the u_i functions we consider, the integrals over $[m_i, M_i]$ remain the same (just because the support of the mixed strategies is restricted to $[m_i, M_i]$). If we do not consider integrals defined over $\mathbb{R}$, then we might have problems when calculating expected payoffs of mixed strategies that put positive probability at m_i. Hence, we use $\int u_i(G_1, \ldots, G_{i-1}, e, G_{i+1}, \ldots, G_n) dG_i(e)$ to mean $\int_{\mathbb{R}} u_i^*(G_1, \ldots, G_{i-1}, e_i, G_{i+1}, \ldots, G_n) dG_i(e_i)$, where u_i^* can be any arbitrarily chosen extension of u_i to $\mathbb{R}$.

Note that the FP-WTA form of a direct game can be defined just as a triple $f := (\{b_i\}_{i\in N}, \{p_i\}_{i\in N}, \{T_i\}_{i\in N})$.

Hereafter, we restrict attention to direct FP-WTA contests and Proposition 2 ensures that all the characterization results we derive in the following sections also apply to general FP-WTA contests. Moreover, we assume, without loss of generality, that $m = 0$, that is, players have to make non-negative efforts.[9] Let $\mathcal{C}^*$ denote the class of direct FP-WTA contests such that $m = 0$ and let $\mathcal{G}$ denote the set of possible mixed strategies, *i.e.*, distribution functions whose support is contained in $[0, M]$.

4 Equilibrium Characterization under ALL-PAY

We assume ALL-PAY throughout this section, where we present the main results we get for the models in which it holds. Hence, regarding the models presented in Section 2, they apply to APA, PCT, MS, LS, MM, and TG. Our results extend, to our unifying model, the strongest equilibrium results that had already been proved for any of the specific models. Since our model is more general than the existing ones, new proofs are needed. Nonetheless, the main intuitions underlying the results remain the same and, therefore, we have omitted most of the proofs. The reader interested in the details is referred to González-Díaz (2010), where the author presents a series of results that parallels and extends the approaches of Baye et al. (1996) and González-Díaz et al. (2007).

First, note that if both ALL-PAY and M-BOUNDING are assumed and $G \in \mathcal{G}^n$ is a Nash equilibrium of C^f, then we have that, for each $i \in N$, $S(G_i) \subseteq [0, \bar{e}_i]$. Moreover, once player 1 knows that none of the other players puts positive probability above $\bar{e}_2$, the efforts in $(\bar{e}_2, \bar{e}_1]$ are strictly dominated for him. Hence, $S(G_1) \subseteq [0, \bar{e}_2]$.

Next Lemma shows that in a Nash equilibrium of C^f no one assigns positive probability to any effort different from 0. This leads to a remarkable simplification in the expression of $u_i^G(e)$.

Lemma 1. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Then, for each $i \in N$,*

i) $J(G_i) \cap (0, M) = \emptyset$,

ii) for each $e \in (0, M)$, $u_i^G(e) = b_i(e) + \prod_{j \neq i} G_j(e) p_i(e)$,

*iii) moreover, under M-*BOUNDING*, we have that* $M \notin J(G_i)$.

Thus, under ALL-PAY, since ties have probability 0 in $(0, M)$, the interpretation of $u_i^G(e)$ becomes very simple.

Now, we introduce a last element into the model. For each $i \in N$ and each $e \in [0, M]$, $b_i(0) - b_i(e)$ is a way of measuring the impact of effort e in i's base payoff; similarly, $p_i(e)$ measures the impact of e in i's prize payoff (if this is finally achieved).

[9] Let $f = (\{b_i\}_{i\in N}, \{p_i\}_{i\in N}, \{T_i\}_{i\in N})$ be an direct FP-WTA form such that, in C^f, $m \neq 0$. Then, we just need to consider the FP-WTA form $f' = (\{b'_i\}_{i\in N}, \{p'_i\}_{i\in N}, \{T'_i\}_{i\in N})$, where each $b'_i : [0, M - m] \to \mathbb{R}$ is defined by $b'_i(e) := b_i(e + m)$, and similarly for the p'_i and T'_i functions. The games C^f and $C^{f'}$ are completely equivalent from the strategic point of view.

Now, for each $i \in N$, let $I_i : [0, \bar{e}_i] \to \mathbb{R}$ be the *impact function* of player i, defined as $I_i(e) := (b_i(0) - b_i(e))/p_i(e)$. The functions I_i provide a way of measuring the aggregate impact of an effort e in i's potential payoff; being this impact measured with respect to i's minimum right $b_i(0)$. Differently, $I_i(e)$ represents the trade-off player i faces when choosing effort e: on the one hand, he faces a loss $b_i(0) - b_i(e)$ but, on the other, he might get an extra payoff of $p_i(e)$. Note that, under ALL-PAY, for each $i \in N$, $I_i(0) = 0$, I_i is a strictly increasing function, and $I_i(e) = 1$ if and only if $e = \bar{e}_i$. Recall that, under ALL-PAY, efforts above $\bar{e}_i$ are strictly dominated for player i.

For the characterization result we present below we assume M-BOUNDING. Moreover, we need to introduce two more assumptions: NO-CROSSING and NO-CROSSING*. These assumptions are complements to M-BOUNDING. The relevance of these assumptions is discussed in Subsection 4.1.

Assumption: NO-CROSSING. For each pair $i, j \in N$, let $\hat{e} = \min\{\bar{e}_i, \bar{e}_j\}$. Then, if there is $e^* \in (0, \hat{e}]$ such that $I_i(e^*) < I_j(e^*)$ then, for each $e \in (0, \hat{e}]$, $I_i(e) < I_j(e)$.

Informally, NO-CROSSING says that, if the impact of a certain effort is higher for i than for j, then the same relation holds for every effort. Note that it also implies that, if there is $e^* \in (0, \hat{e}]$ such that $I_i(e^*) = I_j(e^*)$ then, for each $e \in [0, \hat{e}]$, $I_i(e) = I_j(e)$.[10]

NO-CROSSING is eventually satisfied by all the models at hand in this section and its expression becomes very natural when translated into the different models.[11] Recall that $i < j$ implies that $\bar{e}_i \geq \bar{e}_j$ and now, by NO-CROSSING, for each $e \in [0, \bar{e}_j]$, $I_i(e) \leq I_j(e)$.

Now we introduce the last assumption in the model. Suppose that $\bar{e}_1 > \bar{e}_2$. Since player 1 knows that no one but him is willing to put any effort above $\bar{e}_2$ (they are strictly dominated strategies), he can ensure himself a payoff as close to $b_1(\bar{e}_2) + p_1(\bar{e}_2)$ as desired. So, somehow, he can ensure himself a minimum right of $b_1(\bar{e}_2) + p_1(\bar{e}_2)$. Now, using the idea of the definition of the impact functions we define $I_1^* : [0, \bar{e}_1] \to \mathbb{R}$ by $I_1^*(e) := (b_1(\bar{e}_2) + p_1(\bar{e}_2) - b_1(e))/p_1(e)$. That is, the impact of an effort e in player 1's potential payoff but with respect to $b_1(\bar{e}_2) + p_1(\bar{e}_2)$ instead of $b_1(0)$. Since we are assuming ALL-PAY, $I_1^*(\cdot)$ is strictly increasing. Note that $I_1^*(0) > 0 = I_2(0)$ and $I_1^*(\bar{e}_2) = 1 = I_2(\bar{e}_2)$. Next assumption just says that functions $I_1^*(\cdot)$ and $I_2(\cdot)$ cannot cross below $\bar{e}_2$.

Assumption: NO-CROSSING*. Assumption NO-CROSSING is met and, moreover, if $\bar{e}_1 > \bar{e}_2$ and $\bar{e}_2 = \bar{e}_3$, then, for each $e \in (0, \bar{e}_2)$, $I_2(e) < I_1^*(e)$.

Assumption NO-CROSSING* follows the same motivation than NO-CROSSING and only is more restrictive if $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$. Again, the reader can check that the models discussed above satisfy this property (see footnote 11).

[10] This non-crossing property is quite standard in the literature on contests. In the recent years there have been some efforts trying to dispense with it in different settings; remarkably, Siegel (2006, 2009) introduces a general model of contests, similar to the one we present here, and develops his analysis without assuming any kind of non-crossing property. Yet, he needs some other assumptions to characterize the set of equilibria in his setting.

[11] Model MS might not meet the assumption if we allow simultaneously for different numbers of loyal consumers and different cost functions for the different firms.

Let $G^* \in \mathcal{G}^n$ be the strategy profile defined for players 1, 2, and each $i \in N \backslash \{1,2\}$ as follows:

$$G_1^*(e) = \begin{cases} 0 & e < 0 \\ I_2(e) & 0 \leq e \leq \bar{e}_2 \\ 1 & e > \bar{e}_2 \end{cases}, \quad G_2^*(e) = \begin{cases} 0 & e < 0 \\ I_1^*(e) & 0 \leq e \leq \bar{e}_2 \\ 1 & e > \bar{e}_2 \end{cases}, \quad G_i^*(e) = \begin{cases} 0 & e < 0 \\ 1 & e \geq 0 \end{cases}.$$

The payoffs associated with G^* are $b_1(\bar{e}_2) + p_1(\bar{e}_2)$ for player 1 and, for each $i \neq 1$, $b_i(0)$. If $\bar{e}_1 > \bar{e}_2$, then $b_1(\bar{e}_2) + p_1(\bar{e}_2) > b_1(0)$. On the other hand, if $\bar{e}_1 = \bar{e}_2$, then $b_1(\bar{e}_2)+p_1(\bar{e}_2) = b_1(0)$ and $I_1^*(\cdot) = I_1(\cdot) = I_2(\cdot)$. Hence, in the latter case players 1 and 2 play the same mixed strategy (and get the same payoffs). Now, we are ready to present the characterization result: under Assumption ALL-PAY, the use of mixed strategies allows to recover the existence of the Nash equilibrium with few extra requirements.

Theorem 1. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY, M-BOUNDING, *and* NO-CROSSING*. *Then,*

i) If either $n = 2$ or $\bar{e}_1 \geq \bar{e}_2 > \bar{e}_3$, then G^* is the unique Nash equibrium of C^f.

ii) Otherwise, C^f has a continuum of Nash equibria and G^* is one of them.

All the Nash equilibria of C^f lead to the same payoffs: $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$ and, for each $i \neq 1$, $\eta_i = b_i(0)$.

The Appendix contains a classification of the Nash equilibria of statement ii) in Theorem 1.[12] These equilibria can be described as follows:[13]

Case 1: $\bar{e}_1 = \bar{e}_2 = \bar{e}_3$. For each Nash equilibrium the following statements hold: **i)** players $i \in N$ such that $\bar{e}_2 > \bar{e}_i$ choose effort 0 with probability one, **ii)** at least two players play strategies with support $[0, \bar{e}_2]$, **iii)** each other player i has a strategy with support $\{0\} \cup [d_i, \bar{e}_2]$ and chooses 0 with positive probability whenever $d_i > 0$ ($d_i = \bar{e}_2$ means that i chooses effort 0 with probability one), and **iv)** whenever an effort $e > 0$ belongs to the support of two or more players, their distribution functions coincide at e.

Case 2: $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$. For each NE the following statements hold: **i)** players $i \in N$ such that $\bar{e}_2 > \bar{e}_i$ choose effort 0 with probability one, **ii)** the support of player 1 contains both 0 and $\bar{e}_2$ and either it coincides with the interval $[0, \bar{e}_2]$ or it is strictly contained in it, **iii)** each player $i \neq 1$ chooses 0 with positive probability and the support of his strategy is $\{0\} \cup [d_i, \bar{e}_2]$, with $d_i = 0$ for at least one player $i \neq 1$ ($d_i = \bar{e}_2$ means that i chooses effort 0 with probability one), **iv)** whenever an effort $e > 0$ belongs to the support of at least 2 players different from player 1, their distribution functions coincide at e, and v) for each $e \in [0, \bar{e}_2]$, the product at e of the distribution functions different from that of player 1 is $I_1^*(e)$; intuitively, the players such that $\bar{e}_i = \bar{e}_2$ are substitutes for each other. There is a continuum

[12]The equilibria of the models discussed in this section are particular cases of Theorem 1 and are described by the expressions presented in the Appendix.

[13]With the exception of the possible existence in our general setting of the *gap equilibria* defined below, our classification is the extension of the one made in Baye et al. (1996) to our general framework.

of equilibria where i)-v) hold and, moreover, ii) holds with the support of player 1 being $[0, \bar{e}_2]$ and we get a closed-form characterization of these equilibria. We refer to the Nash equilibria in which i)-v) hold but the support of player 1 is strictly contained in $[0, \bar{e}_2]$ as *gap equilibria*. That is, in a gap equilibrium the strategy profile of player 1 is constant in some interval inside $[0, \bar{e}_2]$, *i.e.*, it has one or more gaps. Providing a closed form characterization of gap equilibria without further assumptions is a hard problem, since in such equilibria the support of a player's strategy may not be "well-behaved" (formally, "constructible" in the sense of Siegel (2006)[14]). The possibility of gap equilibria arises because of the non-linearity of the payoff functions. Indeed, Baye et al. (1996) show that such equilibria do not exist when the payoffs are linear. Finally, we note that in Siegel (2006), where no NO-CROSSING-like property is assumed, gap equilibria are the main equilibria and not only a special case as they are in our framework.

4.1 Implications of the Characterization under ALL-PAY

A remarkable implication of Theorem 1 is the following: the characterization results presented in Baye et al. (1996) can be translated to any other of the models satisfying ALL-PAY, M-BOUNDING, and NO-CROSSING* and they remain valid.[15] Note that, with our general model, we have provided the appropriate language to do such translations and, moreover, to do them in a completely rigorous way.

More specifically, the implications of our characterization result are noteworthy in the MS model. It extends the results included in Narasimham (1988) from two firms to an arbitrary number. That is, even if the number of loyal consumers is different for the different firms, Theorem 1 can be applied regardless of the number of firms. Also the authors in Baye et al. (1992) assert that, as a by-product of their analysis, they generalize the results in Narasimham (1988) to an arbitrary number of firms. Remarkably, we can go even further and allow as well for different cost functions across firms and still apply Theorem 1; in this case we just have to ensure that NO-CROSSING and NO-CROSSING* and are still met. On the other hand, the existing results for models PCT and TG only refer to very specific configurations of the parameters. Theorem 1 extends them to any chosen configuration of the primitives of the two models.

Finally, it is worth to mention that in most of the models satisfying ALL-PAY, it is quite natural to consider asymmetric productivity functions. By Proposition 2 we know that our results can be applied, for instance, to study litigation problems where it is easier to one of the parties to prepare the case, political campaigning with incumbents, tournaments in which players have different skills,... Last but not least, patent races constitute another remarkable application of all-pay auctions in which it

[14] Siegel (2006) shows an example of a "non-constructible" equilibrium in which $S(G_1)$ coincides with the Cantor set.

[15] Yet, one observation has to be made. Namely, for the specific parameter case $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$, the extension of the closed form characterization in Baye et al. (1996) might not completely characterize the set of equilibria. We refer the reader to the Appendix for the details and discussion around this specific parameter configuration.

is quite natural to consider situations where the productivity functions differ across agents (different technologies, more qualified researchers,...).

Discussion of the Assumptions

Setting aside ALL-PAY, which is the assumption that characterizes the type of FP-WTA contests at hand in this Section, it is natural to wonder what happens if M-BOUNDING or NO-CROSSING* are not met. If M-BOUNDING is not satisfied it is possible that, in equilibrium, ties happen with positive probability at M (by Lemma 1 this was not possible under M-BOUNDING). Hence, the role of tie functions becomes more important now. Below, we briefly discuss some of the different possibilities for the Nash equilibria when M-BOUNDING is violated. In particular, the existence of pure Nash equilibria is sometimes recovered:[16]

i) If $M = \bar{e}_1 > \bar{e}_2$, we are almost in the same situation as before. If NO-CROSSING* is met, then the same result as in Theorem 1 still holds (with $\bar{e}_1 > \bar{e}_2$).

ii) If there is $S \subseteq N$, $|S| > 1$, such that $\bar{e}_i = M$ if and only if $i \in S$. Then, for each $S' \subseteq S$ such that

   a) for each $i \in S'$, $T_i(M, S') + b_i(M) \geq b_i(0)$ and

   b) for each $j \in S \backslash S'$, $T_j(M, S' \cup \{j\}) + b_j(M) \leq b_j(0)$,

   there is a Nash equilibrium in which players in S' put probability 1 at M and players in $N \backslash S'$ put probability 1 at 0.

iii) Moreover, depending on the tie functions and on to what extent and NO-CROSSING* holds for the different configurations of the $\bar{e}_i$ parameters, there can exist mixed Nash equilibria similar to those in Theorem 1.

Obtaining a characterization of the equilibria without assumption NO-CROSSING* is an open problem for most of the specific models that FP-WTA contests generalize. Recently, independent research by Siegel (2006, 2009) studies this problem in a unifying model similar to the one we present here. His model is more general because he allows for multiple prizes and does not assume NO-CROSSING*, but he restricts to constant prize functions (they are just valuations) and assumes ALL-PAY. He calls his games all-pay contests and obtains some noteworthy results concerning their equilibria. Siegel (2009) presents a characterization of the equilibrium payoffs and discusses its implications. Moreover, Siegel (2006) needs some extra assumptions to get a complete equilibrium characterization.

5 Equilibrium Characterization under WINNER-PAYS

We assume WINNER-PAYS throughout this Section. Then, we present two characterization results depending on whether M-BOUNDING is met or not. Hence, the result

[16] Now we do not try to give a characterization result as Theorem 1. This is because the set of Nash equilibria would depend on the specific configurations of the $\bar{e}_i$ parameters, on the tie payoff functions, and on whether NO-CROSSING and NO-CROSSING* are met or not. Hence, a clean result as Theorem 1 is not possible here (at least we have not been able to find it).

under M-BOUNDING applies to FPA and BM and the result when M-BOUNDING is not satisfied applies to TC.[17]

First of all, we introduce a preliminary result.

Lemma 2. *Let $C^f \in \mathcal{C}^*$. Assume* WINNER-PAYS. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Let $e \in [0, M]$ and let $i \in N$ be such that $\bar{e}_i > e$. Then, the probability that i participates in a tie at e is 0.*

Proof. Suppose that there is positive probability that i participates in a tie at e. Let $S \subseteq N$ be a coalition containing i for which a tie at e has positive probability. If $\sum_{j \in S} p_j(e) > 0$, by T3, there is $j \in S$ such that $T_j(e, S) < p_j(e)$. Since functions $b_j(\cdot)$ and $p_j(\cdot)$ are continuous, there is $\varepsilon > 0$ such that $b_j(e) + T_j(e, S) < b_j(e+\varepsilon) + p_j(e+\varepsilon)$. Now, it is easy to check that the same holds for $u_j^G(e)$. That is, there is $\varepsilon' > 0$ such that $u_j^G(e) < u_j^G(e + \varepsilon')$.[18] Hence, player j would be better off by moving his probability in e to somewhere in $(e, e + \varepsilon']$. Suppose now that $\sum_{j \in S} p_j(e) \leq 0$. Since $e < \bar{e}_i$, $p_i(e) > 0$. Hence, there is $j \in S$ such that $p_j(e) < 0$. Hence, by T3, $T_j(e, S) < 0$ and $u_j^G(e) = b_j(e) + T_j(e, S) < b_j(e)$. Hence, player j would be better off by moving his probability at e to 0. □

Under WINNER-PAYS, the prize payoff functions are strictly decreasing and the base payoff functions are weakly decreasing. Within this framework, once ALL-PAY is not assumed, the more natural case is the one in which the base payoff functions, apart from not being strictly decreasing, are constant; indeed, this is the case in FPA, BM, and TC. Henceforth, we assume that, for each $i \in N$, there is $b_i \in \mathbb{R}$ such that, for each $e \in [0, M]$, $b_i(e) = b_i$. Recall that, for each player $i \in N$, efforts above $\bar{e}_i$ are weakly dominated strategies (under ALL-PAY, they were strictly dominated). Since we are assuming that the base payoffs are constant, if $\bar{e}_i < M$, then $b_i(0) = b_i(\bar{e}_i) + p_i(\bar{e}_i)$ and, hence, $p_i(\bar{e}_i) = 0$.

Now, given a mixed strategy profile $G \in \mathcal{G}^n$, let $e_G := \inf\{e \in \bigcup_{i \in N} S(G_i) : \prod_{i \in N} G_i(e) > 0\}$. If $e_G < M$, then e_G is the smallest effort such that $e > e_G$ implies that, for each $i \in N$, $G_i(e) > 0$. Similarly, if $e_G < M$, given $G \in \mathcal{G}^n$ and an effort $e \in [0, M]$, the probability of getting the prize at e is: i) zero if $e < e_G$ and ii) strictly positive if $e > e_G$ (at e_G both things might happen). The effort level e_G is essential for the characterization result we present in Theorem 2. Before we introduce a Lemma that almost pins down the value e_G must take in a Nash equilibrium (if any). Moreover, it shows that no one puts positive probability above e_G. Since, by definition of e_G, there is $i \in N$ such that $G_i(e_G^-) = 0$, we have that player i puts probability 1 at e_G. Furthermore, the previous observation implies that, with probability 1, the prize will be awarded at effort e_G.

Lemma 3. *Let $C^f \in \mathcal{C}^*$. Assume* WINNER-PAYS *and that, for each $i \in N$, $b_i(\cdot)$ equals constant $b_i \in \mathbb{R}$. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Then,*

[17] Again, in BM, $(\bar{p} - e)D(\bar{p} - e) - c_i(D(\bar{p} - e))$ has to be strictly decreasing in e.

[18] If there are no discontinuities in $(e, e + \varepsilon]$, then a continuity argument for that interval does the job. If there are discontinuities of $u_j^G(\cdot)$ in $(e, e + \varepsilon]$, then they make the function take even higher values at $e + \varepsilon'$.

i) for each $i \in N$, $G_i(e_G) = 1$,

ii) $e_G \in [\bar{e}_2, \bar{e}_1]$, and

iii) for each $i \in N$, if $\bar{e}_i < e_G$, then $G_i(e_G) - G_i(e_G^-) = 0$.

Proof. **i)** If $e_G = M$, then there is nothing to prove. Hence, we assume that $e_G < M$. We divide the proof in two parts:

There is no $i \in N$ such that $G_i(e_G) < 1$ and, for each $j \neq i$, $G_j(e_G) = 1$: Suppose there is such i. Let $e > e_G$ be such that $e \in S(G_i)$. Now, since for each $j \neq i$, $G_j(e_G) = 1$, there is $\varepsilon > 0$ such that $e_G + \varepsilon < e$ and $u_i^G(e_G + \varepsilon) = b_i + p_i(e_G + \varepsilon) > b_i + p_i(e) = u_i^G(e)$. Contradiction with $e \in S(G_i)$ (e is not a best reply for i against G_{-i}).

There is no $S \subseteq N$, $|S| > 1$, such that, for each $i \in S$, $G_i(e_G) < 1$: Suppose there is such S and assume, without loss of generality, that S is maximal, *i.e.*, if $j \notin S$, then $G_j(e_G) = 1$. For each $i \in S$, $\bar{e}_i > e_G$, since, otherwise, by definition of e_G, strategies above e_G would be strictly dominated for player i. Now, if, for each $i \in S$, $G_i(e_G) - G_i(e_G^-) > 0$, then, by definition of e_G, the probability that players in S participate in a tie at e_G is positive and we get a contradiction with Lemma 2. Hence, there is $j \in S$ such that $G_j(e_G) - G_j(e_G^-) = 0$ and the probability of a tie at e_G is zero. Now, combining the latter with the definition of e_G, the probability of winning with a strategy $e \leq e_G$, is zero. Let $i \in S$. There is $\varepsilon > 0$ such that, if $e \leq e_G$, e is strictly dominated by $e_G + \varepsilon < \bar{e}_i$. Hence, $G_i(e_G) = 0$. Now, $u_i^G(e_G) = b_i$ and, since $|S| > 1$, the function $u_i^G(\cdot) = b_i + \prod_{j \neq i} G_j(\cdot) p_i(\cdot)$ is continuous at e_G. Hence, since $e_G \in S(G_i)$ and $G_i(e_G) = 0$, there is $\varepsilon > 0$ such that $[e_G, e_G + \varepsilon] \in S(G_i)$. Now, $u_i^G(e_G + \varepsilon) > b_i$ and $\lim_{\delta \to 0^+} u_i^G(e_G + \delta) = b_i$. But, since all the strategies in the support of a Nash equilibrium must lead to the same payoff, the latter is not possible.

ii) By i), for each $i \in N$, $G_i(e_G) = 1$. Hence, by definition of e_G, there is $i \in N$ such that $G_i(e_G) = 1$ and $G_i(e_G^-) = 0$, *i.e.*, player i puts probability 1 at e_G. Fix player i. Also by i), the probability of winning at e_G is 1. Hence, we immediately have that $e_G \leq \bar{e}_1$, since, otherwise, player i would be better off by moving his probability at e_G to 0. Now, suppose that $e_G < \bar{e}_2$. Then, there is $j \neq i$ such that $e_G < \bar{e}_j$. By Lemma 2, the probability that j participates in a tie at e_G is 0. Now, since $\bar{e}_j > e_G$ and, for each $e \in [0, e_G)$, $u_j^G(e) = b_j$, there is again $\varepsilon > 0$ such that player j can improve by playing pure strategy $e_G + \varepsilon < \bar{e}_j$, winning the prize for sure and getting a payoff $b_j + p_j(e_G + \varepsilon) > b_j + p_j(\bar{e}_j) = b_j$.

iii) Let $j \in N$ be such that $\bar{e}_j < e_G$ and $G_j(e_G) - G_j(e_G^-) > 0$. Then, by i), the probability that j wins at e_G is positive. Now, for each $S \subseteq N$, $j \in S$, by T3, $T_j(e_G, S) \leq p_j(e_G) < 0$. Hence, $u_j^G(e_G) < b_j$ and player j would be better off moving his probability at e_G to 0. □

Next Theorem fully characterizes the structure of the Nash equilibria under M-BOUNDING.

Theorem 2. *Let $C^f \in \mathcal{C}^*$. Assume* WINNER-PAYS, M-BOUNDING, *and that, for each $i \in N$, $b_i(\cdot)$ equals constant $b_i \in \mathbb{R}$.*

i) Let $\bar{e}_1 > \bar{e}_2$. Then, C^f has no Nash equilibrium in pure strategies but it has a continuum of mixed Nash equilibria. Moreover, the equilibrium payoffs are such that $\eta_1 \in (b_1, b_1 + p_1(\bar{e}_2)]$ and, for each $i \neq 1$, $\eta_i = b_i$.

ii) Let $\bar{e}_1 = \bar{e}_2$. Then, the set of Nash equilibria of C^f is nonempty if and only if there is $S \subseteq N$, $|S| > 1$, such that, for each $i \in S$, $T_i(\bar{e}_2, S) = 0$.

Indeed, whenever players in S play pure strategy $\bar{e}_2$ and players in $N \backslash S$ put probability 0 at $\bar{e}_2$ we have a Nash equilibrium of C^f. Moreover, the equilibrium payoffs are constant across equilibria: for each $i \in N$, $\eta_i = b_i$. Finally, if $n = 2$ and $T_1(\bar{e}_2, \{1,2\}) = T_2(\bar{e}_2, \{1,2\}) = 0$, then the strategy profile in which both players play the pure strategy $\bar{e}_2$ is the unique Nash equilibrium.

Proof. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. By Lemma 3, $e_G \in [\bar{e}_2, \bar{e}_1]$ and there is $i \in N$ such that $G_i(e_G^-) = 0$ and $G_i(e_G) = 1$. That is, player i plays pure strategy e_G.

i) $\bar{e}_1 > \bar{e}_2$. Now, $i = 1$, since, otherwise, either player 1 would like to deviate to $e_G + \varepsilon < \bar{e}_1$ or player i would like to deviate to $e = 0$. Now, we claim that player 1 wins for sure at e_G with no ties. Ties at e_G can only have positive probability when $e_G = \bar{e}_2$. But, since player 1 would participate in such ties and $\bar{e}_1 > \bar{e}_2$, the latter is ruled out by Lemma 2. Hence, when G is played, player 1 gets $b_1 + p_1(e_G)$ and each $i \neq 1$ gets b_i. Now, for each $i \neq 1$ and each $e \in [0, M]$, $u_i^G(e) \leq b_i$. Hence, for each $i \neq 1$, i) $S(G_i) \subseteq [0, e_G]$, ii) since ties at e_G have probability 0, $G_i(e_G) - G_i(e_G^-) = 0$, and iii) i has no incentive to deviate from G.

Now, we show that $e_G < \bar{e}_1$. Suppose $e_G = \bar{e}_1$. Then, player 1 gets payoff $b_1 + p_1(\bar{e}_1) = b_1$. Then, since there is $e < \bar{e}_1$ such that $\prod_{j \neq 1} G_j(e) > 0$, we have $u_1^G(e) = b_1 + \prod_{j \neq 1} G_j(e) p_1(e) > b_1$. Hence, player 1 would move his probability in $\bar{e}_1$ to e. Hence, in a Nash equilibrium, $e_G \in [\bar{e}_2, \bar{e}_1)$. To ensure that player 1 has no incentive to deviate we need that, for each $e \in [0, M]$, $u_1^G(e_G) - u_1^G(e) \geq 0$. If $e > e_G$, since $p(\cdot)$ is strictly decreasing, $u_1^G(e) = b_1 + p_1(e) < b_1 + p_1(e_G) = u_1^G(e_G)$ and we are done. Hence, we need that, for each $e \in [0, \bar{e}_G)$, $u_1^G(e_G) - u_1^G(e) = b_1 + \prod_{j \neq 1} G_j(e_G) p_1(e_G) - b_1 - \prod_{j \neq 1} G_j(e) p_1(e) = p_1(e_G) - \prod_{j \neq 1} G_j(e) p_1(e) \geq 0$. Hence, for each $e \in [0, \bar{e}_G)$, the following equation must hold,

$$\prod_{j \neq 1} G_j(e) \leq \frac{p_1(e_G)}{p_1(e)}. \tag{2}$$

Now, there is a continuum of mixed strategies that satisfy Eq. (2).[19] Note that, once Eq. (2) is met, player 1 has no incentives to put efforts below $\bar{e}_2$. Finally, when e_G takes the different values in $[\bar{e}_2, \bar{e}_1)$ we get the different payoffs for player 1.

ii) $\bar{e}_1 = \bar{e}_2$. Now, $e_G = \bar{e}_2$ and player i is such that $\bar{e}_i = \bar{e}_2 = e_G$. Hence,

[19] Note that Eq. (2) is consistent with the requirement $\prod_{j \neq 1} G_j(e_G) = 1$ imposed by statement i) of Lemma 3.

$p_i(\bar{e}_2) = 0$ and $u_i^G(\bar{e}_2) = b_i + p_i(\bar{e}_2) = b_i$. Let $\bar{S} := \{i \in N : \bar{e}_i = \bar{e}_2\}$. Now, if G is a Nash equilibrium, then it satisfies the following properties:

1) For each $j \in N$, $S(G_j) \subseteq [0, \bar{e}_2]$ (Lemma 3).

2) For each $j \notin \bar{S}$, j puts probability 0 at $\bar{e}_2$, *i.e.*, $G_j(\bar{e}_2) - G_j(\bar{e}_2^-) = 0$ (Lemma 3).

3) There are at least two players that play pure strategy $\bar{e}_2$. Suppose, on the contrary, that only player i plays pure strategy $\bar{e}_2$. Then, there is $e < \bar{e}_2$ such that $\prod_{j \neq i} G_j(e) > 0$. Then, $u_i^G(e) = b_i + \prod_{j \neq i} G_j(e) p_i(e) > b_i$. So player i would move his probability in $\bar{e}_2$ to e.

4) If $S' \subseteq \bar{S}$ is such that the probability of a tie at $\bar{e}_2$ among the players in S' is positive according to G, then, for each $i \in S'$, $T_i(\bar{e}_2, S') = 0$. Suppose there is $j \in S'$ such that $T_j(\bar{e}_2, S') \neq 0$. Since $T_j(\bar{e}_2, S') \leq p_j(\bar{e}_2) = 0$, we have $T_j(\bar{e}_2, S') < 0$. So player j would move his probability in $\bar{e}_2$ to 0.

Clearly, from properties 1)-4), if there is no coalition S, $|S| > 1$, such that, for each $i \in S$, $T_i(\bar{e}_2, S) = 0$, then there is no Nash equilibrium. Moreover, it is straightforward to check that every strategy profile satisfying properties 1)-4) is a Nash equilibrium of C^f. The existence of one such profile is ensured just by defining G as the strategy profile in which players in S play pure strategy $\bar{e}_2$ and all the others play, for instance, a pure strategy $e \in [0, \bar{e}_2)$. Finally, the uniqueness when $n = 2$ follows from property 3) above. $\square$

Remark 3. *Note that the proof of Theorem 2 fully characterizes the set of Nash equilibria in the different cases. In particular, it shows that, if $\bar{e}_1 = \bar{e}_2$, then the Nash equilibrium may fail to exist. Theorem 2 also shows the non-existence of Nash equilibrium in pure strategies when $\bar{e}_1 > \bar{e}_2$. The source of the non-existence is the fact that, when player 2 is playing $\bar{e}_2$, there is no optimal reply for player 1. Here, we can recover existence by discretizing the sets of strategies.*

After the result above it is natural to wonder why, if $\bar{e}_1 > \bar{e}_2$, we do lose the uniqueness of the equilibrium payoffs that we had under ALL-PAY. The answer is as follows. Under ALL-PAY, pure strategies above $\bar{e}_2$ were strictly dominated for players different from player 1. Now, since we are assuming the $b_i(\cdot)$ functions are constant, such strategies are only weakly dominated. Hence, players different from 1 can use them to threaten player 1 and, hence, new equilibrium payoffs can be supported. Nonetheless, it is clear that many refinements of Nash equilibrium concept would allow us to get rid of the equilibria in which player 1 gets less than $b_1 + p_1(\bar{e}_2)$. On the other hand, being the $b_i(\cdot)$ functions constant, it is not possible to coordinate players' mixed strategies over a common support, as we could do under ALL-PAY; this is explicitly used in the proof of Lemma 3 when we show that "There is no $S \subseteq N$, $|S| > 1$, such that, for each $i \in S$, $G_i(e_G) < 1$".

Now, we turn to the characterization result when M-BOUNDING is not met. Let $S^M := \{i \in N : \bar{e}_i = M\}$.

Theorem 3. *Let $C^f \in \mathcal{C}^*$. Assume* WINNER-PAYS*, that M*-BOUNDING *is not satisfied, and that, for each $i \in N$, $b_i(\cdot)$ equals constant $b_i \in \mathbb{R}$.*

i) Let $|S^M| = 1$, i.e., $M = \bar{e}_1 > \bar{e}_2$. *Then,* C^f *has no Nash equilibrium in pure strategies but it has a continuum of mixed Nash equilibria. Moreover, the equilibrium payoffs are such that* $\eta_1 \in [b_1 + p_1(M), b_1 + p_1(\bar{e}_2)] \backslash \{b_1\}$ *and, for each* $i \neq 1$, $\eta_i = b_i$.

ii) Let $|S^M| > 1$, i.e., $M = \bar{e}_1 = \bar{e}_2$. *Then, the set of Nash equilibria of* C^f *is nonempty if there is* $S \subseteq N$ *such that, for each* $i \in S$, $T_i(M, S) \geq 0$*; if* $S = \{i\}$*, then* $T_i(M, S) > 0$*; and, for each* $j \notin S$, $T_j(M, S \cup \{j\}) \leq 0$.

Indeed, whenever players in S *play pure strategy* $\bar{e}_2$ *and players in* $N \backslash S$ *put probability* 0 *at* $\bar{e}_2$ *we have a Nash equilibrium of* C^f*. Finally,* $\eta_i = b_i$ *if either* $\bar{e}_i < M$ *or* $p_i(M) = 0$.

iii) Let $|S^M| > 1$, i.e., $M = \bar{e}_1 = \bar{e}_2$. *If for each* $S \subseteq N$ *and each* $i \in S$, $T_i(M, S) = \frac{p_i(M)}{|S|}$, *then* $G \in \mathcal{G}^n$ *is a Nash equilibrium of* C^f *if and only if*

- *for each* $i \notin S^M$, i *puts probability* 0 *at* M *and*
- *for each* $i \in S^M$, *if* $p_i(M) > 0$, *then* i *puts probability* 1 *at* M.

Proof. i) follows very similar lines to the proof of i) in Theorem 2 and ii) and iii) follow immediately from Lemma 3. □

Remark 4. *There are two important differences between the statement ii) in Theorem 2 and that in Theorem 3. First, although the conditions for existence of Nash equilibria are very similar, the one in Theorem 2 is a necessary and sufficient condition and the one in Theorem 3 is only sufficient. The reason for this is the following. Since* M-BOUNDING *is not met, players can get positive payoffs at* e_G *(under* M-BOUNDING*, if* $\bar{e}_1 = \bar{e}_2$*, then the highest payoff at* e_G *was* 0*). Now, using the freedom we have in the tie payoff functions we can construct games with Nash equilibria that do not satisfy the sufficient condition. Second, if* M-BOUNDING *is not met, then the equilibrium payoffs need not be unique. Just think of a situation in which at* M *all players get 1 if they are the only winners and* 0 *otherwise. Then, whenever a player puts probability 1 at* M *and all the other put probability* 0 *at* M *we have a Nash equilibrium and the payoffs are not always the same. Finally, iii) illustrates that for "natural" tie payoff functions a full characterization is easy to achieve.*

5.1 Implications of the Characterization under WINNER-PAYS

According to the discussion in Remark 3, some of the technical difficulties that arise when studying Nash equilibrium in FP-WTA contests can be overcome by discretizing the sets of strategies of the players. Nonetheless, Theorem 2 shows that mixed strategies can also be used to recover existence of equilibrium in situations where pure equilibria do not exist. Moreover, some of the mixed equibria involve strategic behaviors that had not been noticed before in any of the specific models at hand. The discussion below illustrates this point.

We begin with the FPA model. Theorem 2 says that, when the two highest valuations are different, *i.e.*, $v_1 > v_2$, there is no equilibrium in pure strategies but the existence of equilibrium is always recovered with the use of mixed strategies. In this case, instead of mixing we could also use the discretizing technique to recover existence. Moreover, player 1 does not get $v_1 - v_2$ in all the equilibria. Indeed, all the payoffs in the interval $(0, v_1 - v_2]$ can be achieved as equilibrium payoffs for player 1; the latter leading to different revenues for the auctioneer. Nonetheless, these equilibria in which player 1 gets less that $v_1 - v_2$ are based on "incredible" threats of the other players. Hence, many equilibrium refinements can be used to pin down the equilibria with payoff $v_1 - v_2$ for player 1. On the other hand, when $v_1 = v_2$, since, for each $S \subseteq N$ and each $i \in S$ such that $v_i = v_2$, we have $T_i(v_2, S) = 0 = T_i(v_2, \{i\})$, Theorem 2 ensures the existence of at least one Nash equilibrium. In all these equilibria there are two players that bid v_2 with probability one. Moreover, all the players get a expected payoff of 0; indeed, in this case we might even omit the word expected, *i.e.*, all the players get payoff 0 for sure.

Now, we discuss BM. We begin by identifying Bertrand Paradox as a special implication of Theorem 2. When the cost functions are constant and equal across firms, we have that the marginal cost is common for all firms, say $c = \bar{p} - e$ for some discount e. Marginal cost is chosen with probability one in equilibrium. This is an equilibrium because, when cost functions are constant, we have that, for each $S \subseteq N$ and each $i \in S$, $T_i(e_2, S) = 0 = T_i(e_2, \{i\})$. However, even if the cost functions are equal across firms, if they exhibit strictly decreasing average costs, we have that $T_i(e_2, S) < 0$ whenever $|S| > 1$ and $i \in S$. Hence, Theorem 2 says that in this case there is no Nash equilibrium, even if the firms can use mixed strategies. Next Corollary formally states the sketched result.

Corollary 1. *Take a general Bertrand competition model (BM) with n firms. If the cost function is common for all firms and exhibits strictly decreasing average costs, then the associated FP-WTA contest does not have any Nash equilibrium (neither pure, nor mixed).*

Proof. Immediate from Theorem 2. □

On the other hand, consider a situation in which firms 1 and 2 be the ones with the lowest production costs. Let firm 1 have the lowest of the two. Then, Theorem 2 says that there is an equilibrium in which firm 1 gets the whole market at the price at which firm 2 would get 0 profit if getting all the market. Hence, player 1 would make a positive profit. Nonetheless, as in the FPA model, we can also support in equilibrium all the profits for firm 1 between 0 and the previous one; firms different from 1 always get 0 in equilibrium.

Finally, the TC model. Theorem 3 extends the results presented for Hatfield (2006). This model is special because it is the only one for which M-BOUNDING is not met. This leads to FP-WTA contests for which the payoffs may be different across equilibria. Moreover, our model also allows to extend the results in Hatfield (2006) to situations in which the function of return of capital ρ is not the same for all the districts.

6 More Applications and Further Extensions

So far we have discussed the implications of our main results, Theorems 1 and 2, in already existing models. We want to emphasize that the scope of FP-WTA contests goes beyond that. We think that the main objective of future research would be to look for economic situations that can be modeled as FP-WTA contests and study the implications of our results within those situations. Indeed, most of the models we have discussed in this paper present very simple FP-WTA forms, whereas our model allows for much more general ones. Moreover, it might be interesting to elaborate more on the tightness of the assumptions we have taken in the different results. Below, as a matter of example, we present two different situations that can be modeled using FP-WTA contests.

Combination of FPA and APA: Consider the following auction. First, players submit bids. The player with the highest bid pays his bid and gets the object. Now, for each loser, there is a positive probability that he has to pay his bid (similarly, we could assume that each loser has to pay a given percentage of his bid). Now, it is easy to check that this new auction can be modeled as a FP-WTA contest which, moreover, satisfies ALL-PAY. Hence, the results in Section 4 immediately apply for this type of auctions.

Combination of BM and MS: In Varian's model of sales it is assumed that the number of strategic consumers does not vary as a function of the price. Hence, this model could be extended to a situation in which, as we had in BM, the demand of the strategic consumers decreases as price increases (in BM all the players are strategic). Doing this, we have a common model for BM and MS, being the former the one in which all firms have 0 loyal consumers. Now, this leads to a new FP-WTA contest with the following primitives: i) The demand function for the strategic consumers, ii) the cost functions, iii) the number of loyal consumers of each firm, and i) the upper bound for the price to be set. Hence, it might be worth to study the extent up to which our results can be applied for the different configurations of the latter primitives.

6.1 A Possible Extension

Finally, we also want to discuss on a further generalization of our model. Since the payoff of a player should not increase with his effort unless he gets the prize, it is natural to assume that the base payoff functions are weakly decreasing. On the other hand, there are natural situations in which the prize payoff might increase as a function of the effort; maybe for some intervals or maybe for the whole set of efforts. Indeed, this is what might happen in general Bertrand competition models where, for high prices, the demand function might be very responsive to small changes; although, at the end, for very low prices, they responsiveness would be negligible (the latter might lead to changes in the growth of $(\bar{p}-e)D(\bar{p}-e)-c_i(D(\bar{p}-e))$ as a function of e). Hence, below we present an extension of our model that would account for these situations.

First, we remove the requirement of weakly decreasing prize payoff functions (WDP). Second, consider the following assumptions:

Assumption A1. For each $i \in N$, there is a unique $\bar{e}_i \leq M_i$ such that $b_i(m_i) = b_i(\bar{e}_i) + p_i(\bar{e}_i)$. Moreover, for each $e > \bar{e}_i$, $b_i(m_i) < b_i(\bar{e}_i) + p_i(\bar{e}_i)$.

This assumption follows the same idea of A2, *i.e.*, player i does not want to get the prize with efforts above $\bar{e}_i$. That is, even if the payoff of the prize is increasing in e, at some point the decrease in the base payoff should dominate the increase in the prize payoff.

Assumption A2. If $\bar{e}_1 > \bar{e}_2$, then $b_1(\cdot) + p_1(\cdot)$ is strictly decreasing in $[\bar{e}_2, M]$.

A2 says that, for player 1, the impact of the effort in the base payoff should dominate that in the prize payoff no later than $\bar{e}_2$.

Note that the combination of ALL-PAY, A1, and A2 is weaker than the combination of ALL-PAY, M-BOUNDING, and WDP. Similarly, the combination of A1 and A2 is weaker than the combination of WINNER-PAYS, M-BOUNDING, and WDP. Nonetheless, we claim that, if WDP is not assumed, then A1 and A2 are basically what we need for the essence of the results we have presented in this paper to carry out, *i.e.*, M-BOUNDING has to replaced by A1 and A2, and WINNER-PAYS is removed. The word essence is because, in the situations where there are a continuum of Nash equibria, new equilibria might appear, but no new equilibrium payoffs. The above generalization would allow, in particular, to account for Bertrand competition models without assuming that $(\bar{p} - e)D(\bar{p} - e) - c_i(D(\bar{p} - e))$ is strictly decreasing in e.

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A Equilibrium characterizations under ALL-PAY

In this Appendix we start with a series of auxiliary results that play the same role of the similar lemmata in Baye et al. (1996) and González-Díaz et al. (2007). For each result, we mention the closest result (if any) in each of the previous two papers. As we have already said, most of the arguments in these proofs are not new and, therefore, we omit most of them. All these proofs can be found in González-Díaz (2010). To start with, Lemma 1 in the text corresponds with Lemma 5 in Baye et al. (1996) and Lemma 1 in González-Díaz et al. (2007).

Recall that by Lemma 1 we know that, in a Nash equilibrium $G \in \mathcal{G}^n$, for each $i \in N$ and each $e \in (0, M)$, $G_i(e) = G_i(e^-)$. The latter implies that the payoff functions are continuous on $(0, M)$. Moreover, recall that, under M-BOUNDING, the previous consideration is true also at M. Next we present a series of results that are needed to prove Theorems 4, 5, and 6. The result below corresponds with Lemma 3 in Baye et al. (1996).

Lemma 4. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Then, the probability of a tie under G in $[0, M)$ is 0 and there is $i \in N$ such that $G_i(0) = 0$. Moreover, under M-*BOUNDING *the probability of a tie at M is also* 0.

Next result shows that, in equilibrium, an effort cannot belong to the support of the strategy of exactly one player (Lemma 7 in Baye et al. (1996) and Lemma 2 in González-Díaz et al. (2007)).

Lemma 5. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY *and* M-BOUNDING. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Let $i \in N$ and $e \in S(G_i)$. Then, there is $j \neq i$ such that $e \in S(G_j)$.*

Next Lemma shows that, if some effort e does not belong the the support of any of the equilibrium strategies, then no higher effort does (Lemma 10 in Baye et al. (1996) and Lemma 3 in González-Díaz et al. (2007)).

Lemma 6. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Let $e \in [0, M]$ be such that $e \notin \bigcup_{i \in N} S(G_i)$. Then, $(e, M] \cap \bigcup_{i \in N} S(G_i) = \emptyset$.*

The next Lemma corresponds with Lemma 2 in Baye et al. (1996) and Lemma 7 in González-Díaz et al. (2007).

Lemma 7. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Then, for each $i \in N$, $0 \in S(G_i)$.*

Recall now that whenever we assume M-BOUNDING and NO-CROSSING we also assume, without loss of generality, that players are ordered so that $\bar{e}_1 \geq \bar{e}_2 \geq \ldots \geq \bar{e}_n$. The next result corresponds with Lemma 3 in Baye et al. (1996).

Lemma 8. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY, M-BOUNDING, *and* NO-CROSSING. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. If $\bar{e}_1 > \bar{e}_2$, then $G_1(0) = 0$ and, for each $i \neq 1$, $G_i(0) > 0$.*

In the next results we study how the equilibrium payoffs must be. Since each player can ensure himself a payoff $b_i(0)$ by playing pure strategy 0, we know that for each $i \in N$, his equilibrium payoff must be at least $b_i(0)$. Moreover, recall that, by a straightforward best reply argument, we know that in a Nash equilibrium $G \in \mathcal{G}^n$, for each $i \in N$, the function $u_i^G(\cdot)$ is constant in $S(G_i)$ (Lemma 11 in Baye et al. (1996)).

Lemma 9. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY, M-BOUNDING, *and* NO-CROSSING. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. For each $i \in N$, let η_i denote i's equilibrium payoff. Let $i, j \in N$ be such that $\eta_i = b_i(0)$, $\eta_j \geq b_j(0)$, and $\bar{e}_i > \bar{e}_j$. Let $e > 0$, $e \in S(G_j)$. Then, $G_i(e) > G_j(e)$.*

The next result partially corresponds with Lemma 3 in Baye et al. (1996).

Lemma 10. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY, M-BOUNDING, *and* NO-CROSSING. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. For each $i \in N$, let η_i denote i's equilibrium payoff. If there is $i \in N$ such that $\eta_i > b_i(0)$, then*

i) $G_i(0) = 0$,

ii) for each $j \neq i$, $G_j(0) > 0$ and $\eta_j = b_j(0)$, and

iii) if $j \neq i$ is such that $\bar{e}_j < \max_{k \neq i} \bar{e}_k$, then $G_j(0) = 1$.

Next Proposition pins down what the equilibrium payoffs must be (if any). This result extends the payoff results of Lemmas 3 and 11 in Baye et al. (1996) and Lemma 8 in González-Díaz et al. (2007)). The arguments depend on the specific parameter configurations and, therefore, we present the proof.

Proposition 3. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY, M-BOUNDING, *and* NO-CROSSING*. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. For each $i \in N$, let η_i denote i's equilibrium payoff. Then, $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$ and, for each $i \neq 1$, $\eta_i = b_i(0)$.*

Proof. We distinguish several cases.

Case 1: $\bar{e}_1 > \bar{e}_2$. According to Lemma 8, $G_1(0) = 0$ and, for each $i \neq 1$, $G_i(0) > 0$. Hence, for each $i \neq 1$, $\eta_i = u_i^G(0) = b_i(0)$. Moreover, since for each $i \in N$, $G_i(\bar{e}_2) = 1$, we have $\eta_1 \geq u_1^G(\bar{e}_2) = b_1(\bar{e}_2) + p_1(\bar{e}_2)$. Now, suppose that $\eta_1 > u_1^G(\bar{e}_2)$. Then, we distinguish two cases:

Case 1.1: $\bar{e}_2 > \bar{e}_3$. By the continuity of $u_1^G(\cdot)$ at $\bar{e}_2$, there is $\varepsilon > 0$ such that for each $e \in [\bar{e}_2 - \varepsilon, \bar{e}_2]$, $\eta_1 > u_1^G(e)$. Hence, $S(G_1) \subseteq [0, \bar{e}_2 - \varepsilon]$. Now there is $\delta > 0$ such that player 2 can get more than $b_2(0)$ by putting all his probability at $\bar{e}_2 - \varepsilon + \delta$.

Case 1.2: $\bar{e}_2 = \bar{e}_3$. Let $\hat{e}$ be the maximum effort in $S(G_1)$. Since $\eta_1 > u_1^G(\bar{e}_2)$, $\hat{e} < \bar{e}_2$. Now, by Lemma 5 there is $i \neq 1$ such that $\hat{e} \in S(G_i)$. Now, since $\eta_1 > u_1^G(\bar{e}_2) = b_1(\bar{e}_2) + p_1(\bar{e}_2)$, we have

$$\eta_1 = u_1^G(\hat{e}) = b_1(\hat{e}) + \prod_{j \neq 1} G_j(\hat{e}) p_1(\hat{e}) \quad \Rightarrow \quad \prod_{j \neq 1} G_j(\hat{e}) = \frac{\eta_1 - b_1(\hat{e})}{p_1(\hat{e})} > I_1^*(\hat{e}),$$

$$b_i(0) = \eta_i = u_i^G(\hat{e}) = b_i(\hat{e}) + \prod_{j \neq i} G_j(\hat{e}) p_i(\hat{e}) \quad \Rightarrow \quad \prod_{j \neq i} G_j(\hat{e}) = \frac{b_i(0) - b_i(\hat{e})}{p_i(\hat{e})} = I_i(\hat{e}).$$

Hence, dividing these two expressions we get $\frac{G_i(\hat{e})}{G_1(\hat{e})} > \frac{I_1^*(\hat{e})}{I_i(\hat{e})} \geq \frac{I_1^*(\hat{e})}{I_2(\hat{e})}$. Now, by NO-CROSSING*, since $G_1(\hat{e}) = 1$ and $\hat{e} < \bar{e}_2$, we have $G_i(\hat{e}) > 1$, but this is not possible.

Case 2: $\bar{e}_1 = \bar{e}_2$. Now we have to prove that for each $i \in N$, $\eta_i = b_i(0)$.

Step 1: If $\bar{e}_1 > \bar{e}_i$, then $\eta_i = b_i(0)$. Suppose, on the contrary, that there is $i \in N$ such that $\bar{e}_1 > \bar{e}_i$ and $\eta_i > b_i(0)$. By Lemma 10 we have i) $G_i(0) = 0$, ii) for each $j \neq i$, $G_j(0) > 0$ and $\eta_j = b_j(0)$, and iii) if $j \neq i$ is such that $\bar{e}_j < \bar{e}_1$, then $G_j(0) = 1$. Let $\hat{e}$ be the maximum effort in $S(G_i)$. Note that, by continuity, $G_i(\hat{e}) = 1$. By Lemma 5 there is $j \neq i$ such that $\hat{e} \in S(G_j)$. Indeed, by iii) we must have $\bar{e}_j = \bar{e}_1 > \bar{e}_i$. Now, by Lemma 9, $G_j(\hat{e}) > G_i(\hat{e}) = 1$. Hence, we have a contradiction.

Step 2: There is $i \in N$ such that $\bar{e}_1 = \bar{e}_i$ and $\eta_i = b_i(0)$. Now, let $j \in N$ be such that $\bar{e}_1 = \bar{e}_j$. If j puts positive probability at 0, then $\eta_j = u_j^G(0) = b_j(0)$ and we are done. If $G_j(0) = 0$, let $i \neq j$ be such that $\bar{e}_1 = \bar{e}_i$ (it exists because we are assuming $\bar{e}_1 = \bar{e}_2$). Now, since i) i's payoff function is continuous at 0, ii) by Lemma 7, $0 \in S(G_i)$, and iii) $u_i^G(0) = b_i(0)$, we have that $\eta_i = b_i(0)$.

Step 3: If $\bar{e}_1 = \bar{e}_i$, then $\eta_i = b_i(0)$. Suppose there is $i \in N$ such that $\bar{e}_1 = \bar{e}_i$ and $\eta_i > b_i(0)$. Let $\hat{e}$ be the maximum effort in $S(G_i)$. Note that $\hat{e} < \bar{e}_1$.

Now, let $j \in N$ be the one found in Step 2. That is, $\bar{e}_1 = \bar{e}_i$ and $\eta_j = b_j(0)$. Now,

$$b_i(0) = \eta_i \geq u_i^G(\hat{e}) = b_i(\hat{e}) + \prod_{k \neq i} G_k(\hat{e}) p_i(\hat{e}) \quad \Rightarrow \quad \prod_{k \neq i} G_k(\hat{e}) \leq \frac{b_i(0) - b_i(\hat{e})}{p_i(\hat{e})} = I_i(\hat{e}),$$

$$b_j(0) < \eta_j = u_j^G(\hat{e}) = b_j(\hat{e}) + \prod_{k \neq j} G_k(\hat{e}) p_j(\hat{e}) \quad \Rightarrow \quad \prod_{k \neq j} G_k(\hat{e}) > \frac{b_j(0) - b_j(\hat{e})}{p_j(\hat{e})} = I_j(\hat{e}).$$

But now, by NO-CROSSING and using that $G_i(a) = 1 \geq G_j(a)$, $\prod_{k \neq j} G_k(\hat{e}) > I_j(\hat{e}) = I_i(\hat{e}) = \prod_{k \neq i} G_k(\hat{e}) \geq \prod_{k \neq j} G_k(\hat{e})$. Hence, we have a contradiction. □

Note that, according to the previous result, if $\bar{e}_1 = \bar{e}_2$, then $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2) = b_1(\bar{e}_1) + p_1(\bar{e}_1) = b_1(0)$.

Next Corollary says that, if the players are ordered according to their maximum admissible efforts, then the players whose higher admissible effort is smaller that the one of player 2 put no effort in equilibrium. That is, they play pure strategy 0 and get their minimum right $b_i(0)$. This Corollary combines and extends the results of Lemmas 8 and 11 in Baye et al. (1996) and Lemma 9 in González-Díaz et al. (2007)).

Corollary 2. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY*,* M-BOUNDING*, and* NO-CROSSING**. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f and let $i \in N$. Then,*

i) $G_i(\bar{e}_2) = 1$*,*

ii) if $\bar{e}_2 > \bar{e}_i$*, then* $G_i(0) = 1$*,*

iii) if $\bar{e}_1 > \bar{e}_i$*, then* $G_i(0) > 0$*, and*

iv) if $\bar{e}_1 > \bar{e}_2$*, then* $G_1(0) = 0$*.*

Some of the Lemmas below can be proved without using all the assumptions, but we have tried to find a compromise between the tightness of the partial results and the complexity of the proofs. Next Lemma says that, in equilibrium, every effort in $[0, \bar{e}_2]$ has to belong to the support of the strategy of at least two players (Lemma 7 in Baye et al. (1996)).

Lemma 11. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY*,* M-BOUNDING*, and* NO-CROSSING**. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f and let $i \in N$. Then, each $e \in [0, \bar{e}_2]$ belongs to the support of at least two players.*

The next Lemma partially corresponds with Lemmas 9 and 11 in Baye et al. (1996).

Lemma 12. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY*,* M-BOUNDING*, and* NO-CROSSING**. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Let $i, j \in N$ and $e > 0$ be such that $e \in S(G_i) \cap S(G_j)$. Then,*

i) if $\bar{e}_i = \bar{e}_j$*,* $G_i(e) = G_j(e)$*,*

ii) if $\bar{e}_i > \bar{e}_j$*,* $G_i(e) < G_j(e)$*.*

The next Lemma corresponds with Lemma 10 in Baye et al. (1996).

Lemma 13. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY, M-BOUNDING, *and* NO-CROSSING*. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Let $i \in N$ be such that $\bar{e}_i = \bar{e}_2$ and let $a > 0$ belong to $S(G_i)$. Then, $[a, \bar{e}_2] \subsetneq S(G_i)$.*

Corollary 3. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY, M-BOUNDING, *and* NO-CROSSING*. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Let $i \in N$ and $a > 0$ be such that $a \notin S(G_i)$. Then, for each $e \in (0, a)$, $e \notin S(G_i)$. Moreover, $G_i(0) > 0$.*

Proof. The first part follows from Lemma 13 and the second one from Lemma 7. $\square$

Next Lemma says, among other things, that the support of at least one player coincides with $[0, \bar{e}_2]$; it combines Lemmas 4 and 10 in Baye et al. (1996).

Lemma 14. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY, M-BOUNDING, *and* NO-CROSSING*. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. Then,*

i) there is $i \in N$ with $\bar{e}_i = \bar{e}_2$ such that $S(G_i) = [0, \bar{e}_2]$,

ii) if $\bar{e}_1 = \bar{e}_2$ there is $j \in N$, $j \neq i$, with $\bar{e}_j = \bar{e}_2$ such that $S(G_j) = [0, \bar{e}_2]$. Moreover, $G_i(0) = G_j(0) = 0$ and hence, both G_i and G_j are continuous,

iii) if $\bar{e}_1 > \bar{e}_2$, then both 0 and $\bar{e}_2$ belong to $S(G_1)$.

Lemma 15. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY, M-BOUNDING, *and* NO-CROSSING*. *Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. If $\bar{e}_1 > \bar{e}_2$, then for each $e \in S(G_1)$, $\prod_{i \neq 1} G_i(e) = I_1^*(e)$.*

Proof. First, let $e > 0, e \in S(G_1)$. By Proposition 3, $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$. Hence, $u_1^G(e) = \eta_1 = b_1(e) + \prod_{j \neq 1} G_j(e) p_1(e)$ and $\prod_{j \neq 1} G_j(e) = I_1^*(e)$. The result for $e = 0$ comes from the fact that distribution functions are right-continuous and $I_1^*(\cdot)$ is continuous in $[0, M]$. $\square$

Now we are ready to present the main theorems. For simplicity, we present them assuming that $n \geq 3$. The case $n = 2$ and $\bar{e}_1 > \bar{e}_2$ is covered by Theorem 4. The case $n = 2$ and $\bar{e}_1 = \bar{e}_2$ is covered by Theorem 6. These results show that, under ALL-PAY, M-BOUNDING, NO-CROSSING, and NO-CROSSING*, a Nash equilibrium always exists. Moreover, they provide a classification of the set of Nash equilibrium depending on the $\bar{e}_i$ values. Nonetheless, note that because of Proposition 3 we already know that all the equilibria lead to the same payoffs.

Theorem 4 states that there is a unique Nash equilibrium when $\bar{e}_1 > \bar{e}_2 > \bar{e}_3$. This result is the generalization of Theorem 2 in González-Díaz et al. (2007) to our framework.

Theorem 4. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY, M-BOUNDING, *and* NO-CROSSING. *Let $G^* = (G_i^*)_{i \in N}$ be the strategy profile defined for players 1, 2, and $i \notin N \backslash \{1, 2\}$ as follows:*

$$G_1^*(e) = \begin{cases} 0 & e < 0 \\ I_2(e) & 0 \leq e \leq \bar{e}_2 \\ 1 & e > \bar{e}_2 \end{cases}, \quad G_2^*(e) = \begin{cases} 0 & e < 0 \\ I_1^*(e) & 0 \leq e \leq \bar{e}_2 \\ 1 & e > \bar{e}_2 \end{cases}, \quad G_i^*(e) = \begin{cases} 0 & e < 0 \\ 1 & e \geq 0 \end{cases}.$$

If $\bar{e}_1 > \bar{e}_2 > \bar{e}_3$, then G^* is the unique Nash equilibrium of C^f. Moreover, the equilibrium payoffs are $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$ and, for each $i \neq 1$, $\eta_i = b_i(0)$.

Proof. "$\Rightarrow$" Note that in this Theorem we do not need NO-CROSSING* because the case $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$ is ruled out by assumption. Suppose $G \in \mathcal{G}^n$ is a Nash equilibrium of C^f. By Corollary 2, for each $i > 2$, $G_i = G_i^*$. Hence, by Lemma 11, $S(G_1) = S(G_2) = [0, \bar{e}_2]$. By Proposition 3, $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$ and $\eta_2 = b_2(0)$. Now, for each $e \in (0, \bar{e}_2]$, $\eta_1 = u_1^G(e) = b_1(e) + G_2(e)p_1(e)$. Hence, $G_2(e) = I_1^*(e) = G_1^*(e)$. Similarly, using $u_2^G(e)$ we get $G_2(e) = G_2^*(e)$. Hence, G^* is the unique possible Nash equilibrium.

"$\Leftarrow$" Straightforward computations show that for each $i \in N$, if $e \in S(G_i^*)$, then $u_i^{G^*}(e) = \eta_i$. Now we check that no player has incentives to deviate from G^*. Let $i \in N$.

Case 1: $\bar{e}_i = \bar{e}_1$. We have $i = 1$. Player 1 cannot get more than $b_1(\bar{e}_2) + p_1(\bar{e}_2)$ outside $S(G_1^*) = [0, \bar{e}_2]$.

Case 2: $\bar{e}_i = \bar{e}_2$. We have $i = 2$. Player 2 cannot get more than $b_2(0)$ outside $S(G_2^*) = [0, \bar{e}_2]$.

Case 3: $\bar{e}_i < \bar{e}_2$. We have $i \geq 3$. Since $\bar{e}_i < \bar{e}_2$, player i cannot get more that $b_i(0)$ with strategies in $[e_2, M]$. Let $e \in (0, \bar{e}_2)$. Then,

$$u_i^G(e) = b_i(e) + \prod_{j \neq i} G_j^*(e)p_i(e) = b_i(e) + G_1^*(e)G_2^*(e)p_i(e) = b_i(e) + I_2(e)I_1^*(e)p_i(e).$$

Suppose now that $u_i^G(e) > b_i(0)$. Then, $b_i(e) + I_2(e)I_1^*(e)p_i(e) > b_i(0)$. Hence, $I_i(e) < I_2(e)I_1^*(e)$. Recall that for each $e \in (0, \bar{e}_2)$, $J(e) < 1$. Hence, $I_i(e) < I_2(e)I_1^*(e) < I_2(e)$, contradiction with $\bar{e}_i < \bar{e}_2$. $\square$

Now, we formally define a set of strategy profiles: NE^1. All of them lead to the equilibrium payoffs and, indeed, Theorem 5 shows that all of them are Nash equilibria and that any equilibrium that does not belong to NE^1 is a gap equilibrium. The forthcoming descriptions of the equilibrium strategy profiles and the statements of Theorems 5 and 6 are the generalizations to our framework of the ones in Baye et al. (1990), Theorem 1, and in Baye et al. (1996), Theorem 2, respectively.

Henceforth, let $m \in \{2, \ldots, n\}$ denote the player such that $\bar{e}_m = \bar{e}_2 > \bar{e}_{m+1}$. Assume that $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$ ($m \geq 3$). By Lemma 14, in a Nash equilibrium both 0 and $\bar{e}_2$ belong to the support of player 1's strategy. We want to characterize equilibria that are not gap equilibria and hence, we assume that the support of player 1's strategy is the whole interval $[0, \bar{e}_2]$. On the other hand, by Corollary 2, for each $i \in N$ such that $\bar{e}_i < \bar{e}_2$, i plays the pure strategy 0. Hence, we only need to worry about the players in $\{2, \ldots, m\}$. A strategy profile in NE^1 is characterized by a vector $d = (d_2, \ldots, d_m) \in [0, \bar{e}_2]^{m-1}$ such that there is $i \in \{2, \ldots, m\}$ with $d_i = 0$. Let D denote the set of all such vectors. Now, let $d \in D$. The interpretation of the vector d is the following: i) if $d_i < \bar{e}_2$, then $S(G_i) = \{0\} \cup [d_i, \bar{e}_2]$ and ii) if $d_i = \bar{e}_2$, $S(G_i) = \{0\}$, *i.e.* d_i can be seen as the "delay" chosen by a player to "enter" in the game. For each $j \in \{2, \ldots, m\}$, let $H(d_j)$ and $L(d_j)$ be the sets of players $i \in \{2, \ldots, m\}$ such that

$d_i > d_j$ and $d_i \leq d_j$, respectively. For simplicity, assume that $0 = d_2 \leq \ldots \leq d_m$. We define $G^d = (G^d_i)_{i\in N}$ as follows. If $i > m$, player i plays the pure strategy 0. Now,

$\boldsymbol{e \in [d_m, \bar{e}_2]}$**:**

For each $i \in \{2,\ldots,m\}$, $G^d_i(e) = I^*_1(e)^{\frac{1}{m-1}} = I^*_1(e)^{\frac{1}{|L(d_j)|}}$.

For player 1, $G^d_1(e) = I_2(e)I^*_1(e)^{-\frac{m-2}{m-1}} = I_2(e)I^*_1(e)^{-\frac{|L(d_j)|-1}{|L(d_j)|}}$.

$\boldsymbol{e \in [d_j, d_{j+1})}$**,** $j \in \{m-1,\ldots,2\}$**:**

For each $i \in H(d_j)$, $G^d_i(e) = G^d_i(d_i)$.

For each $i \in L(d_j)$, $G^d_i(e) = I^*_1(e)^{\frac{1}{|L(d_j)|}}\left(\prod_{k\in H(d_j)} G^d_k(d_k)\right)^{-\frac{1}{|L(d_j)|}}$.

For player 1, $G^d_1(e) = I_2(e)I^*_1(e)^{-\frac{|L(d_j)|-1}{|L(d_j)|}}\left(\prod_{k\in H(d_j)} G^d_k(d_k)\right)^{-\frac{1}{|L(d_j)|}}$.

Let $G \in \mathcal{G}^n$. Then, $G \in NE^1$ if and only if there is $d \in D$ such that $G = G^d$. It is straightforward to check that the strategy profiles in NE^1 satisfy satisfying the five points in the discussion after Theorem 1 for the case $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$.

Theorem 5. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY*, M-*BOUNDING*, and* NO-CROSSING**. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. If $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$, then G is a Nash equilibrium of C^f if and only if either $G \in NE^1$ or G is a gap equilibrium. Again, the equilibrium payoffs are $\eta_1 = b_1(\bar{e}_2) + p_1(\bar{e}_2)$ and, for each $i \neq 1$, $\eta_i = b_i(0)$.*

Proof. "⇒" First we show that, if $G \in \mathcal{G}^n$ is Nash equilibrium, then it belongs to either NE^1 or the set of gap equilibria. We show that the five points in the discussion after Theorem 1 for the case $\bar{e}_1 > \bar{e}_2 = \bar{e}_3$ are satisfied: **i)** is implied by Corollary 2, **ii)** is implied by Lemma 14, **iii)** is implied by the combination of Corollary 2, Lemma 13, and Lemma 14 (the latter being necessary for the part $d_i = 0$ for at least one $i \neq 1$), **iv)** is Lemma 12, and, finally, **v)** is Lemma 15. Now, if G is not a gap equilibrium, there is $d \in D$ such that, for each $i \in \mathbb{N}$, $S(G_i) = S(G^d_i)$. The specific expressions for the strategies of the players different from 1 come from iv) and v). The strategy of player 1 is pinned down by the equation $b_i(0) = \eta_i = b_i(e) + \prod_{j\neq 1} G_j(e)p_i(e)$, where i is a player such that $d_i = 0$.

"⇐" Now we show that all the profiles in NE^1 are equilibria. Let $d \in D$. Straightforward computations show that for each $i \in N$, if $e \in S(G^d_i)$, then $u^{G^d}_i(e) = \eta_i$. We check that no player has incentives to deviate from G^d. Let $i \in N$.

Case 1: $\boldsymbol{\bar{e}_i = \bar{e}_1}$**.** We have $i = 1$. Player 1 cannot get more than $b_1(\bar{e}_2) + p_1(\bar{e}_2)$ outside $S(G^*_1) = [0, \bar{e}_2]$.

Case 2: $\boldsymbol{\bar{e}_i = \bar{e}_2}$**.** We have $i \in \{2,\ldots,m\}$. Player i cannot get more than $b_i(0)$ with strategies in $[e_2, M]$. Suppose that there is $a \in [0, \bar{e}_2]$ such that $u^{G^d}_i(a) > b_i(0)$. We already know that for each $e \in S(G^d_i)$, $u^{G^d}_i(e) = b_i(0)$ and also that $u^{G^d}_i(0) = b_i(0)$. Hence, $a \in (0, d_i)$. Now, by iii) above, there is $j \in \{2,\ldots,m\}$, $j \neq i$ be such that $[a, \bar{e}_2] \in S(G_j)$. Now, $u^{G^d}_i(d_i) = u^{G^d}_j(d_i)$ and G_j is strictly increasing in (a, d_i). Hence, $G_j(a) < G_j(d_i) = G_i(d_i) = G_i(a)$. Now, $u^{G^d}_i(a) = b_i(a) + \prod_{k\neq i} G_k(a)p_i(a)$ and, since $\bar{e}_i = \bar{e}_j$ and $G_j(a) < G_i(a)$, we have $u^{G^d}_i(a) < b_j(a) +$

$\prod_{k\neq j} G_k(a)p_j(a) = \eta_j = b_j(0) = b_i(0)$, and we have a contradiction.

Case 3: $\bar{e}_i < \bar{e}_2$. Analogous to Case 3 in the proof of Theorem 4. $\square$

Now, we turn to the case $\bar{e}_1 = \bar{e}_2$ and define the set of strategy profiles NE^2. By Corollary 2, for each $i \in N$ such that $\bar{e}_i < \bar{e}_2$, i plays the pure strategy 0. We need to worry about the players in $\{1, \ldots, m\}$. Again, a strategy profile in NE^2 is characterized by a vector $d = (d_1, \ldots, d_m) \in [0, \bar{e}_2]^{m-1}$ such that there are $i, j \in \{1, \ldots, m\}$, $i \neq j$, such that $d_i = d_j = 0$. Let $\bar{D}$ denote the set of all such vectors. Now, let $d \in \bar{D}$. The interpretation of the vector d is the same as above. Now, for each $j \in \{1, \ldots, m\}$, let $\bar{H}(d_j)$ and $\bar{L}(d_j)$ be the sets of players $i \in \{1, \ldots, m\}$ such that $d_i > d_j$ and $d_i \leq d_j$, respectively. For simplicity, assume that $0 = d_1 \leq \ldots \leq d_m$. We define $\bar{G}^d = (\bar{G}_i^d)_{i\in N}$ as follows. If $i > m$, player i plays the pure strategy 0. Now,[20]

$\boldsymbol{e \in [d_m, \bar{e}_2]}$**:**

For each $i \in \{1, \ldots, m\}$, $\quad \bar{G}_i^d(e) = I_1(e)^{\frac{1}{m}} = I_1(e)^{\frac{1}{|\bar{L}(d_j)|}}$.

$\boldsymbol{e \in [d_j, d_{j+1})}$**,** $j \in \{m-1, \ldots, 1\}$**:**

For each $i \in \bar{H}(d_j)$, $\quad \bar{G}_i^d(e) = \bar{G}_i^d(d_i)$.

For each $i \in \bar{L}(d_j)$, $\quad \bar{G}_i^d(e) = I_1^*(e)^{\frac{1}{|\bar{L}(d_j)|}} \left(\prod_{k\in\bar{H}(d_j)} \bar{G}_k^d(d_k)\right)^{-\frac{1}{|\bar{L}(d_j)|}}$.

Let $G \in \mathcal{G}^n$. Then, $G \in NE^2$ if and only if there is $d \in \bar{D}$ such that $G = \bar{G}^d$. Again, it is straightforward to check that the strategy profiles in NE^2 satisfy the four points in the discussion after Theorem 1 for the case $\bar{e}_1 = \bar{e}_2 = \bar{e}_3$.

Theorem 6. *Let $C^f \in \mathcal{C}^*$. Assume* ALL-PAY*,* M-BOUNDING*, and* NO-CROSSING*. Let $G \in \mathcal{G}^n$ be a Nash equilibrium of C^f. If $\bar{e}_1 = \bar{e}_2$, then G is a Nash equilibrium of C^f if and only if $G \in NE^2$. Moreover, the equilibrium payoffs are, for each $i \in N$, $\eta_i = b_i(0)$.*

Proof. "$\Rightarrow$" First we show that, if $G \in \mathcal{G}^n$ is Nash equilibrium, then it belongs to NE^2. We show that the four points in the discussion after Theorem 1 for the case $\bar{e}_1 = \bar{e}_2 = \bar{e}_3$ are satisfied: i) is implied by Corollary 2, ii) is is implied by Lemma 14, iii) is implied by the combination of Corollary 2 and Lemma 13, and, finally, iv) is Lemma 12. Hence, there is $d \in \bar{D}$ such that, for each $i \in \mathbb{N}$, $S(G_i) = S(\bar{G}_i^d)$. The specific expressions for the strategies of the players different from 1 come from iv) and the equilibrium payoffs.

"$\Leftarrow$" Let $d \in \bar{D}$. Straightforward computations show that for each $i \in N$, if $e \in S(G_i^d)$, then $u_i^{G^d}(e) = \eta_i$. We check that no player has incentives to deviate from G^d. Let $i \in N$.

Case 1: $\bar{e}_i = \bar{e}_2$. Analogous to Case 2 in the proof of Theorem 5.

Case 2: $\bar{e}_i < \bar{e}_2$. Analogous to Case 3 in the proof of Theorem 4. $\square$

[20] In the corresponding expressions for the equilibrium strategies when $\bar{e}_1 = \bar{e}_2$ included in Baye et al. (1990, 1996) there is a minor typo. They wrote $|\bar{L}(d_j)| - 1$ instead of $|\bar{L}(d_j)|$ in the expressions of the G_i functions.