Understanding the Coincidence of Allocation Rules: Symmetry and Orthogonality in TU-Games

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Abstract

The main goal of this paper is to understand the reasons driving the coincidence of different allocation rules for different classes of games. We define a new symmetry property, reverse symmetry, and study its geometric and game theoretic implications. In particular, we show that most classic allocation rules satisfy it. Then, we introduce and study a notion of orthogonality between TU-games, which allows to establish a restricted additivity property for the nucleolus. Also, in our analysis we identify different classes of games for which all allocation rules satisfying some sets of basic properties coincide. These properties are satisfied, among others, by the Shapley value and the nucleolus.

Keywords. TU games, symmetry, orthogonality, coincidence of allocation rules
MSC2010 classification codes. 91A12
OR/MS classification words. cooperative games

1 Introduction

An important goal of cooperative game theory is to improve the understanding of the different allocation rules and, in particular, the similarities and differences among them. Within this line of research it is common to look for classes of games in which various allocation rules coincide.

Over the last few years, this topic has become quite active and several coincidence results have been obtained, mainly for the Shapley value and the (pre)nucleolus, as in Kar et al. (2009) and Chang and Tseng (2011). Part of this literature has been motivated by the study, from a game theoretical perspective, of different operations research problems such as graph related problems (Deng and Papadimitriou, 1994), telecommunication problems (van den Nouweland et al., 1996), and queueing problems (Chun and Hokari, 2007; Maniquet, 2003). This research has led to classes of cooperative games with a special structure; sometimes sufficient to get the aforementioned coincidence results. It is worth noting that these coincidence results have also been pursued in the power indices literature (see, for instance, Dragan (1996)).

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The main contribution of this paper is not so much to provide new classes of games where different allocation rules coincide, but to uncover the driving factors for some already existing classes. We follow an approach similar to that in Shapley (1971) and Maschler et al. (1979), studying the geometric implications of different properties and how different solution concepts behave with respect to them.

The analysis in the first part of the paper builds upon the geometric implications of different symmetry properties. The main contribution here is not so much about the new results we get, but to show how these geometric insights can be used to get simple proofs and generalizations of existing results. We start with a brief overview of the implications of the classic symmetry and then introduce the notion of *reverse symmetry*. Roughly speaking, a game is reverse symmetric if, given an ordering of the players, the vectors of contributions associated with this ordering and with the reverse one cancel out. An allocation rule satisfies reverse symmetry if all players get the same in reverse symmetric games. We show that most classic allocation rules are reverse symmetric. Moreover, we prove that PS-games (Kar et al., 2009), a class of games where the prenucleolus and the Shapley value coincide, can be generated by translations of *reverse symmetric games*. These games encompass several classes of games associated with operations research problems mentioned above, such as 2-games and queueing games (Chun and Hokari, 2007; Maniquet, 2003; van den Nouweland et al., 1996). Then, we show that there is a unique allocation rule satisfying efficiency, translation covariance, and reverse symmetry in the class of PS-games. This delivers the coincidence of Shapley value, prenucleolus, nucleolus, core-center, and τ value for PS-games, extending the results in Kar et al. (2009). Maybe more importantly, we uncover the reason for this coincidence result: the underlying geometric symmetry of the relevant set-valued solutions: core, Weber set, and core cover. Since the classic symmetry and reverse symmetry are not logically related, we present another symmetry property, *squareness*, which is a strengthening of both and that, combined with translation covariance and efficiency, leads to a new characterization for the Shapley value.

Chang and Tseng (2011) derive a class of simple games where the Shapley value and the nucleolus coincide and use these games as "generators" to construct two larger classes of games with the coincidence property: those with the classic symmetry property and another one very related to the class of PS-games, which, as we show in this paper, are also fairly symmetric.

In the second part of the paper we introduce the notion of orthogonality of TU-games and use it to define a restricted version of the additivity property (Shapley, 1953), which is not satisfied by many allocation rules beyond the Shapley value.[1] We provide a definition of orthogonality between TU-games that, ultimately, requires that there is at most one player who is not dummy in any of the two games. Then, we require allocation rules to be additive with respect to (weakly) orthogonal games. We call this property *orthogonal additivity* and show that it is satisfied, for instance, by the prenucleolus and the nucleolus. Finally, we build upon this property to obtain a new class of games where various allocation rules coincide. Again, this result can be better understood by looking at the geometric implications of orthogonality: to some extent, the cores (and Weber sets) associated to orthogonal games are orthogonal to each other.

One of our main results, the orthogonal additivity of prenucleolus and nucleolus, is an important result on its own. The behavior of these two allocation rules with respect to additivity has already been studied before. Kohlberg (1971) showed that the set of TU-games is a union of a finite number of closed and convex cones, on each of which the nucleolus is a linear function. In this sense, the orthogonal additivity we establish for the nucleolus implies that, in the presence of (weak) orthogonality, the above linearity may also be preserved across cones.

Finally, we would like to emphasize that, although the formal analysis on symmetry and

[1]For a couple of exceptions see van den Brink (2007), where the author obtains characterizations of the equal division and equal surplus division solutions in which, essentially, the null player property in Shapley's characterization is replaced with a nullifying player property (and translation covariance for equal surplus).

orthogonality are independent of each other, both use geometric tools to get a better understanding of the the coincidence of allocation rules. Yet, there is another important connection that lies in the so called *tent game*. The tent game is a 4-player game for which several allocation rules coincide, and this paper started as an attempt to understand why. The last result of the paper, Proposition 9, combines insights from symmetry and orthogonality properties to uncover the reasons for such a coincidence.

The paper is structured as follows. In Section 2 we present the basic notations. In Section 3 we devote our analysis to the study of symmetry properties. In Section 4 we define the notion of orthogonality of TU-games and discuss its geometric and game theoretic implications.

2 Basic notations and definitions

In this paper we cover a wide range of set-valued solutions, allocation rules, and properties, as well as several classes of games. In order to facilitate the flow in the rest of the paper, we define now most of the standard concepts, trying to be as concise as possible.[2]

A transferable utility or TU-game is a pair (N, v), where $N = \{1, \dots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a function assigning, to each coalition $S \subseteq N$, a payoff $v(S)$.[3] By convention, $v(\emptyset) = 0$. Throughout the paper we restrict attention to games where $v(N) \geq \sum_{i \in N} v(i)$. When no confusion arises we use i to denote $\{i\}$ (*e.g.*, $v(i)$, $v(S \cup i)$, $N \backslash i, \dots$). Let G^n be the set of all n-player TU-games. For the sake of exposition, hereafter we assume that the set N is fixed and we use v as a shorthand for (N, v). Given $S \subseteq N$, let $|S|$ be the number of players in S.

Two players i and j are *symmetric* if, for each $S \subseteq N \backslash \{i, j\}$, $v(S \cup i) = v(S \cup j)$. The *contribution* of a player i to a coalition $S \subseteq N \backslash i$ is defined by $\Delta_i(v, S) := v(S \cup i) - v(S)$. A player i is a *null player* if, for each $S \subseteq N \backslash i$, $\Delta_i(v, S) = 0$. A player i is a *dummy player* if, for each $S \subseteq N \backslash i$, $\Delta_i(v, S) = v(i)$; let $D(v)$ denote the set of dummy players of game v. Let $\Pi(N)$ denote the set of all permutations of the elements in N and, for each $\pi \in \Pi(N)$, let $P^\pi(i)$ denote the set of predecessors of i under the ordering given by π, *i.e.*, $j \in P^\pi(i)$ if and only if $\pi(j) < \pi(i)$. Moreover, $-\pi$ represents the reverse ordering to the one given by π. Given a game v and $\pi \in \Pi(N)$, the *vector of contributions* associated with π, $m^\pi(v) \in \mathbb{R}^N$, is defined, for each $i \in N$, by $m_i^\pi(v) := \Delta_i(v, P^\pi(i))$. The *utopia* and *minimum right* vectors, $\overline{M}(v)$ and $\underline{M}(v)$ are defined, for each $i \in N$, by $\overline{M}_i(v) := v(N) - v(N \backslash i)$ and $\underline{M}_i(v) := \max_{S \subseteq N, i \in S} \{v(S) - \sum_{j \in S \backslash i} \overline{M}_j(v)\}$, respectively (Tijs and Lipperts, 1982).

Classes of TU-games

A game is *symmetric* if all players are symmetric to each other. A game v is *convex* if, for each $i \in N$ and each S and T such that $S \subseteq T \subseteq N \backslash i$, $\Delta_i(v, S) \leq \Delta_i(v, T)$. A game v is *monotonic* if, for each pair $S, T, \subseteq N$ with $S \subseteq T$, $v(S) \leq v(T)$. A game v is *zero-normalized* if, for each $i \in N$, $v(i) = 0$. Given a game v, its *zero-normalization* v^0 is defined, for each $S \subseteq N$, by $v^0(S) = v(S) - \sum_{i \in S} v(i)$. A game v is *zero-monotonic* if its zero-normalization is a monotonic game. Given $S \subseteq N$, $S \neq \emptyset$, the *unanimity game of coalition* S, u_S, is defined as follows: for each $T \subseteq N$, $u_S(T) := 1$ if $S \subseteq T$ and $v_S(T) := 0$ otherwise. A game $v \in G^n$ is *additive* if, for each player $i \in N$ and each coalition $S \subseteq N \backslash i$, $v(S) + v(i) = v(S \cup i)$; in particular, for each $S \subseteq N$, $v(S) = \sum_{i \in S} v(i)$.

[2] Some references on the topic are Peters (2008) and González-Díaz et al. (2010).
[3] We use $\subset$ for strict set inclusions and $\subseteq$ for weak set inclusions.

Allocations, sets of allocations, and allocation rules

Given $x \in \mathbb{R}^n$ and $S \subseteq N$, $x(S) := \sum_{i\in S} x_i$. A (feasible) *allocation* is a vector $x \in \mathbb{R}^n$ such that $x(N) \leq v(N)$. An *allocation rule* defined on some domain $\Omega \subseteq G^n$ is a function that, for each game $v \in \Omega$, selects an allocation.

An allocation x is *efficient* if $x(N) = v(N)$; x is *individually rational* if, for each $i \in N$, $x_i \geq v(i)$; x is *coalitionally rational* if for each $S \subset N$, $x(S) \geq v(S)$.

The *set of preimputations* is the set of all efficient allocations. The *set of imputations* is the set of all efficient and individually rational allocations. The *core* is the set of all efficient and coalitionally rational allocations (Gillies, 1953). The *Weber set* is the convex hull of the set of vectors of contributions (Weber, 1988). The *core cover* is the set of all efficient allocations in which every player receives at least his minimum right and at most his utopia payoff (Tijs and Lipperts, 1982).

Given an allocation x and a coalition $S \subseteq N$, the *excess* of S with respect to x is $e(S,x) := v(S) - x(S)$. The *vector of ordered excesses* $\theta(x)$ is constructed by arranging the excesses corresponding to the coalitions in $2^N\backslash\{\emptyset, N\}$ in non-increasing order. We use $\prec_L$ and $\preceq_L$ to compare vectors with respect to the lexicographic ordering. The *prenucleolus*, $\mathrm{PNu}(v)$, is the unique allocation in the set $\{x : x$ is a preimputation and, for each preimputation y, $\theta(x) \preceq_L \theta(y)\}$ (Maschler et al., 1979). The *nucleolus*, $\mathrm{Nu}(v)$, is the unique allocation in the set $\{x : x$ is an imputation and, for each imputation y, $\theta(x) \preceq_L \theta(y)\}$ (Schmeidler, 1969).

The *Shapley value*, $\mathrm{Sh}(v)$, is the average of the $n!$ vectors of contributions (Shapley, 1953). The τ value of a game with a nonempty core cover, $\tau(v)$, is the unique efficient point on the line segment joining $\underline{M}(v)$ and $\overline{M}(v)$ (Tijs, 1981). The *core-center* of a game with a nonempty core, $\mathrm{corecenter}(v)$, is the center of gravity of the core (González-Díaz and Sánchez-Rodríguez, 2007). The *equal division* and the *equal surplus division* solutions, $\mathrm{ED}(v)$ and $\mathrm{ESD}(v)$, assign, to each $i \in N$, $\mathrm{ED}_i(v) := v(N)/n$ and $\mathrm{ESD}_i(v) := v(i) + \big(v(N) - \sum_{j\in N} v(j)\big)/n$, respectively (see, for instance, van den Brink (2007)).

Properties

Let φ be an allocation rule: φ is *efficient*, EFF, if it always selects efficient allocations; φ is *individually rational*, IR, if it always selects individually rational allocations; φ satisfies *additivity*, ADD, if for each two games (N,v) and (N,w), $\varphi(v+w) = \varphi(v) + \varphi(w)$; φ satisfies *translation covariance*, TC, if for each two games (N,v) and (N,w), and each $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$ such that for each $S \subseteq N$, $w(S) = v(S) + \sum_{i\in S}\alpha_i$, then $\varphi(N,w) = \varphi(N,v) + \alpha$; φ is *symmetric*, SYM, if for each pair $\{i,j\}$ of symmetric players, $\varphi_i(v) = \varphi_j(v)$; φ is *weakly symmetric*, WSYM, if for each symmetric game and each pair $\{i,j\}$ of players, $\varphi_i(v) = \varphi_j(v)$; φ satisfies the *null player property*, NPP, if for each null player i, $\varphi_i(N,v) = 0$; φ satisfies the *dummy player property*, DPP, if for each dummy player i, $\varphi_i(N,v) = v(i)$. In Table 1 we present a summary of how the different allocation rules that we discuss in this paper behave with respect to the standard properties.

3 Symmetry properties and geometric implications

3.1 Classic symmetry

We start by discussing some straightforward implications of the classic definition of symmetry, probably familiar to most of the readers. Yet, it is convenient to undergo this discussion to facilitate the contextualization of the new symmetry property that we introduce in Section 3.2.

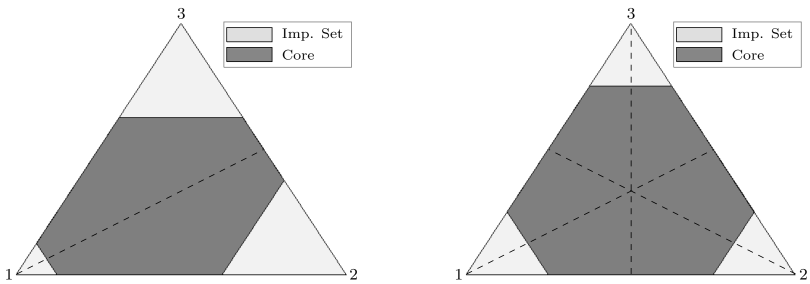
	EFF	IR	ADD	TC	SYM	WSYM	NPP	DPP
Prenucleolus	✓	✓*	X	✓	✓	✓	✓*	✓*
Nucleolus	✓	✓	X	✓	✓	✓	✓	✓
Shapley value	✓	✓*	✓	✓	✓	✓	✓	✓
τ value	✓	✓	X	✓	✓	✓	✓	✓
Core-center	✓	✓	X	✓	✓	✓	✓	✓
Equal division	✓	X	✓	X	✓	✓	X	X
Equal surplus division	✓	✓*	✓	✓	✓	✓	X	X

* This property holds if we restrict attention to zero-monotonic games; which ensures that the imputations set is nonempty and, further, that the prenucleolus and the nucleolus coincide.

Table 1: Allocation rules and properties.

Lemma 1. *In the class of symmetric games, any allocation rule satisfying* EFF *and* WSYM *coincides with the equal division solution.*

Next, we look at symmetry properties from the geometric point of view. First, we define a pair of standard symmetry notions for sets. Let $A \subseteq \mathbb{R}^n$. The set A has *reflection symmetry* with respect to hyperplane H if, for each $x \in A$, the mirror image of x with respect to H belongs to A.The set A has *inversion or point symmetry* if there is $y \in \mathbb{R}^n$ such that, for each $x \in A$, the mirror image of x with respect to y belongs to A. Formally, $x \in A$ if and only if $-x + 2y \in A$.



(a) A game with two players that are symmetric.

(b) A symmetric game.

Figure 1: Geometric implications of symmetries.

Given two players i and j, we define the hyperplane $H^{ij} := \{x \in \mathbb{R}^n : x_i = x_j\}$. In general, if i and j are symmetric in a given game, most classic sets of allocations are reflection symmetric with respect to H^{ij}. Figure 1(a) represents the imputations set and core of a game in which players 2 and 3 are symmetric, leading to a core symmetric with respect to hyperplane H^{23}.[4]

Lemma 2. *Let $v \in G^n$. If players i and j are symmetric then, the core, the Weber set, and the core cover are reflection symmetric with respect to hyperplane H^{ij}.*

Given a symmetric game, Lemma 2 implies that the equal division solution is the center of gravity of the core, the Weber set, and the core cover. From the geometric point of view, one can go a bit beyond symmetric games and still preserve symmetric sets. When taking a translation of a given game, the above sets are translated in the same way. Since reflection symmetries are preserved by translations, the respective centers of gravity also coincide for games that are translations of symmetric games. This is summarized in the following well known result.

[4]Although throughout the paper we use the core to illustrate and motivate our analysis, in most of the figures we could have also used the Weber set or the core cover.

Lemma 3. *Let $v \in G^n$ be the translation of a symmetric game. Then, any allocation rule satisfying* EFF*,* WSYM*, and* TC *coincides with the equal surplus division solution.*

3.2 Reverse symmetry

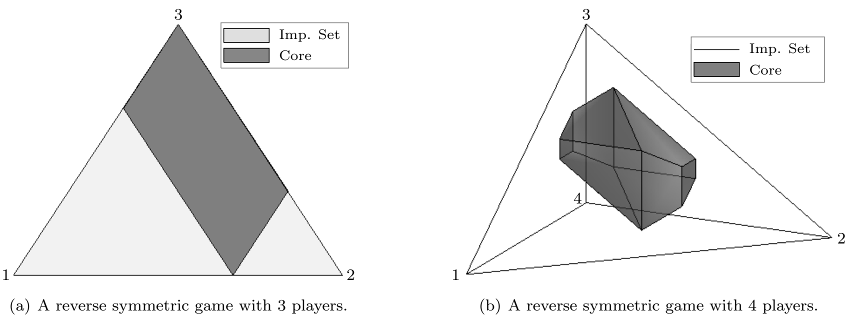
In this section we define a symmetry property that delivers new coincidence results and, in particular, allows to get a better understanding of some already existing ones. The main contribution lies in the tool, *reverse symmetry*, that was not identified by previous works and that allows to get cleaner intuitions and very simple proofs of the coincidence results.

Definition 1. A game $v \in G^n$ is *reverse symmetric* if there is $k \in \mathbb{R}$ such that, for each $\pi \in \Pi(N)$, $m^\pi + m^{-\pi} = (k, \ldots, k)$.

In a reverse symmetric game, for each player and each ordering in which his contribution is "good" with respect to utility level k, there is another ordering (the reverse one) in which his contribution is as "bad" as "good" was before. That is, there is a symmetry across players in the sense that their contributions across orderings even up. Thus, it is sensible to consider the following property for allocation rules.

Definition 2. An allocation rule satisfies *reverse symmetry*, RSYM, if all players get the same in reverse symmetric games.

The combination of EFF and RSYM implies that all players get $v(N)/n$ in an reverse symmetric game. We show below that, with the exception of the equal surplus division solution, all the allocation rules under study satisfy reverse symmetry. First, we study the geometric implications of this property. In Figure 2 we show the imputations set and the core of two reverse symmetric games. Clearly, both cores have a symmetric structure, but they exhibit no reflection symmetry with respect to the H^{ij} hyperplanes. Instead, both of them are inversion symmetric. We define now a strengthening of the reverse symmetry property that is helpful for the ensuing analysis.



(a) A reverse symmetric game with 3 players. (b) A reverse symmetric game with 4 players.

Figure 2: Two reverse symmetric games.

Definition 3. A game $v \in G^n$ is *strongly reverse symmetric* if, for each $\pi \in \Pi(N)$, $m^\pi + m^{-\pi} = 0$, *i.e.*, $m^\pi = -m^{-\pi}$.

Note that we can transform an reverse symmetric in a strongly reverse symmetric game via the translation with vector $(-k/2, \ldots, -k/2)$.

Lemma 4. *If $v \in G^n$ is a strongly reverse symmetric game, then $v(N) = 0$. Moreover, for each $S \subset N$, $v(S) = v(N \backslash S)$.*

Proof. Let $\pi \in \Pi(N)$. Then, $v(N) = m^\pi(N)$ and, further, $2v(N) = m^\pi(N) + m^{-\pi}(N) = 0$. Therefore, $v(N) = 0$. Now, let $S \subset N$ and let $\pi \in \Pi(N)$ be an ordering in which the players in S go first (and then, in $-\pi$, the players in $N \backslash S$ go first). Therefore, we have $v(S) = m^\pi(S)$ and $v(N \backslash S) = m^{-\pi}(N \backslash S)$. Now, we get that $v(S) = v(N \backslash S)$, since

$$0 = v(N) = m^\pi(N) = m^\pi(S) + m^\pi(N \backslash S) = v(S) - m^{-\pi}(N \backslash S) = v(S) - v(N \backslash S). \qquad \square$$

Proposition 1. *If $v \in G^n$ is a strongly reverse symmetric game, then the core and the Weber set are inversion symmetric with respect to the allocation $(0, \ldots, 0)$. If, moreover, v is zero-monotonic, also the core cover is inversion symmetric with respect to $(0, \ldots, 0)$.*

Proof. A set is inversion symmetric with respect to the origin if, for each element x in the set, also $-x$ belongs to the set. By Lemma 4, for each $S \subset N$, $v(S) = v(N \backslash S)$ and $v(N) = 0$.

For the Weber set note that this set is the convex hull of the vectors of contributions and, for each such vector, say m^π, $m^{-\pi} = -m^\pi$. Let $x \in C(v)$ and suppose $-x \notin C(v)$. By efficiency, $x(N) = v(N) = 0$ and so $-x(N) = 0 = v(N)$. Then, there is $S \subset N$ such that $v(S) > -x(S)$. Hence, $v(N \backslash S) = v(S) > -x(S) = -(v(N) - x(N \backslash S)) = x(N \backslash S)$, which contradicts that $x \in C(v)$. Suppose now that v is zero-monotonic and so we have that, for each $S \subseteq T$, $v(S) + \sum_{j \in T \backslash S} v(j) \leq v(T)$. Let $i \in N$. Then, $0 = v(N) \geq v(i) + v(N \backslash i) = 2v(i)$, so $v(i) \leq 0$. Also, $\overline{M}_i = v(N) - v(N \backslash i) = 0 - v(i) = -v(i) \geq 0$. Finally,

$$\underline{M}_i = \max_{S \subseteq N, i \in S} \{v(S) - \sum_{j \in S \backslash i} \overline{M}_j(v)\} = \max_{S \subseteq N, i \in S} \{v(N \backslash S) + \sum_{j \in S \backslash i} v(j)\}.$$

By zero-monotonicity, for each $S \subset N$ with $i \in S$, $v(N \backslash S) + \sum_{j \in S \backslash i} v(j) \leq v(N \backslash i)$. Thus, the above maximum is attained when $S = \{i\}$ and takes the value $v(N \backslash i) = v(i)$. Therefore, $\overline{M} = -(v(1), \ldots, v(n))$ and $\underline{M} = (v(1), \ldots, v(n))$, from which it is straightforward to check that the core cover is inversion symmetric with respect to $(0, \ldots, 0)$. $\square$

Proposition 2. *Let $v \in G^n$ be a strongly reverse symmetric game. Then, the Shapley value, the prenucleolus, the nucleolus, the core-center, and the equal division solution select the allocation $(0, \ldots, 0)$.*[5] *If, moreover, v is zero-monotonic, also the τ value selects $(0, \ldots, 0)$.*

Proof. The Shapley value is the average of the vectors of contributions. For each ordering π, since v is reverse symmetric, $m^\pi + m^{-\pi} = 0$, so the average across all orderings is $(0, \ldots, 0)$.

By Lemma 4, for each $S \subset N$, $v(S) = v(N \backslash S)$. Let x be an efficient allocation and let $S \subset N$. Then, $x(S) = v(N) - x(N \backslash S) = -x(N \backslash S)$ and, hence, $e(S, x) = v(S) - x(S) = v(N \backslash S) + x(N \backslash S) = e(N \backslash S, -x)$. Therefore, $\theta(x) = \theta(-x)$. In particular, $\theta(\mathrm{PNu}(v)) = \theta(-\mathrm{PNu}(v))$; but the prenucleolus is the unique allocation minimal with respect to the ordering $\preceq_L$, so $\mathrm{PNu}(v) = -\mathrm{PNu}(v) = (0, \ldots, 0)$. The argument for the nucleolus is analogous.

The core-center is the center of gravity of the core, which we have shown is $(0, \ldots, 0)$. By definition, the equal division solution also delivers $(0, \ldots, 0)$. If the game is zero-monotonic, the τ value is the unique efficient allocation in the line joining $\underline{M} = (v(1), \ldots, v(n))$ and $\overline{M} = -(v(1), \ldots, v(n))$, *i.e.*, $(0, \ldots, 0)$. $\square$

Corollary 1. *The Shapley value, the prenucleolus, the nucleolus, the core-center, and the equal division solution satisfy* RSYM. *Moreover, the τ value satisfies* RSYM *for zero-monotonic games.*

[5] The statement for the core-center only applies to games where the core is nonempty. Since v is strongly reverse symmetric, nonemptyness of the core is equivalent to require that, for each $S \subset N$, $v(S) \leq 0$.

Proof. The result is trivially true for the equal division solution. Now, recall that an reverse symmetric game with respect to constant k can be transformed into a strongly reverse symmetric game using the translation with respect to the vector $(-k/2, \ldots, -k/2)$. Hence, since all the allocation rules apart from equal division satisfy TC, the result follows from Proposition 2. $\square$

	SYM	WSYM	RSYM
Prenucleolus	✓	✓	✓
Nucleolus	✓	✓	✓
Shapley Value	✓	✓	✓
τ value	✓	✓	✓*
Core-center	✓	✓	✓
Equal division	✓	✓	✓
Equal surplus division	✓	✓	X

* This property holds for zero-monotonic games.

Table 2: Allocation rules and symmetry properties.

Table 2 summarizes the behavior of the allocation rules we are considering in this paper with respect to the different symmetry properties. The example below illustrates that the equal surplus division solution does not satisfy RSYM and that zero-monotonicity is needed for the τ value.

Example 1. Let $v \in G^3$ be the reverse symmetric game defined by $v(N) = v(3) = v(12) = 0$, $v(1) = v(23) = -1$, $v(2) = v(13) = -2$. This game is not zero-monotonic. The utopia vector is $\overline{M} = (1, 2, 0)$ and the minimum rights vector is $\underline{M} = (-1, -1, 0)$. The τ value is $\tau(v) = (-0.2, 0.2, 0)$. The equal surplus division solution is $\mathrm{ESD}(v) = (0, -1, 1)$. It is worth noting that, for three player games, the core and the core cover coincide (Tijs and Lipperts, 1982) and, therefore, although the core of v, and so the core cover as well, is inversion symmetric with respect to $(0, \ldots, 0)$, the τ value is not selecting this allocation, the center of symmetry. $\diamond$

Example 2. Zero-monotonicity is not enough to recover RSYM for the equal surplus division solution. Let $v \in G^3$ be the game defined by $v(N) = v(3) = v(12) = 0$, $v(1) = v(2) = v(23) = v(13) = -3$. This game is reverse symmetric and zero-monotonic and $\mathrm{ESD}(v) = (-1, -1, 2)$. $\diamond$

As we did with the classic symmetry, we present a result that combines RSYM with EFF and TC to get a class of games in which most allocation rules coincide. The proof is straightforward.

Proposition 3. *Let $v \in G^n$ be the translation of a reverse symmetric game. Then, all the allocation rules satisfying* EFF, RSYM, *and* TC *select the same allocation.*

Note that, with respect to Proposition 2, this result only leaves out the equal division solution, which does not satisfy TC. It is now natural to wonder how large the above class of games is. What we show below is that it coincides exactly with the class of PS-games.

Definition 4. A game $v \in G^n$ is a *PS-game* if there is $c \in \mathbb{R}^n$ such that, for each $i \in N$ and each $S \subseteq N \backslash i$, we have that $\Delta_i(v, S) + \Delta_i(v, N \backslash (S \cup i)) = c_i$.

These games were introduced in Kar et al. (2009) as an example of a class of games where Shapley value and prenucleolus coincide. They showed they encompass, for instance, the queueing games defined in Maniquet (2003) and those in Chun (2006); the coincidence of different allocation rules for the latter had already been studied in Chun and Hokari (2007).

Proposition 4. *A game $v \in G^n$ is a PS-game if and only if it is the translation of a reverse symmetric game.*

Proof. First of all note that a game is the translation of a reverse symmetric game if and only if there is $c \in \mathbb{R}^n$ such that, for each $\pi \in \Pi(N)$, $m^\pi + m^{-\pi} = c$.

"$\Rightarrow$" Let v be a PS-game. By definition, there is $c \in \mathbb{R}^n$ such that, for each $i \in N$ and each $S \subseteq N \backslash i$, $\Delta_i(v, S) + \Delta_i(v, N \backslash (S \cup i)) = c_i$. Now, let $\pi \in \Pi(N)$ and $i \in N$. Then,

$$m_i^\pi(v) = \Delta_i(v, P^\pi(i)) = -\Delta_i(v, N \backslash (P^\pi(i) \cup i)) + c_i = -\Delta_i(v, P^{-\pi}(i)) + c_i = -m_i^{-\pi}(v) + c_i,$$

so $m_i^\pi(v) + m_i^{-\pi}(v) = c_i$ and v is the translation of a reverse symmetric game.

"$\Leftarrow$" Let v be the translation of a reverse symmetric game. Let $i \in N$ and $S \subseteq N \backslash i$. Let $\pi \in \Pi(N)$ be an ordering in which $P^\pi(i) = S$. Then,

$$\begin{aligned} \Delta_i(v, S) &= \Delta_i(v, P^\pi(i)) = m_i^\pi(v) = -m_i^{-\pi}(v) + c_i = \\ &= -\Delta_i(v, P^{-\pi}(i)) + c_i = -\Delta_i(v, N \backslash (S \cup i)) + c_i, \end{aligned}$$

so $\Delta_i(v, S) + \Delta_i(v, N \backslash (S \cup i)) = c_i$ and v is a PS-game. □

Corollary 2. *Let $v \in G^n$ be a PS-game. Then, all the allocation rules satisfying* EFF, RSYM, *and* TC *select the same allocation.*

Proof. Immediate from the combination of Propositions 3 and 4. □

In view of the last two results, Proposition 3 can be seen as an extension of the coincidence result for the Shapley value and the prenucleolus in Kar et al. (2009). Maybe more importantly, our approach has uncovered the elements driving the coincidence of the different allocation rules for the games in this class: PS-games are translations of reverse symmetric games.

3.3 Beyond symmetry and reverse symmetry: squareness

So far we have presented two classes of games where most classic allocation rules coincide. Both build upon symmetry properties but there is no logical relation between them. We introduce now another symmetry property that implies both WSYM and RSYM and study its implications. Let $v \in G^n$. A player $i \in N$ is an *average player* if his average contribution is $v(N)/n$, *i.e.*, $\sum_{\pi \in \Pi(N)} m_i^\pi / n! = v(N)/n$. A game $v \in G^n$ is *square* if all the players are average players. Since in a square game all the players are average players or, put differently, their contributions even up, it seems natural to consider the following property: An allocation rule satisfies *squareness*, SQ, if all players get the same in square games.

Example 3. Consider the game v given by $v(1) = 3.5$, $v(2) = 1.5$, $v(3) = 0$, $v(12) = 6$, $v(13) = 7.5$, $v(23) = 9.5$, and $v(N) = 15$. Table 3 shows the vectors of contributions of v and their sum, which is equal across players and so v is a square game. ◇

It is easy to check that symmetric and reverse symmetric games are square games. Further, as we show in the proof of Proposition 5 below, every game in G^n can be transformed in a square game by using a translation. Then, since being square is a relatively mild requirement on a game, one should expect SQ to be a strong requirement on an allocation rule. Actually, replacing RSYM with SQ in the properties of Proposition 4 and Corollary 2 has a dramatic effect.

Proposition 5. *In G^n, the Shapley value is the unique allocation rule satisfying* EFF, TC, *and* SQ. *Moreover, if $n \geq 3$, each of the above properties is logically independent of the other two.*

Order π	Player 1	2	3	Eff.
123	3.5	2.5	9	15
132	3.5	7.5	4	15
213	4.5	1.5	9	15
231	5.5	1.5	8	15
312	7.5	7.5	0	15
321	5.5	9.5	0	15
Squareness	30	30	30	90

Table 3: Vectors of contributions in the square game of Example 3.

Our Characterization		**Shapley's Characterization**
Efficiency	$=$	Efficiency
Translation covariance	$\Leftarrow$	Null player + Additivity
Squareness	$\not\Leftarrow$	Symmetry

Table 4: Characterizations' comparison.

Proof. We already know that the Shapley value satisfies EFF and TC and, since it gives each player his average contribution, it trivially satisfies SQ as well. We show now that every game in G^n is one translation away from being square. Let $v \in G^n$ and, for each $i \in N$, let $\bar{m}_i := \sum_{\pi\in\Pi(N)} m_i^\pi(v)/n!$, that is, $\bar{m}$ is the vector of average contributions. Now, let w be the game obtained from v after translating using vector $-\bar{m}$. Clearly, w is a square game. Hence, by SQ and EFF, all players get $w(N)/n$ in game w and, by TC, player i gets $w(N)/n + \bar{m}_i$ in game v.

Independence: If we drop EFF, $\mathrm{Sh}(v) + (1, \dots, 1)$ satisfies TC and SQ; if we drop TC, equal division satisfies EFF and SQ; if we drop SQ, the prenucleolus satisfies EFF and TC. □

In Table 4 we compare this new characterization with the classic one in Shapley (1953). On the one hand, we replace ADD and DPP with the much weaker property of TC. On the other hand, we include a symmetry requirement with a broader scope, SQ instead of SYM.

3.4 Dummy players and symmetry

To conclude this section we present a last class of "symmetric" games along with a couple of straightforward results. We come back to this class of games when we discuss the motivation and implications of the orthogonality notions defined in Section 4.

Definition 5. A game $v \in G^n$ is *dummy-symmetric* if in its zero-normalization, v^0, each pair of players that are not dummy are symmetric.

An important subclass of dummy-symmetric games is the class of unanimity games. In the unanimity game of coalition S, all the players in S are symmetric to each other and all the players outside S are dummy players. The next two results are now straightforward.

Proposition 6. *Let $v \in G^n$ be a dummy-symmetric game. Then, all the allocation rules satisfying* EFF, DPP, SYM, *and* TC *select the same allocation.*

Corollary 3. *The Shapley value, the nucleolus, the τ value, and the core-center coincide for dummy symmetric games. If, moreover, the games are zero-monotonic, also the prenucleolus coincides with them.*

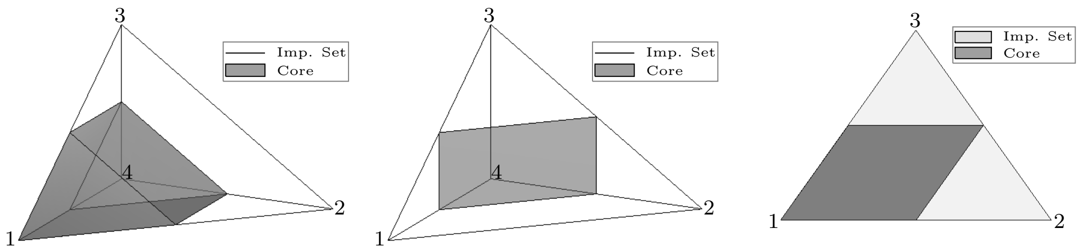
Geometrically, the Weber set, the core, and the core cover of dummy symmetric games are as symmetric as they are for symmetric games. In general, these sets are full dimensional in $\mathbb{R}^{n-1}$ (not in $\mathbb{R}^n$ because of the efficiency constraint). However, when there are dummy players, these sets become degenerate (a dummy player i always gets $v(i)$) and they look like the corresponding sets in the game where the dummy players have been left out. Thus, there are enough reflection symmetries to pin down the center of gravity of the set as the intersection of the H^{ij} hyperplanes.

4 Orthogonality in TU-games

In this section we keep studying the coincidence of different allocation rules from a geometrical point of view, but with a different angle. We introduce the notion of orthogonality of TU-games and show that most allocation rules satisfy additivity with respect to orthogonal games. We present below an example for which several allocation rules coincide. The reasons for this become clear at the end of this section, where Proposition 9 builds upon both symmetry and orthogonality notions to obtain a coincidence result that includes the tent game as a particular case.

4.1 A motivating example: the tent game

Consider the game $v \in G^4$ defined as the sum of the unanimity games u_{123} and u_{14}. This game has some symmetries (players 2 and 3 are symmetric) but does not fall into any of the categories discussed in the previous section. Therefore, none of the coincidence results presented there applies to this game. Although we know that most allocation rules coincide for the dummy symmetric games u_{123} and u_{14}, there is no result that tells whether or not they still coincide for the game $v = u_{123}+u_{14}$. In this example we get that $\mathrm{Sh}(v) = \mathrm{PNu}(v) = \mathrm{Nu}(v) = \mathrm{corecenter}(v) = (5/6, 1/3, 1/3, 1/2)$, the sum of what these allocation rules would select for the two unanimity games, but $\tau(v) = (4/5, 2/5, 2/5, 2/5)$. Then, even if we have lost the coincidence with the τ value, there may be something special about game v. In Figure 3(a) we have depicted the imputations set and the core of v. It resembles a tent and, therefore, we refer to this game as the *tent game* from now onwards. Clearly, the core of v is fairly symmetric and, further, it somehow looks like the Cartesian product of $C(u_{123})$ (a triangle) and $C(u_{12})$ (a segment); there seems to be some orthogonality between the segments joining the two triangular faces of the core and the two triangles. In the rest of this section we uncover what is so special about games like v.



(a) The game $v = u_{123} + u_{14}$. (b) The game $v = u_{12} + u_{34}$. (c) The game $v = u_{12} + u_{13}$.

Figure 3: Sums of games that, under our definitions, are (weakly) orthogonal.

4.2 Orthogonality

A vector $\alpha \in \mathbb{R}^n$ can be seen as an additive game in which the worth of each coalition S is just $\alpha(S)$. Now, DPP implies that in an additive game each player i gets $v(i)$ and, hence, the combination of ADD and DPP implies TC. Similarly, an allocation rule that satisfies DPP and TC satisfies additivity when one of the involved games is an additive game. This restricted version of additivity is quite natural. One of the main criticisms to the standard additivity property is that it does not allow for any kind of synergies or externalities when adding two games together. However, additive games, where all the players are dummy players, generate no synergies when being added to other games and so imposing additivity when these games are involved does not seem very restrictive. The idea of the orthogonality notion we define in this paper is to go one step beyond additive games and identify other situations where it is natural to assume that the two games being added generate no synergies to one another. The main contribution in this section is to introduce an *orthogonal additivity* property that lies in between ADD and TC and that is satisfied by most classic allocation rules.

It is natural to consider that two games v and w are orthogonal if, whenever a contribution of a player in v is different from 0, the corresponding contribution in w is 0, *i.e.*, for each $i \in N$ and each $S \subseteq N\backslash i$, $\Delta_i(v, S)\, \Delta_i(w, S) = 0$. This idea has very strong implications for the relation between the dummy players of games v and w. We present now our general definition of orthogonality and then elaborate on its connections with the products $\Delta_i(v, S)\, \Delta_i(w, S)$.

Definition 6. Two games v and w in G^n are *orthogonal* if $|D(v) \cup D(w)| = n$. They are *weakly orthogonal* if $|D(v) \cup D(w)| \geq n - 1$.

The notion of weak orthogonality is crucial for our approach, since orthogonality is too strong for the tent game. There we had the games u_{123} and u_{14}, which are not orthogonal because player 1 is not a dummy player in any of them. Actually, if a game is the sum of orthogonal games, the core and the Weber set will be degenerate since they will lie in the hyperplanes $\{x \in \mathbb{R}^n : x(D(v)) = w(N)\}$ and $\{x \in \mathbb{R}^n : x(D(w)) = v(N)\}$. Figure 3(b) shows the (degenerate) core of the sum of two orthogonal games: u_{12} and u_{34}. The core is a rectangle, the Cartesian product of the two segments corresponding with the cores of u_{12} and u_{34}.[6] Weak orthogonality can account for games that have a full dimensional Weber set or core.

Lemma 5. *Let v and w be two zero-normalized convex games. Then, v and w are orthogonal if and only if, for each $i \in N$ and each $S \subseteq N\backslash i$, $\Delta_i(v, S)\, \Delta_i(w, S) = 0$. Similarly, v and w are i-*weakly orthogonal *if for each $j \neq i$ and each $S \subseteq N\backslash j$, $\Delta_j(v, S)\, \Delta_j(w, S) = 0$.*

Proof. We only present the proof for orthogonal games, since the one for weakly orthogonal games is analogous. "$\Rightarrow$" Straightforward.

"$\Leftarrow$" Let $i \notin D(v)$. Then, there is $S \subseteq N\backslash i$ such that $\Delta_i(v, S) \neq 0$. Suppose that $i \notin D(w)$. Then, there is $S' \subseteq N\backslash i$ such that $\Delta_i(w, S') \neq 0$. Then, by convexity, $\Delta_i(v, S' \cup S) \neq 0$ and $\Delta_i(w, S' \cup S) \neq 0$, which contradicts the orthogonality of v and w. □

The above result implies that two convex games v and w are *orthogonal* if, for each $i \in N$ and each $S \subseteq N\backslash i$, $(\Delta_i(v, S) - v(i))\, (\Delta_i(w, S) - w(i)) = 0$. According to our discussion earlier in this section, one can somewhat say that there is not much room for "synergies" when adding two orthogonal games or even two weakly orthogonal games. This motivates the following definition.

Definition 7. An allocation rule φ satisfies *orthogonal additivity*, ADD$^{\perp}$, if for each pair of weakly orthogonal games v and w, $\varphi(v + w) = \varphi(v) + \varphi(w)$.

Since the Shapley value, the equal division solution, and the equal surplus division solutions satisfy additivity, they also satisfy ADD$^{\perp}$. The rest of this section is devoted to study how other allocation rules behave with respect to ADD$^{\perp}$. Once these results are established, we will have uncovered the reasons underlying the coincidence of the different allocation rules for the tent game, since the games u_{123} and u_{14} are weakly orthogonal. The example below shows that the τ value does not satisfy ADD$^{\perp}$, not even for orthogonal convex games.

[6] It is worth noting that orthogonality is related to the notion of decomposability introduced in Shapley (1971). Roughly speaking, a convex game is decomposable if and only if it is the sum of orthogonal games. Further, it is not hard to check that two zero-normalized games that are weakly orthogonal are also disjoint in the sense of van den Brink et al. (2006) and, therefore, orthogonal additivity will typically be weaker than disjoint additivity.

Example 4. Let v and w be following two orthogonal zero-normalized convex games in G^5. In game v, $D(v) = \{4, 5\}$, $v(12) = 4$, $v(13) = v(23) = 0$, and $v(123) = 12$; in game w, $D(w) = \{1, 2, 3\}$ and $w(45) = 4$; the remaining coalitions are formed adding dummy players, so the worth of each of them can be trivially computed. The corresponding τ values are $\tau(v) = (4.5, 4.5, 3, 0, 0)$, $\tau(w) = (0, 0, 0, 2, 2)$, and $\tau(v + w) = (4.8, 4.8, 3.2, 1.6, 1.6) \neq \tau(v) + \tau(w)$. $\diamond$

We present now some extra notations. Hereafter, given two weakly orthogonal games v and w, we assume, without loss of generality, that $N\backslash 1 \subseteq D(v) \cup D(w)$, *i.e.*, player 1 is the only player that may not be dummy in any of the two games. Let $N^v := N\backslash D(v)$ and $N^w := N\backslash(D(w)\backslash D(v))$. So defined, given two weakly orthogonal games v and w, $N^v \cup N^w = N$ and $N^v \cap N^w \subseteq \{1\}$. Given $S \subseteq N$, we define $S^v := S \cap N^v$ and $S^w := S \cap N^w$. Since all the solution concepts under consideration satisfy TC we often restrict to zero-normalized games. Suppose that v and w are zero-normalized weakly orthogonal games and let z be an efficient allocation in game $v + w$. Then, we associate with z one allocation for v and one allocation for w; z^v and z^w respectively. They are defined as follows. For each $i \notin N^v$, $z^v_i := 0$ and, for each $i \in N^v$, $i \neq 1$, $z^v_i := z_i$; $z^v_1 := v(N) - z^v(N\backslash 1)$. Similarly, for each $i \notin N^w$, $z^w_i := 0$ and, for each $i \in N^w$, $i \neq 1$, $z^w_i := z_i$; $z^w_1 := w(N) - z^w(N\backslash 1)$. Clearly, the efficiency of z ensures that $z = z^v + z^w$. Moreover, so defined, also z^v and z^w are efficient in v and w, respectively. Note that weak orthogonality is crucial to be able to uniquely define the allocations z^v and z^w above, since we can use efficiency to pin down the share of z_1 between z^v_1 and z^w_1.

4.3 Orthogonality: the core and the core-center

Since both the core and the core-center satisfy TC, throughout this section we assume, without loss of generality, that the games under consideration are zero-normalized.

We show below that, in some sense, the orthogonality between two games translates into a more geometrical orthogonality of their cores. Let v and w be two orthogonal games with a nonempty core, let x^1 and x^2 be two allocations in $C(v)$, and let y^1 and y^2 be two allocations in $C(w)$. Then, the vectors x^1-x^2 and y^1-y^2 are orthogonal to each other, *i.e.*, $(x^1-x^2)(y^1-y^2) = 0$. To see this, let $i \in N$ and recall that, since v and w are orthogonal, $D(v) \cup D(w) = N$. If $i \in D(v)$, for each $x \in C(v)$, $x_i = v(i)$ and, hence, $x^1_i - x^2_i = 0$ and $(x^1 - x^2)_i(y^1 - y^2)_i = 0$. An analogous argument applies if $i \in D(w)$. Given a game v with a nonempty core, let $\mathrm{Vol}(C(v)) := \int_{C(v)} dx$ denote the volume of $C(v)$ (in the highest dimensional space in which it has a nonempty interior).[7] Then, the previous discussion leads to the following result for orthogonal games.

Lemma 6. *Let v and w be two orthogonal games with a nonempty core. Then,*

$$\mathrm{Vol}(C(v + w)) = \mathrm{Vol}(C(v)) \cdot \mathrm{Vol}(C(w)).$$

Proof. Since $C(v)$ and $C(w)$ are orthogonal in the sense described above, we can find orthogonal subspaces V and W of $\mathbb{R}^n$ such that $C(v) \subset V$ and $C(w) \subset W$. Then, by Fubini's theorem,

$$\mathrm{Vol}(C(v+w)) = \int_{C(v+w)} dx = \int_{C(v)} \int_{C(w)} dydz = \int_{C(v)} dy \int_{C(w)} dz = \mathrm{Vol}(C(v))\,\mathrm{Vol}(C(w)). \quad \square$$

The above result suffices to show that the core-center is additive with respect to orthogonal games. Proposition 7 below is a particular case of Proposition 8, which establishes the result for weakly orthogonal games. Yet, we present a separate proof of this result because it helps to illustrate that "full" orthogonality allows for simpler arguments.

[7] If the core is a singleton we define its (0-dimensional) volume to be 1.

Proposition 7. *Let v and w be two orthogonal games with a nonempty core. Then, we have that* $\operatorname{corecenter}(v+w) = \operatorname{corecenter}(v) + \operatorname{corecenter}(w)$.

Proof. By definition, the core-center of a game v is the expectation of the uniform distribution defined over $C(v)$. Then,

$$\operatorname{corecenter}(v+w) = \frac{\int_{C(v+w)} x dx}{\operatorname{Vol}(C(v+w))} = \frac{\int_{C(v+w)} x^v dx + \int_{C(v+w)} x^w dx}{\operatorname{Vol}(C(v+w))}.$$

Then, by the orthogonality between $C(v)$ and $C(w)$,

$$\int_{C(v+w)} x^v dx = \int_{C(v)} x \operatorname{Vol}(C(w)) dx \quad \text{and} \quad \int_{C(v+w)} x^w dx = \int_{C(w)} x \operatorname{Vol}(C(v)) dx,$$

and, therefore,

$$\begin{aligned} \operatorname{corecenter}(v+w) &= \frac{\int_{C(v)} x \operatorname{Vol}(C(w)) dx + \int_{C(w)} x \operatorname{Vol}(C(v)) dx}{\operatorname{Vol}(C(v+w))} \\ &= \frac{\operatorname{Vol}(C(v)) \operatorname{Vol}(C(w)) \Big(\operatorname{corecenter}(v) + \operatorname{corecenter}(w) \Big)}{\operatorname{Vol}(C(v+w))} \\ &\overset{\text{Lem. } 6}{=} \operatorname{corecenter}(v) + \operatorname{corecenter}(w). \end{aligned}$$

□

However, some extra work is needed for weakly orthogonal games since Lemma 6 may not hold anymore. Just consider the game $v = u_{12} + u_{13}$, whose core is represented in Figure 3(c). The core of u_{12} is the line joining $(1,0,0)$ and $(0,1,0)$ and the core of u_{13} is the line joining $(1,0,0)$ and $(0,0,1)$. Both have length $\sqrt{2}$. Thus, $\operatorname{Vol}(C(v)) \operatorname{Vol}(C(w)) = 2$. Yet, the area of $C(v+w)$ is not the product of the lengths of the two lines, but the product of the base and the height of the rhomboid. In this case, this area coincides with half the area of the imputations set, an equilateral triangle with edges of length $2\sqrt{2}$. Hence, $\operatorname{Vol}(C(v+w)) = \sqrt{3} < 2$.

In general, given two games v and w with a nonempty core and given $x \in C(v)$ and $y \in C(w)$, we have that $x + y \in C(v+w)$, *i.e.*, $C(v) + C(w) \subseteq C(v+w)$.[8] When working with weakly orthogonal games the reverse inclusion also holds.

Lemma 7. *Let v and w be two weakly orthogonal games with a nonempty core. Then,*

$$C(v) + C(w) = C(v+w).$$

Proof. "$\subseteq$" (always true, no orthogonality needed) Let $x \in C(v)$ and $y \in C(w)$ and suppose $x + y \notin C(v+w)$. Then, since $x+y$ is efficient in $v+w$, there is $S \subset N$ such that $v(S) + w(S) > x(S) + y(S)$ so, either $v(S) > x(S)$ or $w(S) > y(S)$ and we have a contradiction, since both x and y are core elements of v and w, respectively.

"$\supseteq$" Let $x \in C(v+w)$. We claim that $x^v \in C(v)$ and $x^w \in C(w)$, which, since $x = x^v + x^w$, suffices to prove the result. Suppose that $x^v \notin C(v)$. Then, there is $S \subset N^v$ such that $v(S) > x^v(S)$. We distinguish two cases.

$1 \notin S$: Since $S \subset N^v$, $(v+w)(S) = v(S) + w(S) = v(S) > x^v(S) = x(S)$, which contradicts that $x \in C(v+w)$.

$1 \in S$: Now, $(v+w)(S \cup N^w) = v(S) + w(N^w) > x^v(S) + w(N^w) = x^v(S) + x^w(N^w) = x(S \cup N^w)$, which again contradicts that $x \in C(v+w)$.

[8] The operation $A + B$ denotes the Minkowski sum of the sets A and B, *i.e.*, $A + B := \{a + b : a \in A, b \in B\}$.

Therefore, $x^v \in C(v)$. The argument for x^w is analogous. □

The following lemma shows that an even stronger property holds: different allocations in $C(v)$ can never be combined with allocations in $C(w)$ to get the same allocation in $C(v+w)$; that is, $C(v+w)$ is the "direct sum" of $C(v)$ and $C(w)$. Put differently, each allocation in $C(v+w)$ can be uniquely decomposed as the sum of two allocations in $C(v)$ and $C(w)$.

Lemma 8. *Let v and w be two weakly orthogonal games. If $x^1, x^2 \in C(v)$ and $y^1, y^2 \in C(w)$ are such that $x^1 + y^1 = x^2 + y^2$, then $x^1 = x^2$ and $y^1 = y^2$.*

Proof. Suppose $x^1 \neq x^2$. Since $x^1(N) = x^2(N) = v(N)$, there are i, j with $i \neq j$ such that $x^1_i \neq x^2_i$ and $x^1_j \neq x^2_j$. Thus, we can assume, without loss of generality, that $i \neq 1$. Since i) the core satisfies DPP, ii) both x^1 and x^2 belong to $C(v)$, and iii) $x^1_i \neq x^2_i$, we have that $i \notin D(v)$. Then, since $i \neq 1$, $i \in D(w)$. Using again DPP, we have $y^1_i = y^2_i = w(i)$. Therefore, $x^1_i + y^1_i = x^1_i + w(i) \neq x^2_i + w(i) = x^2_i + y^2_i$, which contradicts that $x^1 + y^1 = x^2 + y^2$. Thus, $x^1 = x^2$. The argument to show that $y^1 = y^2$ is analogous. □

Proposition 8. *Let v and w be two weakly orthogonal games with a nonempty core. Then,* $\operatorname{corecenter}(v+w) = \operatorname{corecenter}(v) + \operatorname{corecenter}(w)$.

Proof. Let $i \in D(w)$. We show now that $\operatorname{corecenter}(v+w)_i = \operatorname{corecenter}(v)_i + \operatorname{corecenter}(w)_i = \operatorname{corecenter}(v)_i$. By Lemma 8, if we let $\bigsqcup$ denote the disjoint union of sets,

$$C(v+w) = \bigsqcup_{a \in C(v)} (a + C(w)).$$

Roughly speaking, all the elements of $C(v)$ are used the same number of times to form elements of $C(v+w)$. Since $i \in D(w)$, for each $y \in C(w)$, $y_i = 0$. Hence, the expected value of the i-th component of an allocation in $C(v+w)$ coincides with the expected value of the same component in $C(v)$, *i.e.*, $\operatorname{corecenter}(v+w)_i = \operatorname{corecenter}(v)_i$. Similarly, we can show that, for each $i \in D(v)$, $\operatorname{corecenter}(v+w)_i = \operatorname{corecenter}(v)_i + \operatorname{corecenter}(w)_i = \operatorname{corecenter}(w)_i$. If v and w are orthogonal we are done, since $D(v) \cup D(w) = N$. If they are just weakly orthogonal, the equality of the remaining coordinate is pin down by efficiency. □

In the above proof, in some sense, the extra degree of freedom allowed under weak orthogonality is controlled by the efficiency constraint. If two games are weakly orthogonal, we use orthogonality to pin down all payoffs but one, which is pin down by efficiency. This is also crucial for the orthogonal additivity of the prenucleolus and the nucleolus, which we discuss below.

4.4 Orthogonality: the prenucleolus and the nucleolus

Theorem 1. *Let v and w be two weakly orthogonal games in G^n such that at least one of them has a nonempty core. Then,* $\operatorname{PNu}(v+w) = \operatorname{PNu}(v) + \operatorname{PNu}(w)$.

As we argued in the Introduction, this result adds to the literature on the additivity of the nucleolus. In this context, Kohlberg (1971, Theorem 3) showed the following. Suppose that games v and w are such that the most "unhappy" coalitions according to $\operatorname{Nu}(v)$ and $\operatorname{Nu}(w)$ coincide, the second most "unhappy" ones also coincide, and so on. Then, $\operatorname{Nu}(\alpha v + \beta w) = \alpha \operatorname{Nu}(v) + \beta \operatorname{Nu}(w)$. Theorem 1 implies that linearity can also hold under conditions independent from those in Kohlberg (1971). We need to go over some preliminaries to prove Theorem 1. Given a game v, an efficient allocation x, and $\alpha \in \mathbb{R}$, let $\mathcal{D}^v(\alpha, x) = \{S \subseteq N \backslash \{\emptyset\} : e^v(S, x) \geq \alpha\}$. A *side-payment* is a vector $y \in \mathbb{R}^n$ such that $y(N) = 0$. Below we prove Theorem 1 building upon the following version of Kohlberg's criterion (Peters, 2008, Theorem 19.5).[9]

[9] This version is just the adaptation of the approach taken in Kohlberg (1971) for the nucleolus.

Theorem 2. *Let $v \in G^n$ and let x be an efficient allocation. Then, the following two statements are equivalent:*

i) $x = \mathrm{PNu}(v)$.

ii) *For each $\alpha \in \mathbb{R}$ such that $\mathcal{D}^v(\alpha, x) \neq \emptyset$ and for each side-payment y with $y(S) \geq 0$ for every $S \in \mathcal{D}^v(\alpha, x)$, we have $y(S) = 0$ for every $S \in \mathcal{D}^v(\alpha, x)$.*

Corollary 4. *Let $v \in G^n$, $i \in N$ and $\alpha = \theta_1^v(\mathrm{PNu}(v))$.*

i) *There is $S \in \mathcal{D}^v(\alpha, \mathrm{PNu}(v))$ such that $i \in S$.*

ii) *Unless $\mathcal{D}^v(\alpha, \mathrm{PNu}(v)) = \{N\}$, there is $S \in \mathcal{D}^v(\alpha, \mathrm{PNu}(v))$ such that $i \notin S$.*

Proof. i) Suppose that there is $i \in N$ such that, for each $S \in \mathcal{D}^v(\alpha, \mathrm{PNu}(v))$, $i \notin S$. Then, we can violate condition ii) in Theorem 2 by defining a side-payment in which player i transfers some (possibly very small) utility to the players in $N \backslash i$.

ii) Suppose that $\mathcal{D}^v(\alpha, \mathrm{PNu}(v)) \neq \{N\}$ and that there is $i \in N$ such that, for each $S \in \mathcal{D}^v(\alpha, \mathrm{PNu}(v))$, $i \in S$. Let $S \in \mathcal{D}^v(\alpha, \mathrm{PNu}(v))$, $S \neq N$. Let $j \in N \backslash S$. Then, we we can violate condition ii) in Theorem 2 by defining a side-payment in which player j transfers some (possibly very small) utility to player i. $\square$

Proof of Theorem 1. Let v and w be two weakly orthogonal games. Since the prenucleolus satisfies TC, NPP and the *reduced game property* (Sobolev, 1975),[10] we may assume with out loss of generality that v and w are zero-normalized (all dummies are null players), and that $D(v) \cap D(w) = \emptyset$, so $v + w$ has no dummy players. For each efficient allocation x,

$$e^{v+w}(S, x) = v(S) + w(S) - x(S) = v(S) - x^v(S) + w(S) - x^w(S) = e^v(S, x^v) + e^w(S, x^w),$$

but, clearly, $e^v(S, x^v) = e^v(S^v, x^v)$ and $e^w(S, x^w) = e^w(S^w, x^w)$. Hence,

$$e^{v+w}(S, x) = e^v(S^v, x^v) + e^w(S^w, x^w).$$

In particular, the above implies that $\theta_1^{v+w}(x) = \max_{S \subseteq N} e^v(S^v, x^v) + \max_{S \subseteq N} e^w(S^w, x^w)$.

Now, suppose that game v is the one with a nonempty core. Let $\bar{x} = \mathrm{PNu}(v + w)$ and $\bar{z} = \mathrm{PNu}(v) + \mathrm{PNu}(w)$, so $\bar{z}^v = \mathrm{PNu}(v)$ and $\bar{z}^w = \mathrm{PNu}(w)$. We divide the proof in several claims. The first ones crucially exploit the nonemptyness of the core of v.

Claim 1: $\max_{S \subseteq N} e^v(S^v, \bar{x}^v) \leq 0$. Suppose that $\max_{S \subseteq N} e^v(S^v, \bar{x}^v) > 0$. Let $\hat{x} = \bar{z}^v + \bar{x}^w$. Since v has a nonempty core, the prenucleolus belongs to it and, thus, $\max_{S \subseteq N} e^v(S^v, \bar{z}^v) = e^v(N^v, \bar{z}^v) = 0$. Therefore, we get a contradiction with the fact that $\bar{x} = \mathrm{PNu}(v + w)$, since

$$\theta_1^{v+w}(\bar{x}) = \max_{S \subseteq N} e^v(S^v, \bar{x}^v) + \max_{S \subseteq N} e^w(S^w, \bar{x}^w) > \max_{S \subseteq N} e^v(S^v, \bar{z}^v) + \max_{S \subseteq N} e^w(S^w, \bar{x}^w) = \theta_1^{v+w}(\hat{x}).$$

Claim 2: $\bar{x}^w = \bar{z}^w = \mathrm{PNu}(w)$. Suppose, on the contrary, that $\bar{x}^w \neq \mathrm{PNu}(w)$. Then, we can apply Theorem 2 to the restriction of the game w to the players in N^w. Thus, by negating statement ii) in Theorem 2 we obtain: i) an excess $\alpha \in \mathbb{R}$, ii) a side-payment $y \in \mathbb{R}^n$ such that $y(S) \geq 0$ for all $S \in \mathcal{D}^w(\alpha, \bar{x}^w)$ and $y_i = 0$ for each $i \in D(w)$, and iii) a coalition $T \subset N^w$, $T \in \mathcal{D}^w(\alpha, \bar{x}^w)$ such that $y(T) > 0$. By Claim 1 we have that, for each $S \in \mathcal{D}^{v+w}(\alpha, \bar{x})$,

$$\alpha \leq e^{v+w}(S, \bar{x}) = e^v(S^v, \bar{x}^v) + e^w(S^w, \bar{x}^w) \leq e^w(S^w, \bar{x}^w). \tag{1}$$

Therefore, $S^w \in \mathcal{D}^w(\alpha, \bar{x}^w)$. This implies that the side-payment y is such that, for each $S \in \mathcal{D}^{v+w}(\alpha, \bar{x})$, $y(S) \geq 0$. Now, we distinguish two cases:

[10] We could as well rely on the less standard *strong null player property* (see, for instance, Peleg and Sudhölter (2003)).

$1 \in T$. Let $S = T \cup N^v$. Then, $e^{v+w}(S, \bar{x}) = e^v(N^v, \bar{x}^v) + e^w(T, \bar{x}^w) = e^w(T, \bar{x}^w) \geq \alpha$. Thus, $S \in \mathcal{D}^{v+w}(\alpha, \bar{x})$ and we have $y(S) = y(T) + \sum_{i \in N^v \cap D(w)} y_i = y(T) > 0$.

$1 \notin T$. Now, $e^{v+w}(T, \bar{x}) = e^v(\emptyset, \bar{x}^v) + e^w(T, \bar{x}^w) = e^w(T, \bar{x}^w) \geq \alpha$. Thus, $T \in \mathcal{D}^{v+w}(\alpha, \bar{x})$ and $y(T) > 0$.

In both cases we can apply Theorem 2 to reach a contradiction with the fact that $\bar{x} = \mathrm{PNu}(v+w)$.

Claim 3: $\bar{x}^v = \bar{z}^v = \mathbf{PNu}(v)$. If the core of w is nonempty we can repeat the arguments of Claim 2, interchanging the roles of v and w. Otherwise, let $\beta = \max_{S \subseteq N} e^w(S^w, \bar{x}^w) = \theta_1^w(\bar{x}^w) = \theta_1^w(\bar{z}^w) < 0$. By Corollary 4, since $\bar{x}^w = \bar{z}^w = \mathrm{PNu}(w)$, there are $R \subset N^w$ and $\hat{R} \subset N^w$, with $1 \in R$ and $1 \notin \hat{R}$, such that both R and $\hat{R}$ belong to $\mathcal{D}^w(\beta, \bar{x}^w)$. The proof now is analogous to that of Claim 2, with coalition R playing the role of N^v when $1 \in T$, $\hat{R}$ playing the role of $\emptyset$ when $1 \notin T$, and with $\alpha + \beta$ instead of just α in Eq. (1).

The combination of Claims 2 and 3 establishes that $\bar{x} = \bar{z}$. □

Corollary 5. *Let v and w be two weakly orthogonal zero-monotonic games in G^n such that at least one of them has a nonempty core. Then,* $\mathrm{Nu}(v + w) = \mathrm{Nu}(v) + \mathrm{Nu}(w)$.

Proof. Follows immediately from Theorem 1 and the fact that the nucleolus and the prenucleolus coincide for zero-monotonic games. □

We present below an example that illustrates that the nonemptyness condition for the core of one of the two games in Theorem 1 cannot be dropped.

Example 5. Let v and w be two orthogonal zero-normalized and zero-monotonic games in G^7 defined as follows. In game v, $D(v) = \{4, 5, 6, 7\}$, $v(12) = v(13) = v(23) = 3$, and $v(123) = 4$; in game w, $D(w) = \{1, 2, 3\}$ and $w(46) = w(47) = w(56) = w(57) = 2$, $w(45) = w(67) = w(456) = w(457) = w(467) = w(567) = 3$, and $w(4567) = 4$; the remaining coalitions are formed adding dummy players, so the worth of each of them can be trivially computed. The corresponding values for the prenucleolus and the nucleolus are $\mathrm{PNu}(v) = \mathrm{Nu}(v) = (4/3, 4/3, 4/3, 0, 0, 0, 0)$, $\mathrm{PNu}(w) = \mathrm{Nu}(w) = (0, 0, 0, 1, 1, 1, 1)$, and $\mathrm{PNu}(v + w) = \mathrm{Nu}(w + w) = (3/2, 3/2, 3/2, 7/8, 7/8, 7/8, 7/8) \neq \mathrm{PNu}(v) + \mathrm{PNu}(w) = \mathrm{Nu}(v) + \mathrm{Nu}(w)$.

In game $v + w$ in this example, according to the nucleolus and the prenucleolus, coalition $\{1, 2, 3\}$ gets $4.5 > 4 = v(123)$ and coalition $\{4, 5, 6, 7\}$ gets $3.5 < 4 = w(4567)$. With respect to the allocation $\mathrm{Nu}(v)+\mathrm{Nu}(w)$, in order to minimize the vector of ordered excesses, the four players in $\{4, 5, 6, 7\}$ "transfer" some utility to the players in $\{1, 2, 3\}$; by doing so, since $|\{4, 5, 6, 7\}| > |\{1, 2, 3\}|$, the coalitions containing players $\{1, 2, 3\}$ are made "happier" than "unhappier" the coalitions containing players $\{4, 5, 6, 7\}$ which, in this example, suffices to improve the vector of excesses according to the lexicographic order. ◇

The next example shows that zero-monotonicity cannot be dropped either to get the orthogonal additivity of the nucleolus: even if one of the two orthogonal games has a nonempty core, the nucleolus may fail to satisfy $\text{ADD}^{\perp}$ if the other one is not zero-monotonic.

Example 6. Let v and w be two zero-normalized orthogonal games in G^5 defined as follows. In game v, $D(v) = \{4, 5\}$, $v(12) = v(13) = -10$ $v(23) = 10$, and $v(123) = 5$; in game w, which has a nonempty core, $D(w) = \{1, 2, 3\}$ and $w(45) = 20$; the remaining coalitions are formed adding dummy players, so the worth of each of them can be trivially computed. The corresponding values for the nucleolus are $\mathrm{Nu}(v) = (0, 5/2, 5/2, 0, 0)$ and $\mathrm{Nu}(w) = (0, 0, 0, 10, 10)$. We argue now that $\mathrm{Nu}(v) + \mathrm{Nu}(w) = (0, 5/2, 5/2, 10, 10)$ cannot be the nucleolus of $v + w$. In the vector $\theta(\mathrm{Nu}(v) + \mathrm{Nu}(w))$, the largest excess is 5, and corresponds to coalitions $\{2, 3\}$ and $\{2, 3, 4, 5\}$. Therefore, we can get an imputation that dominates lexicographically $\mathrm{Nu}(v)+\mathrm{Nu}(w)$ by transferring some utility from coalition $\{4, 5\}$ to coalition $\{2, 3\}$. ◇

4.5 Orthogonal additivity and the coincidence of allocation rules

Table 5 summarizes the results for TC and ADD, along with our findings for ADD$^\perp$. In the next result, which is now straightforward, we use ADD$^\perp$ to present a last coincidence result which combines symmetry and orthogonality properties and, in particular, applies to the tent game.

	TC	ADD$^\perp$	ADD
Prenucleolus	✓	✓*	X
Nucleolus	✓	✓*	X
Shapley Value	✓	✓	✓
τ value	✓	X	X
Core-center	✓	✓	X
Equal division	X	✓	✓
Equal surplus division	✓	✓	✓

* At least one of the two games must have a nonempty core and, for the nucleolus, both must be zero-monotonic.

Table 5: Allocation rules and orthogonal additivity.

Proposition 9. *Let $v \in G^n$ and let $i \in N$. Suppose that $v = v^1 + \ldots + v^p$, where the games $v^1, \ldots, v^p$ are dummy symmetric and (pairwise) i-weakly orthogonal, with all of them having a nonempty core. Then, the Shapley value, the prenucleolus, the nucleolus, and the corecenter coincide for game v, and so does any allocation rule satisfying* EFF, SYM, DPP, *and* ADD$^\perp$.

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