Monotonicity of the core-center of the airport game

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Abstract

One of the main goals of this paper is to improve the understanding of the way in which the core of a specific cooperative game, the airport game (Littlechild and Owen, 1973), responds to monotonicity properties. Since such properties are defined for single-valued allocation rules, we use the core-center (González-Díaz and Sánchez-Rodríguez, 2007) as a proxy for the core. This is natural, since the core-center is the center of gravity of the core and its behavior with respect to a given property can be interpreted as the "average behavior" of the core. We also introduce the lower-cost increasing monotonicity and higher-cost decreasing monotonicity properties that reflect whether or not a variation in a particular agent's cost is beneficial to the other agents.

Keywords: cooperative TU games, monotonicity, core, core-center, airport games.

1 Introduction

The airport problem, introduced by Littlechild and Owen (1973), is a classic cost allocation problem that has been widely studied. To get a better idea of the attention it has generated one can refer to the survey by Thomson (2007). The core, introduced by Gillies (1953), stands as one of the most studied solution concepts in the theory of cooperative games. Its properties have been thoroughly analyzed. Airport games are concave games and therefore balanced games, games with a nonempty core. There is an important family of properties one often studies when working with single-valued solution concepts: the monotonicity properties. These properties are very hard to adapt to set-valued solutions. In this respect, there is virtually no paper that studies the monotonicity of the core. A related issue deals with the question of defining allocation rules that meet some monotonicity requirement and always selects a core allocation. This issue goes back to Young (1985) and was followed by series of other results such as Housman and Clark (1998) and, more recently, Arin (2013).[1] This type of studies have also given rise to new set-valued solution concepts that aim to integrate the stability requirements of the core with some monotonicity properties (see, for instance, Calleja et al. (2009) and Getán et al. (2009)).

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[1]Similar studies for the bargaining set and the kernel go even further back, to Megiddo (1974).

Importantly, the cooperative game associated with an airport problem has a special structure that can be exploited to facilitate the analysis of different solutions. In particular, $2^n - 1$ parameters are needed to define a general n-player cooperative game, whereas for an airport game one just needs n. This special structure simplifies the geometry of the core of such games, since it turns out to be defined by $2n - 1$ inequality constraints instead of the usual $2^n - 2$.

When the core of a game is nonempty, there is a set of alternatives at which agents' payoffs differ that are coalitionally stable. Studying the center of gravity of such set may be interesting in many cases. The core-center (González-Díaz and Sánchez-Rodríguez, 2007) selects the mathematical expectation of the uniform distribution over the core of the game. The intuition provided by its definition is a good reason to be interested in it and to justify the study of its properties.

As we already have mentioned, on the domain of balanced games the properties of core stability and monotonicity are not compatible. However, the existence of monotonic core selectors is known on subdomains of the domain of balanced games, such as the airport games. The Shapley value and the nucleolus are the best known monotonic core selectors. A consequence of the results in this paper is that the core-center is also a monotonic core selector.

Throughout this paper we exploit the aforementioned structure of the core of an airport game to gain insights in the monotonicity properties of its core-center. More precisely, since most of these properties are only defined for single-valued allocation rules, we use one special allocation in the core of an airport game as a proxy to improve our understanding of the core. Naturally, the behavior of the core-center with respect to a monotonicity property can be interpreted as the "average" behavior of the core with respect to it.

The formal definition of the core-center is given in terms of integrals over the core of the airport game. In fact, (González-Díaz et al., 2014) show that the j-th coordinate of the core-center of an airport game is the ratio of the volumes of the core of the airport game obtained by replicating agent j and the core of the original game. A variation, no matter how small, in any cost parameter of an airport game, produces a change in the shape of the core and therefore in its volume. The core-center is very sensitive to such changes. This approach leads, in a natural way, to the definition of two new monotonicity properties that restrict how a change on a cost parameter of a given agent affects the payoffs of the other agents. They are called "higher-cost decreasing monotonicity" and "lower-cost increasing monotonicity". The first one says that if a cost c_i increases, then the payoffs of all players with costs higher than c_i decrease. The second property is a kind of reciprocal, the payoffs of all players with costs lower than c_i increase. The study of these properties allows us to check whether or not the core-center satisfies the usual monotonicity properties in the literature.

The paper is structured as follows. In Section 2 we present the basic concepts and notations. In Section 3 we derive several ways of decomposing the core volume as a sum of volumes of airport games with some cloned costs. Higher-cost decreasing monotonicity is introduced in Section 4. We develop our main mathematical results, which build upon a thorough exploration of the derivatives of the volumes of the core of an airport game, to prove that the core-center satisfies this property. Similarly, in Section 5 we introduce lower-cost increasing monotonicity and prove that the core-center satisfies it. Relying on the previous properties, we analyze in Section 6 which of the classical monotonicity properties are satisfied or violated by the core-center.

2 Preliminaries

There is an infinite set of potential agents, indexed by the natural numbers. Then, in each given problem, only a finite number of those players are present. Let $\mathcal{N}$ be the set of all finite subsets of $\mathbb{N} = \{1, 2, \ldots\}$.

An *airport problem* (Littlechild and Owen (1973)) with set of agents $N \in \mathcal{N}$ is a non-negative vector $c \in \mathbb{R}^N_+$. Let $\mathcal{C}^N$ denote the domain of all airport problems with agent set N. We make the standard assumption that, for each pair of agents $i, j \in N$, if $i < j$ then $c_i \leq c_j$. A *cost allocation* for $c \in \mathcal{C}^N$ is a non-negative vector $x \in \mathbb{R}^N_+$ such that $0 \leq x_i \leq c_i$ for all $i \in N$ and $\sum_{i \in N} x_i = c_n$. A basic requirement is that at an allocation x no group $N' \subset N$ of agents should contribute more that what it would have to pay on its own, $\max\{c_i : i \in N'\}$. Otherwise, the group would unfairly "subsidize" the other agents. The constraints $\sum_{j \leq i} x_j \leq c_i$, $i \in N$, are called the *no-subsidy constraints*. A rule ψ associates to each airport problem $c \in \mathcal{C}^N$ a cost allocation $\psi(c)$ satisfying the no-subsidy constraints. For a complete survey on airport problems the reader is referred to Thomson (2007).

A cooperative *cost game* with transferable utility is a pair (N, c), where $N \in \mathcal{N}$ is the set of players and the characteristic function $c\colon 2^N \to \mathbb{R}$ is a function assigning, to each coalition $S \subset N$, its cost $c(S)$. By convention $c(\emptyset) = 0$. Let $\mathcal{V}^N$ be the domain of all cooperative cost games with player set N. Given a coalition of players $S \subset N$, $|S|$ denotes its cardinality. Given $N \in \mathcal{N}$, a vector $x \in \mathbb{R}^N$ is referred to as an *allocation*. For every $S \subset N$ denote $x(S) = \sum_{i \in S} x_i$. An allocation is *efficient* if $x(N) = c(N)$. A cost game $c \in \mathcal{V}^N$ is *concave* if $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$, for each $i \in N$ and each S and T such that $S \subseteq T \subseteq N \backslash \{i\}$. A *solution* defined on some subdomain of cost games is a correspondence ψ that associates to each cost game $c \in \mathcal{V}^N$ in the subdomain a subset $\psi(c)$ of efficient allocations. Given a cost game $c \in \mathcal{V}^N$, the *imputation set* comprises the individually rational and efficient allocations, $I(c) = \{x \in \mathbb{R}^N : x(N) = c(N),\ x_i \leq c(\{i\}) \text{ for all } i \in N\}$. The *core* (Gillies (1953)) is defined as $C(c) = \{x \in I(c) : x(S) \leq c(S) \text{ for all } S \subset N\}$. If a solution is single-valued then it is referred to as a *rule*.

To each airport problem $c \in \mathcal{C}^N$ one can associate a cost game $c \in \mathcal{V}^N$ with N as the set of players and the characteristic function given, for each $S \subseteq N$, by $c(S) = \max_{i \in S}\{c_i\}$. Such a game is called an *airport game*. We denote by the same letter c both the airport problem and the associated cost game. It should be clear from the context to which one we are referring to. Naturally, any given solution to coalitional games provides a rule for the airport problem when applied to the associated coalitional airport game. Existing rules are evaluated and compared in terms of the properties they satisfy or violate. There is an important family of properties that specify how a rule should respond to changes in the cost parameters of an airport problem: the monotonicity properties. Generally, these properties are concerned with the effect of a variation of an agent cost parameter, or the cost parameters of a particular group, on the contribution of that agent, or of that group of agents. We say that a rule ψ satisfies:

- *Individual cost monotonicity* if, for each pair $c, c' \in \mathcal{C}^N$ and each $i \in N$ such that $c'_i \geq c_i$ and, for all $j \in N \backslash \{i\}$, $c'_j = c_j$, then $\psi_i(c') \geq \psi_i(c)$.
- *Others-oriented cost monotonicity* if, under the assumptions of individual cost monotonicity, for each $j \in N \backslash \{i\}$, $\psi_j(c') \leq \psi_j(c)$.
- *Population monotonicity* if, for each N and N' with $N' \subset N$, $\psi_{N'}(c) \leq \psi(c_{N'})$.
- *Strong cost monotonicity* if, for each pair $c, c' \in \mathcal{C}^N$ such that $c \leq c'$, then $\psi(c) \leq \psi(c')$.
- *Weak cost monotonicity* if, for each pair $c, c' \in \mathcal{C}^N$ such that $c' = c + c''$ for some $c'' \in \mathcal{C}^N$, then $\psi(c') \geq \psi(c)$.
- *Downstream cost monotonicity* if, for each pair $c, c' \in v$ and each $i \in N$, such that for each $j \in N$ with $c_j < c_i$, $c'_j = c_j$ and for each $j \in N$ with $c_j \geq c_i$, $c'_j - c_j = c'_i - c_i \geq 0$, then for each $j \in N$ such that $c_j \geq c_i$, $\psi_j(c') \geq \psi_j(c)$.
- *Marginalism* if, under the hypotheses of downstream cost monotonicity, for each $j \in N$ such that $c_j < c_i$, $\psi_j(c') = \psi_j(c)$.

When the core of a cooperative game is nonempty, there is a set of alternatives that are coalitionally stable at which agents' payoffs differ. If one considers that all of the core alternatives are equally preferable, then selecting the average stable payoff seems to be a natural choice. Given a balanced game $v \in \mathcal{V}^N$, the core-center $\mu(v)$ (González-Díaz and Sánchez-Rodríguez, 2007) is defined as the mathematical expectation of the uniform distribution over the core of the game, i.e., the center of gravity of $C(v)$. Given a convex polytope $K \subset I(v)$, denote its center of gravity by $\mu(K)$. Then $\mu(v) = \mu(C(v))$. Any airport game $c \in \mathcal{V}^N$ is a concave game and its core coincides with the set of allocations satisfying the no-subsidy constraints:

$$C(c) = \big\{x \in \mathbb{R}^N : x \geq 0,\ x(N) = c_n, \sum_{j \leq i} x_j \leq c_i \text{ for all } i \in N\big\}.$$

The core of the airport game is contained in the efficiency hyperplane $x_1 + \cdots + x_n = c_n$ and it is defined by, at most, $2n-2$ inequality constraints, so, whenever $c_1 > 0$, it is a $(n-1)$-dimensional convex polytope. Therefore, the core-center, when applied to airport games, provides an rule for the airport problem. Exploiting the special structure of the core of an airport game, (González-Díaz et al., 2014) obtained an integral expression for the core-center. Given $0 < a_1 \leq \cdots \leq a_k$, let $V_k(a_1, \ldots, a_k) = \int_0^{a_1} \int_0^{a_2 - x_1} \cdots \int_0^{a_k - \sum_{j=1}^{k-1} x_j} dx_k \ldots dx_2 dx_1$ and let $\hat{\mu}_i(a_1, \ldots, a_k) = \dfrac{V_{k+1}(a_1, \ldots, a_i, a_i, \ldots a_k)}{V_k(a_1, \ldots, a_k)}$, $i = 1, \ldots, k$. Then, for any airport problem $c \in \mathcal{C}^N$ the core-center $\mu(c)$ is given by: $\mu_i(c) = \hat{\mu}_i(c_1, \ldots, c_{n-1})$ if $i \in N \setminus \{n\}$, and $\mu_n(c) = c_n - \sum_{i=1}^{n-1} \mu_i(c)$. The value $V_{n-1}(c_1, \ldots, c_{n-1})$ coincides, up to a scaling factor $\sqrt{n}$, with the volume of the core, $C(c)$, see Figure 1. Then, what the core-center assigns to agent j in the original problem is the percentage of stable allocations in the game with a clone of player j over the original stable allocations. Mathematically, the core-center is the ratio of the volumes of the core of the airport game obtained by replicating agent j and the core of the original game.

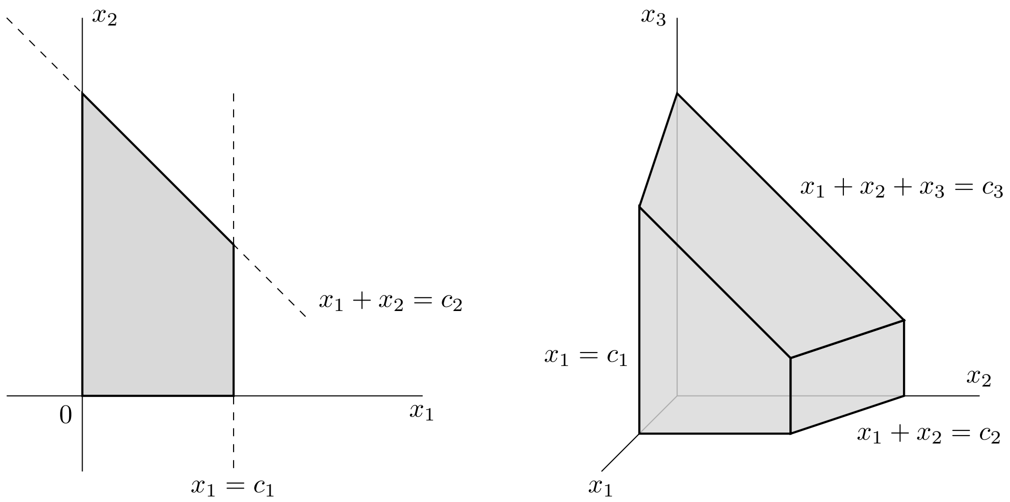

Figure 1: The domain of integration of $V_2(c_1, c_2)$ (left) and $V_3(c_1, c_2, c_3)$ (right).

The main goal of this paper is to study which monotonicity properties are satisfied or violated by the core-center. For most of the following discussion, we start with a fixed n-player set $N = \{1, 2, \ldots, n\}$. The following result (González-Díaz et al., 2014), is a key tool to understanding how the core-center varies with respect to changes in the cost parameters.

Proposition 1. *Let $c \in \mathcal{C}^N$ be an airport game with $0 < c_1$ and $i, j \in N \setminus \{n\}$. Then, if $i \leq j$, $\mu_j(c)$ is increasing with respect to c_i if and only if $\mu_j(c_1, \ldots, c_n) \leq \mu_j(c_1, \ldots, c_i)$. Conversely, if $i < j$, $\mu_j(c)$ is decreasing with respect to c_i if and only if $\hat{\mu}_j(c_1, \ldots, c_{n-1}) \geq \hat{\mu}_{j-i}(c_{i+1} - c_i, \ldots, c_{n-1} - c_i)$.*

Therefore, the monotonicity properties of the core-center of an airport game can be studied by comparing the core-center assignments given to a player in the original game and in some particular "truncated" games[2]. These truncated games suggest the introduction of two new monotonicity properties. Assume that airline i increases its cost from c_i to c'_i. We consider two groups: the first group is formed by the airlines with cost larger than

[2]Given an airport problem $c \in \mathcal{C}^N$ denote by $\hat{C}(c)$ the projection of the core $C(c)$ onto $\mathbb{R}^{n-1}$ that simply "drops" the n-th coordinate. Figure 1 shows $\hat{C}(c)$ for a 3-agent and 4-agent problem. The face $F_i = \hat{C}(c) \cap \{x \in \mathbb{R}^{n-1} : x_1 + \cdots + x_i = c_i\}$, $i \in N \setminus \{n\}$, of the polytope $\hat{C}(c)$ is a cross product of the cores of two airport games: $F_i = C(c_1, \ldots, c_i) \times \hat{C}(c_{i+1} - c_i, \ldots, c_n - c_i)$.

c_i, and the second group is formed by the airlines with cost lower than or equal to c_i. Higher-cost decreasing monotonicity requires that only the higher-cost agents (individually) should benefit (or should not be harmed) from agent's i cost increase, because the differences $c_k - c_i' < c_k - c_i$ have diminished for all $k > i$. Conversely, lower-cost increasing monotonicity requires that the lower-cost agents should pay more (should not be favored), because the differences $c_i' - c_k > c_i - c_k$ have increased for all $k < i$. We devote Sections 4 and 5 to formally define both properties and to prove that the core-center satisfies them.

3 Decompositions of the core volume

Let $N = \{1, 2, \ldots, n\}$ and let $c \in \mathcal{C}^N$ be an airport problem. We devote this section to derive some technical decompositions of the volume functions V_k. Easily, one can prove (González-Díaz et al., 2014), that $V_1(c_1) = c_1$, $V_2(c_1, c_2) = \frac{c_2^2}{2} - \frac{(c_2-c_1)^2}{2}$, and, for all $k \geq 3$,

$$V_k(c_1, \ldots, c_k) = \frac{c_k^k}{k!} - \frac{(c_k - c_1)^k}{k!} - \sum_{i=2}^{k-1} \frac{(c_k - c_i)^{k-i+1}}{(k-i+1)!} V_{i-1}(c_1, \ldots, c_{i-1}). \quad (1)$$

First, we introduce some notations. By convention, $V_0 = 1$. An expression like $V_{p+s-1}(c_1, \ldots, c_p, \overset{s}{\ldots}, c_p)$ means that cost c_p is repeated s times. When all the costs are equal, we write $V_k(c_1, \ldots, c_1)$ instead of $V_k(c_1, \overset{k}{\ldots}, c_1)$.

Lemma 1. *For all $k \in \mathbb{N}$ and $\alpha \geq 0$, $V_k(\alpha, \ldots, \alpha) = \frac{\alpha^k}{k!}$.*

Proof. Clearly, the property holds for $k = 1$. Assume that the result is true for $k - 1$ and proceed by induction. Then, $V_k(\alpha, \ldots, \alpha) = \int_0^\alpha V_{k-1}(\alpha - x_1, \ldots, \alpha - x_1) dx_1 = \int_0^\alpha \frac{(\alpha - x_1)^{k-1}}{(k-1)!} dx_1 = \dfrac{\alpha^k}{k!}$. □

Lemma 2. *Given $k \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq c_1 \leq \cdots \leq c_k$, we have that*

$$\int_\alpha^\beta V_k(c_1 - x_1, \ldots, c_k - x_1) dx_1 = V_{k+1}(\beta - \alpha, c_1 - \alpha, \ldots, c_k - \alpha).$$

Proof. The result is true for $k = 1$ since $\int_\alpha^\beta (c_1 - x_1) dx_1 = \frac{(c_1-\alpha)^2}{2} - \frac{(c_1-\beta)^2}{2}$. Assume that the equality holds for all $i < k$. Then,

$$\begin{aligned}
&\int_\alpha^\beta V_k(c_1 - x_1, \ldots, c_k - x_1) dx_1 \\
&= \int_\alpha^\beta \frac{(c_k - x_1)^k}{k!} dx_1 - \int_\alpha^\beta \frac{(c_k - c_1)^k}{k!} dx_1 - \sum_{i=2}^{k-1} \frac{(c_k - c_i)^{k-i+1}}{(k-i+1)!} \int_\alpha^\beta V_{i-1}(c_1 - x_1, \ldots, c_{i-1} - x_1) dx_1 \\
&= \frac{(c_k - \alpha)^{k+1}}{(k+1)!} - \frac{(c_k - \beta)^{k+1}}{(k+1)!} - \frac{(c_k - c_1)^k}{k!}(\beta - \alpha) - \sum_{i=2}^{k-1} \frac{(c_k - c_i)^{k-i+1}}{(k-i+1)!} V_i(\beta - \alpha, c_1 - \alpha \ldots, c_{i-1} - \alpha),
\end{aligned}$$

where the first equality holds by Equality (1) and the second by the induction hypothesis. Again by (1), the last expression equals $V_{k+1}(\beta - \alpha, c_1 - \alpha, \ldots, c_k - \alpha)$. □

The following result allows to decompose any given volume in terms of volumes with repeated costs.

Proposition 2. *If $0 < \alpha \leq c_1 \leq \cdots \leq c_k$, $k \in \mathbb{N}$, then*

1. $V_k(c_1, \ldots, c_k) = \sum_{i=0}^{k} V_i(\alpha, \ldots, \alpha) V_{k-i}(c_{i+1} - \alpha, \ldots, c_k - \alpha).$

2. $V_k(c_1-\alpha,\dots,c_k-\alpha)=\sum_{i=1}^{k}V_i(c_1-\alpha,\dots,c_1-\alpha)V_{k-i}(c_{i+1}-c_1,\dots,c_k-c_1)$.

3. $V_k(c_1-\alpha,\dots,c_k-\alpha)=\sum_{i=2}^{k}V_i(c_1-\alpha,c_2-\alpha,\overset{i-1}{\dots},c_2-\alpha)V_{k-i}(c_{i+1}-c_2,\dots,c_k-c_2)$.

Proof. Assume that $0<\alpha\le c_1\le\cdots\le c_k$ and let us prove the three statements above.

1. Let $s\in\mathbb{N}$ such that $1\le s\le k-2$ and denote

$$I_s=\int_0^{c_{s+1}-\sum_{j=1}^{s}x_j}\dots\int_0^{c_k-\sum_{j=1}^{k-1}x_j}dx_k\dots dx_{s+1}=V_{k-s}\Big(c_{s+1}-\sum_{j=1}^{s}x_j,\dots,c_k-\sum_{j=1}^{s}x_j\Big).$$

We claim that

$$\int_0^{\alpha}\dots\int_0^{\alpha-\sum_{j=1}^{s-1}x_j}I_s\,dx_s\dots dx_1=$$
$$V_s(\alpha,\dots,\alpha)V_{k-s}(c_{s+1}-\alpha,\dots,c_k-\alpha)+\int_0^{\alpha}\dots\int_0^{\alpha-\sum_{j=1}^{s}x_j}I_{s+1}\,dx_{s+1}\dots dx_1 \quad (2)$$

Indeed,

$$\int_0^{\alpha}\dots\int_0^{\alpha-\sum_{j=1}^{s-1}x_j}I_s\,dx_s\dots dx_1=\int_0^{\alpha}\dots\int_0^{\alpha-\sum_{j=1}^{s}x_j}I_{s+1}\,dx_{s+1}\dots dx_1+\int_0^{\alpha}\dots\int_{\alpha-\sum_{j=1}^{s}x_j}^{c_{s+1}-\sum_{j=1}^{s}x_j}I_{s+1}\,dx_{s+1}\dots dx_1.$$

Then, in order to prove the claim, we just have to decompose the second summand of the last expression. But, since $I_{s+1}=V_{k-s-1}\Big(c_{s+2}-\sum_{j=1}^{s+1}x_j,\dots,c_k-\sum_{j=1}^{s+1}x_j\Big)$, we have, by Lemma 2, $\int_{\alpha-\sum_{j=1}^{s}x_j}^{c_{s+1}-\sum_{j=1}^{s}x_j}I_{s+1}\,dx_{s+1}=V_{k-s}\big(c_{s+1}-\alpha,c_{s+2}-\alpha\dots,c_k-\alpha\big)$. Consequently,

$$\begin{aligned}\int_0^{\alpha}\dots\int_{\alpha-\sum_{j=1}^{s}x_j}^{c_{s+1}-\sum_{j=1}^{s}x_j}I_{s+1}\,dx_{s+1}\dots dx_1&=\int_0^{\alpha}\dots\int_0^{\alpha-\sum_{j=1}^{s-1}x_j}V_{k-s}\big(c_{s+1}-\alpha,\dots,c_k-\alpha\big)\,dx_s\dots dx_1\\&=V_{k-s}\big(c_{s+1}-\alpha,\dots,c_k-\alpha\big)\int_0^{\alpha}\dots\int_0^{\alpha-\sum_{j=1}^{s-1}x_j}dx_s\dots dx_1\\&=V_{k-s}(c_{s+1}-\alpha,\dots,c_k-\alpha)V_s(\alpha,\dots,\alpha).\end{aligned}$$

Then, Equation (2) holds. Next, observe that

$$\begin{aligned}V_k(c_1,\dots,c_k)&=\int_0^{c_1}I_1dx_1=\int_0^{\alpha}I_1dx_1+\int_{\alpha}^{c_1}I_1dx_1\\&=\int_0^{\alpha}I_1dx_1+\int_{\alpha}^{c_1}V_{k-1}(c_2-x_1,\dots,c_k-x_1)dx_1=\int_0^{\alpha}I_1dx_1+V_k(c_1-\alpha,\dots,c_k-\alpha),\end{aligned}$$

where the last equality holds by Lemma 2. But, now, according to Equation (2),

$$\int_0^\alpha I_1 dx_1 = \int_0^\alpha \int_0^{\alpha - x_1} I_2 dx_2 dx_1 + V_{k-1}(c_2-\alpha,\dots,c_k-\alpha)V_1(\alpha).$$

Then, $V_k(c_1,\dots,c_k) = V_k(c_1-\alpha,\dots,c_k-\alpha)+V_1(\alpha)V_{k-1}(c_2-\alpha,\dots,c_k-\alpha)+\int_0^\alpha\int_0^{\alpha-x_1} I_2 dx_2dx_1$. Decompose $\int_0^\alpha\int_0^{\alpha-x_1} I_2 dx_2dx_1$ applying Equation (2), and repeat the process until the intended equality is reached.

2. To prove the second statement, just take $A = c_1 - \alpha$, so that $(c_i - \alpha) - A = c_i - c_1$, and apply statement 1.

$$V_k(c_1-\alpha,\dots,c_k-\alpha) = \sum_{i=0}^{k} V_i(c_1-\alpha,\dots,c_1-\alpha)V_{k-i}(c_{i+1}-c_1,\dots,c_k-c_1).$$

Now, simply observe that, for $i = 0$, $V_k(c_1-c_1, c_2-c_1,\dots,c_k-c_1) = 0$.

3. From Lemma 2 and statement 2.

$$\begin{aligned} V_k(c_1-\alpha,\dots,c_k-\alpha) &= \int_\alpha^{c_1} V_{k-1}(c_2-x_1,\dots,c_k-x_1)dx_1 \\ &= \sum_{i=2}^{k} V_{k-i}(c_{i+1}-c_2,\dots,c_k-c_2)\int_\alpha^{c_1} V_{i-1}(c_2-x_1,\dots,c_2-x_1)dx_1. \end{aligned}$$

But, by Lemma 2, $\int_\alpha^{c_1} V_{i-1}(c_2-x_1,\dots,c_2-x_1)dx_1 = V_i(c_1-\alpha, c_2-\alpha, \overset{i-1}{\dots}, c_2-\alpha)$. □

The next step consists of providing a way to decompose any given volume in terms of volumes involving only the costs up to a fixed c_p.

Proposition 3. *Let $p, k \in \mathbb{N}$ be such that $p < k$ and $0 < c_1 \le \dots \le c_k$. Then,*

$$V_k(c_1,\dots,c_k) = \sum_{i=0}^{k-p} V_{k-p-i}(c_{p+1+i}-c_p,\dots,c_k-c_p)V_{p+i}(c_1,\dots,c_p,\overset{i+1}{\dots},c_p).$$

Proof. First we prove that

$$V_k(c_1,\dots,c_k) = V_{k-p}(c_{p+1}-c_p,\dots,c_k-c_p)V_p(c_1,\dots,c_p) + V_k(c_1,\dots,c_p,c_p,c_{p+2},\dots,c_k). \tag{3}$$

Indeed, we know that

$$V_k(c_1,\dots,c_k) = \int_0^{c_1}\dots\int_0^{c_p - \sum_{j=1}^{p-1} x_j} V_{k-p}\Big(c_{p+1}-\sum_{j=1}^{p} x_j,\dots,c_k-\sum_{j=1}^{p} x_j\Big)dx_p\dots dx_1. \tag{4}$$

If $u_{k-p}(x_p) = V_{k-p}\Big(c_{p+1}-\sum_{j=1}^{p} x_j,\dots,c_k-\sum_{j=1}^{p} x_j\Big)$ then[3], $\dfrac{du_{k-p}(x_p)}{dx_p} = -V_{k-p-1}\Big(c_{p+2}-\sum_{j=1}^{p} x_j,\dots,c_k-\sum_{j=1}^{p} x_j\Big)$.

[3] In general, let $0 < c_1 \le \dots \le c_k$, $k \in \mathbb{N}$, $x_1 \le c_1$, and denote $u_k(x_1) = V_k(c_1-x_1,\dots,c_k-x_1)$. Then, $\frac{du_k}{dx_1}(x_1) = -V_{k-1}(c_2-x_1,\dots,c_k-x_1)$.

Integrating by parts,

$$\int_0^{c_p-\sum\limits_{j=1}^{p-1}x_j} V_{k-p}\Big(c_{p+1}-\sum_{j=1}^{p}x_j,\dots,c_k-\sum_{j=1}^{p}x_j\Big)dx_p =$$

$$V_{k-p}(c_{p+1}-c_p,\dots,c_k-c_p)\Big(c_p-\sum_{j=1}^{p-1}x_j\Big)+\int_0^{c_p-\sum\limits_{j=1}^{p-1}x_j} x_pV_{k-p-1}\Big(c_{p+2}-\sum_{j=1}^{p}x_j,\dots,c_k-\sum_{j=1}^{p}x_j\Big)\,dx_p. \quad (5)$$

Analogously, integrating by parts,[4]

$$\int_0^{c_p-\sum\limits_{j=1}^{p-1}x_j} V_{k-p}\Big(c_p-\sum_{j=1}^{p}x_j,c_{p+2}-\sum_{j=1}^{p}x_j,\dots,c_k-\sum_{j=1}^{p}x_j\Big)dx_p =$$

$$\int_0^{c_p-\sum\limits_{j=1}^{p-1}x_j} x_pV_{k-p-1}\Big(c_{p+2}-\sum_{j=1}^{p}x_j,\dots,c_k-\sum_{j=1}^{p}x_j\Big)\,dx_p. \quad (6)$$

Then, combining equations (4), (5), and (6),

$$\begin{aligned}V_k(c_1,\dots,c_k) &= V_{k-p}(c_{p+1}-c_p,\dots,c_k-c_p)\int_0^{c_1}\dots\int_0^{c_{p-1}-\sum\limits_{j=1}^{p-2}x_j}\Big(c_p-\sum_{j=1}^{p-1}x_j\Big)dx_{p-1}\dots dx_1\\ &+\int_0^{c_1}\dots\int_0^{c_p-\sum\limits_{j=1}^{p-1}x_j} V_{k-p}\Big(c_p-\sum_{j=1}^{p}x_j,c_{p+2}-\sum_{j=1}^{p}x_j,\dots,c_k-\sum_{j=1}^{p}x_j\Big)dx_p\dots dx_1\\ &= V_{k-p}(c_{p+1}-c_p,\dots,c_k-c_p)V_p(c_1,\dots,c_p)+V_k(c_1,\dots,c_p,c_p,c_{p+2},\dots,c_k).\end{aligned}$$

Therefore, Equation (3) holds. Now, applying Equation (3) to $V_k(c_1,\dots,c_p,c_p,c_{p+2},\dots,c_k)$, we find that

$$V_k(c_1,\dots,c_p,c_p,c_{p+2},\dots,c_k) = V_{k-p+1}(c_{p+2}-c_p,\dots,c_k-c_p)V_{p+1}(c_1,\dots,c_p,c_p)+V_k(c_1,\dots,c_p,c_p,c_p,c_{p+3},\dots,c_k).$$

Repeating this process, the result eventually follows. □

4 Higher-cost decreasing monotonicity

The higher-cost decreasing monotonicity property states that if a single cost c_i increases, then the contributions requested by the rule for the agents with cost higher than c_i should not increase.

Definition 1. *A rule ψ satisfies higher-cost decreasing monotonicity if for each pair $c,c'\in\mathcal{C}^N$ and each $i\in N$ such that $c'_i\geq c_i$ and $c'_j=c_j$ for all $j\in N\backslash\{i\}$, then $\psi_j(c')\leq\psi_j(c)$ whenever $c_j>c_i$.*

As already pointed out, since the core-center is very sensible to changes in the cost parameters, proving monotonicity properties of the core-center is not a simple task. Basically, one has to check an inequality of the type $\hat{\mu}_p(c_1,\dots,c_k)\leq\hat{\mu}_{p+1}(d_1,\dots,d_{k+1})$, $p\leq k$, which in turn, since the core-center is a ratio of airport core volumes, is equivalent to an inequality such as $\Gamma=V_{k+1}(c_1,\dots,c_p,c_p,\dots,c_k)V_{k+1}(d_1,\dots,d_{k+1})-$

[4] In fact, this equality is the key result to express the core-center as a ratio of volumes, see (González-Díaz et al., 2014).

$V_k(c_1,\ldots,c_k)V_{k+2}(d_1,\ldots,d_{p+1},d_{p+1},\ldots,d_{k+1}) \leq 0$. Then, one uses the volume decompositions developed in Section 3 to write each of the volumes involved in expression Γ in terms of volumes of certain "manageable" types. Finally, one has to rearrange Γ as a sum of expressions involving these types of "manageable" volumes and study, by induction, their sign.

Let us see how this general scheme works to prove that the core-center satisfies higher-cost decreasing monotonicity. First, we study one type of "manageable" volumes. We need to introduce some notations. Given $k \in \mathbb{N}$ and $0 < c_1 \leq \cdots \leq c_k$, let $Z_0 = 1$ and $Z_s^\alpha = V_s(c_{k-s+1}-\alpha,\ldots,c_k-\alpha)$, for all $s = 1,\ldots,k$, $\alpha < c_{k-s+1}$. When no confusion arises, we write Z_s instead of Z_s^α.

Remark 1. *Let $q,k \in \mathbb{N}$, $q < k$, $0 < c_1 \leq \cdots \leq c_k$, and fix $\alpha < c_{k-q}$. Clearly, $Z_1^\alpha = V_1(c_k-\alpha) = c_k-\alpha$. Now, let $A_0 = 1$ and $A_r = Z_r^{c_{k-q+1}} = V_r(c_{k-r+1}-c_{k-q+1},\ldots,c_k-c_{k-q+1})$, for $r = 1,\ldots,q-1$. Then, by statement 2 and statement 3 of Proposition 2, respectively,*

$$Z_q^\alpha = \sum_{i=1}^{q} \mathcal{V}_i A_{q-i}, \text{ with } \mathcal{V}_i = V_i(c_{k-q+1}-\alpha,\ldots,c_{k-q+1}-\alpha),\ i = 1,\ldots,q,$$

$$Z_{q+1}^\alpha = \sum_{i=2}^{q+1} \bar{\mathcal{V}}_i A_{q+1-i}, \text{ with } \bar{\mathcal{V}}_j = V_j(c_{k-q}-\alpha, c_{k-q+1}-\alpha, \overset{j-1}{\ldots}, c_{k-q+1}-\alpha),\ j = 2,\ldots,q+1.$$

Lemma 3. *For all $q,k \in \mathbb{N}$, $q < k$, and $0 < c_1 \leq \cdots \leq c_k$, fix $\alpha < c_{k-q}$. Then, $Z_1^\alpha Z_q^\alpha - Z_{q+1}^\alpha \geq 0$.*

Proof. We use the notation and decompositions of Remark 1. Clearly, by Lemma 1,

$$\mathcal{V}_i = \frac{(c_{k-q+1}-\alpha)^i}{i!},\ i = 1,\ldots,q. \tag{7}$$

Besides, applying the definition of $\bar{\mathcal{V}}_i$ and Lemma 1,

$$\begin{aligned}\bar{\mathcal{V}}_i &= \int_0^{c_{k-q}} \frac{(c_{k-q-1}-\alpha-x_1)^{i-1}}{(i-1)!}\,dx_1 = \frac{1}{i!}\big((c_{k-q+1}-\alpha)^i - (c_{k-q+1}-c_{k-q})^i\big)\\ &= \mathcal{V}_i - X_i, \quad \text{where } X_i = \frac{1}{i!}(c_{k-q+1}-c_{k-q})^i,\ i = 2,\ldots,q+1.\end{aligned} \tag{8}$$

In order to prove that $Z_1 Z_q - Z_{q+1} = \sum_{i=0}^{q-1}\big((c_k-\alpha)\mathcal{V}_{q-i} - \bar{\mathcal{V}}_{q-i+1}\big)A_i \geq 0$, we check that, for all $i = 0,\ldots,q-1$, $(c_k-\alpha)\mathcal{V}_{q-i} - \bar{\mathcal{V}}_{q-i+1} \geq 0$. Certainly,

$$(c_k-\alpha)\mathcal{V}_{q-i} - \bar{\mathcal{V}}_{q-i+1} = \frac{(c_k-\alpha)(c_{k-q+1}-\alpha)^{q-i}}{(q-i)!} - \frac{(c_{k-q+1}-\alpha)^{q-i+1}}{(q-i+1)!} + \frac{(c_{k-q+1}-c_{k-q})^{q-i+1}}{(q-i+1)!} \geq 0,$$

since $(c_k-\alpha) \geq (c_{k-q+1}-\alpha)$ and $(q-i+1)! \geq (q-i)!$. □

Proposition 4. *Let $t,q,k \in \mathbb{N}$ be such that $t \leq q < k$ and $c_1 \leq \cdots \leq c_k$. Fix $\alpha < c_{k-q}$. Then, $Z_t^\alpha Z_q^\alpha - Z_{t-1}^\alpha Z_{q+1}^\alpha \geq 0$.*

Proof. We proceed by induction on $t \in \mathbb{N}$. The case $t = 1$ has been proved in Lemma 3. Now, assume that the result holds for any $i \leq t-1$, *i.e.*,

$$Z_i^\beta Z_j^\beta - Z_{i-1}^\beta Z_{j+1}^\beta \geq 0,\ i \leq j < k,\ \beta < c_{k-j}, \tag{9}$$

and then, let us prove that it also holds for $t<k$. According to the notation and decompositions of Remark 1,

$$\begin{aligned} Z_tZ_q - Z_{t-1}Z_{q+1} &= \Big(\sum_{i=0}^{t}\mathcal{V}_iA_{t-i}\Big)\Big(\sum_{i=1}^{q}\mathcal{V}_iA_{q-i}\Big) - \Big(\sum_{i=0}^{t-1}\mathcal{V}_iA_{t-1-i}\Big)\Big(\sum_{i=2}^{q+1}\bar{\mathcal{V}}_iA_{q+1-i}\Big) \\ &= \sum_{s=0}^{t-1}\sum_{r=0}^{q-1}A_sA_r\big(\mathcal{V}_{t-s}\mathcal{V}_{q-r} - \mathcal{V}_{t-1-s}\bar{\mathcal{V}}_{q+1-r}\big) + A_t\sum_{r=1}^{q}A_{q-r}\mathcal{V}_r. \end{aligned}$$

Certainly, $A_t\sum_{r=1}^{q}A_{q-r}\mathcal{V}_r \geq 0$. Then, it suffices to prove that

$$S = \sum_{s=0}^{t-1}\sum_{r=0}^{q-1}A_sA_r\Delta_{s,r} \geq 0, \text{ where } \Delta_{s,r} = \mathcal{V}_{t-s}\mathcal{V}_{q-r} - \mathcal{V}_{t-s-1}\bar{\mathcal{V}}_{q-r+1}. \tag{10}$$

First, we claim that

$$\mathcal{V}_i\mathcal{V}_j - \mathcal{V}_{i-1}\bar{\mathcal{V}}_{j+1} \geq 0, \text{ if } i \leq j+1. \tag{11}$$

Indeed, applying Equality (8) in Lemma 3, we have to prove that $\mathcal{V}_i\mathcal{V}_j - \mathcal{V}_{i-1}\mathcal{V}_{j+1} + \mathcal{V}_{i-1}X_{j+1} \geq 0$. Since $\mathcal{V}_{i-1}X_{j+1} \geq 0$, it suffices to prove that $\mathcal{V}_i\mathcal{V}_j - \mathcal{V}_{i-1}\mathcal{V}_{j+1} \geq 0$ whenever $i \leq j+1$. Let $B = (c_{k-q+1} - \alpha)$ and apply Equation (7) in Lemma 3,

$$\mathcal{V}_i\mathcal{V}_j - \mathcal{V}_{i-1}\mathcal{V}_{j+1} = \frac{B^i}{i!}\frac{B^j}{j!} - \frac{B^{i-1}}{(i-1)!}\frac{B^{j+1}}{(j+1)!} = \Big(\frac{1}{i!j!} - \frac{1}{(i-1)!(j+1)!}\Big)B^{i+j}.$$

Now Equation (11) is straightforward, since $\frac{1}{i!j!} - \frac{1}{(i-1)!(j+1)!} \geq 0$ if and only if $i \leq j+1$.

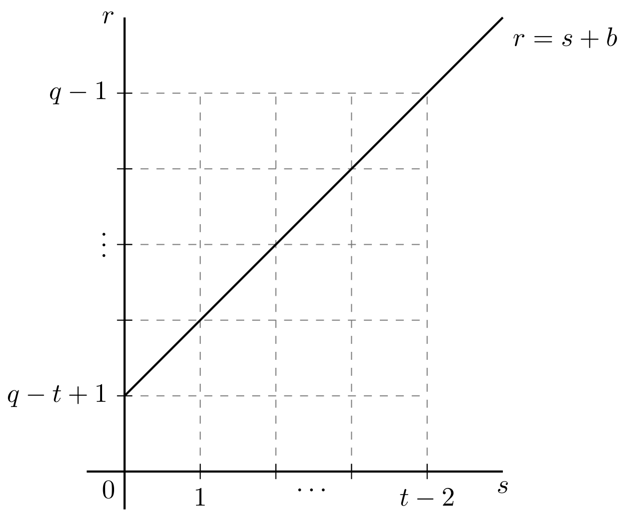

Figure 2: The straight line $r = s+b$, with $b = q-t+1$.

Let $b = q-t+1 > 0$ and $T = \{(s,r) \in [0,t-2]\times[0,q-1] : r > s+b\} \subset \mathbb{N}^2$ be the set depicted in Figure 2. According to Equation (11), if $(s,r) \in [0,t-1]\times[0,q-1]$ but $(s,r) \notin T$ then $A_sA_r\Delta_{s,r} \geq 0$. Now, take $(r,s) \in T$ such that $A_sA_r\Delta_{s,r} \leq 0$, then $h = (r-s)-b > 0$ and $(s+h, r-h) \notin T$, because $(r-h) \leq (s+h)+b$. In addition, $t-s-h=q-r+1$ and $q-r+h=t-s-1$. Therefore, each negative addend $A_sA_r\Delta_{s,r}\leq 0$ in Equation (10) can be paired with the corresponding $A_{s+h}A_{r-h}\Delta_{s+h,r-h}\geq 0$ in the following way

$$\begin{aligned}
&A_sA_r\Delta_{s,r}+A_{s+h}A_{r-h}\Delta_{s+h,r-h}=\\
&A_sA_r\Big(\mathcal{V}_{t-s}\mathcal{V}_{q-r}-\mathcal{V}_{t-s-1}\bar{\mathcal{V}}_{q-r+1}\Big)+A_{s+h}A_{r-h}\Big(\mathcal{V}_{t-s-h}\mathcal{V}_{q-r+h}-\mathcal{V}_{t-s-h-1}\bar{\mathcal{V}}_{q-r+h+1}\Big)=\\
&A_sA_r\Big(\mathcal{V}_{t-s}\mathcal{V}_{q-r}-\mathcal{V}_{t-s-1}\big(\mathcal{V}_{q-r+1}-X_{q-r+1}\big)\Big)+A_{s+h}A_{r-h}\Big(\mathcal{V}_{q-r+1}\mathcal{V}_{t-s-1}-\mathcal{V}_{q-r}\big(\mathcal{V}_{t-s}-X_{t-s}\big)\Big)=\\
&A_sA_r\big(\mathcal{V}_{t-s}\mathcal{V}_{q-r}-\mathcal{V}_{t-s-1}\mathcal{V}_{q-r+1}\big)+A_{s+h}A_{r-h}\big(\mathcal{V}_{q-r+1}\mathcal{V}_{t-s-1}-\mathcal{V}_{q-r}\mathcal{V}_{t-s}\big)+\\
&A_sA_r\mathcal{V}_{t-s-1}X_{q-r+1}+A_{s+h}A_{r-h}\mathcal{V}_{q-r}X_{t-s}=\\
&\big(A_{s+h}A_{r-h}-A_sA_r\big)\big(\mathcal{V}_{t-s-1}\mathcal{V}_{q-r+1}-\mathcal{V}_{q-r}\mathcal{V}_{t-s}\big)+A_sA_r\mathcal{V}_{t-s-1}X_{q-r+1}+A_{s+h}A_{r-h}\mathcal{V}_{q-r}X_{t-s}.
\end{aligned}$$

Therefore, if we prove that the last expression is positive whenever $(s,r)\in T$, then $S\geq 0$. Clearly, the last two terms are positive. Since $(q-r+1)<(t-s-1)+1$ then, applying Equation (11), we get that $\mathcal{V}_{t-s-1}\mathcal{V}_{q-r+1}-\mathcal{V}_{q-r}\mathcal{V}_{t-s}\geq 0$. It remains to show that for all $(s,r)\in T$, $A_{s+h}A_{r-h}-A_sA_r\geq 0$. Now, if $(s,r)\in T$, then

1. $s\leq r-h\leq s+h\leq r$, since $s\leq s+h=r-b\leq r$, $s\leq s+b=r-h\leq r$ and $(r-h)<(s+h)+b\leq(s+h)$.
2. $s+h\leq t-2$, since $s+h=r-b\leq q-1-b=t-2$.

Thus, $A_{r-h}A_{s+h}-A_sA_r=\big(A_{r-h}A_{s+h}-A_{r-h-1}A_{s+h+1}\big)+\big(A_{r-h-1}A_{s+h+1}-A_{r-h-2}A_{s+h+2}\big)+\cdots+\big(A_{s+1}A_{r-1}-A_sA_r\big)$. All the expressions in parentheses are of the form $A_iA_j-A_{i-1}A_{j+1}=Z_i^{\beta}Z_j^{\beta}-Z_{i-1}^{\beta}Z_{j+1}^{\beta}$, with $i\leq t-2$, $i\leq j$ and $\beta=c_{k-q+1}$. Therefore, we can apply Equation (9), the induction hypotheses, and conclude that all the addends $A_iA_j-A_{i-1}A_{j+1}\geq 0$ are positive and so $A_{s+h}A_{r-h}-A_sA_r\geq 0$ as well. □

Lemma 4. *Let $m\in\mathbb{N}$. Given real numbers H^j,G^j, $j=1,\dots,m+2$, and Z_i, $i=1,\dots,m+1$, then*

$$\Big(\sum_{i=0}^{m+1}G^{i+1}Z_{m+1-i}\Big)\Big(\sum_{i=0}^{m}H^{i+1}Z_{m-i}\Big)-\Big(\sum_{i=0}^{m}G^{i+1}Z_{m-i}\Big)\Big(\sum_{i=0}^{m+1}H^{i+1}Z_{m+1-i}\Big)=$$
$$=\sum_{i=0}^{m}\sum_{j=i}^{m}\big(G^{m+2-i}H^{m+1-j}-G^{m+1-j}H^{m+2-i}\big)\big(Z_iZ_j-Z_{i-1}Z_{j+1}\big).$$

where $G^r=H^r=0$ for all $r\neq 1,\dots,m+2$, $Z_0=1$ and $Z_r=0$ for all $r\neq 0,\dots,m+1$.

Proof. Let

$$D=\Big(\sum_{i=0}^{m+1}G^{i+1}Z_{m+1-i}\Big)\Big(\sum_{i=0}^{m}H^{i+1}Z_{m-i}\Big)-\Big(\sum_{i=0}^{m}G^{i+1}Z_{m-i}\Big)\Big(\sum_{i=0}^{m+1}H^{i+1}Z_{m+1-i}\Big).$$

Straightforward computations show that

$$\begin{aligned}
D=&\sum_{i=0}^{m}\big(G^{m+2-i}H^{m+1-i}-G^{m+1-i}\;H^{m+2-i}\big)Z_iZ_i+\sum_{i=0}^{m}\big(G^1H^{m+1-i}-G^{m+1-i}\;H^1\big)Z_iZ_{m+1}\\
&+\sum_{i=0}^{m-1}\sum_{j=i+1}^{m}\big(G^{m+2-i}H^{m+1-j}+G^{m+2-j}H^{m+1-i}-G^{m+1-i}H^{m+2-j}-G^{m+1-j}H^{m+2-i}\big)Z_iZ_j.
\end{aligned}$$

Next, we group all terms of the type $\Delta(q,t) = G^q H^t - G^t H^q$, with $t < q$. Let $A(i,j)$, $i \le j$, be the coefficient of $Z_i Z_j$ in the last expression and let $A^+(i,j) = \Delta(m+2-i, m+1-j)$ and $A^-(i,j) = \Delta(m+1-i, m+2-j)$. For all $i = 0, \dots, m$, $A^-(i, i+1) = 0$, so, in particular, $A(m, m+1) = A^-(m, m+1) = 0$. Then,

$$A(i,j) = \begin{cases} A^+(i,i) & \text{if } i = j \in \{0,\dots,m\} \\ A^+(i,i+1) & \text{if } j = i+1 \in \{1,\dots,m\} \\ -A^-(i,m+1) & \text{if } j = m+1, i \in \{0,\dots,m-1\} \\ A^+(i,j) - A^-(i,j) & \text{if } i \in \{0,\dots,m-2\},\ i+2 \le j \le m. \end{cases}$$

Observe that $m+2-i \ge m+1-j$ whenever $i \le j$ and also $m+1-i \ge m+2-j$ whenever $j \ge i+2$. All the coefficients $A^+(i,j)$ and $A^-(i,j)$ involved are of the type $\Delta(q,t)$ with $t < q$. But, clearly, $A^-(i,j) = A^+(i+1, j-1)$. Then,

$$D = \sum_{i=0}^{m} A^+(0,i) Z_i + \sum_{i=0}^{m-1} \sum_{j=i+2}^{m+1} A^+(i+1, j-1)(Z_{i+1} Z_{j-1} - Z_i Z_j).$$

Rearranging the indices and setting, if necessary, $G^r = H^r = 0$ for all $r \ne 1, \dots, m+2$ and $Z_r = 0$ for all $r \ne 0, \dots, m+1$, we obtain the expression of the statement of the theorem. □

Next, we use the decomposition of Proposition 3 to extend, by induction, the sign inequalities of the "manageable" volumes Z_t to volumes of a more general type. We need some extra notation. Given $p, s \in \mathbb{N}$ and $0 < \delta \le d_1 \le \dots \le d_p$, we write $g_p^s = (d_1 - \delta, \dots, d_p - \delta, \overset{s}{\dots}, d_p - \delta)$, $G_p^s = V_{p+s-1}(g_p^s)$, $h_p^s = (\delta, d_1, \dots, d_p, \overset{s}{\dots}, d_p)$ and $H_p^s = V_{p+s}(h_p^s)$.

Lemma 5. *For all $s \in \mathbb{N}$ and $0 < \delta \le d_1$, we have $\hat{\mu}_1(d_1 - \delta, \overset{s}{\dots}, d_1 - \delta) \le \hat{\mu}_2(\delta, d_1, \overset{s}{\dots}, d_1)$.*

Proof. By Lemma 1, $\hat{\mu}_1(d_1 - \delta, \overset{s}{\dots}, d_1 - \delta) = \frac{d_1 - \delta}{s+1}$. We have to prove that $\Lambda = \frac{(d_1-\delta)}{s+1} V_{s+1}(\delta, d_1, \overset{s}{\dots}, d_1) - V_{s+2}(\delta, d_1, \overset{s+1}{\dots}, d_1) \le 0$. Now, by Proposition 3 with $p = 1$ and $k = s+1$, and Lemma 1, $V_{s+2}(\delta, d_1, \overset{s+1}{\dots}, d_1) = \sum_{i=0}^{s+1} \frac{(d_1-\delta)^{s-i+1}}{(s-i+1)!} \frac{\delta^{i+1}}{(i+1)!}$ and

$$\tfrac{(d_1-\delta)}{s+1} V_{s+1}(\delta, d_1, \overset{s}{\dots}, d_1) = \tfrac{(d_1-\delta)}{s+1} \sum_{i=0}^{s} \frac{(d_1-\delta)^{s-i}}{(s-i)!} \frac{\delta^{i+1}}{(i+1)!} = \sum_{i=0}^{s} \frac{(d_1-\delta)^{s-i+1}}{(s+1)(s-i)!} \frac{\delta^{i+1}}{(i+1)!}.$$

Then, since $(s-i+1)! = (s-i+1)(s-i)!$ and $s+1 \ge s+1-i$, for all $0 \le i \le s$,

$$\Lambda = \sum_{i=0}^{s} \frac{(d_1-\delta)^{s-i+1} \delta^{i+1}}{(i+1)!} \left(\frac{1}{(s+1)(s-i)!} - \frac{1}{(s-i+1)!} \right) - \frac{\delta^{s+2}}{(s+2)!} \le 0,$$

as we wanted to prove. □

Proposition 5. *Let $p, s, t, q \in \mathbb{N}$ and $0 < \delta \le d_1 \le \dots \le d_p$. It holds that*

1. *$\hat{\mu}_p(d_1 - \delta, \dots, d_p - \delta, \overset{s}{\dots}, d_p - \delta) \le \hat{\mu}_{p+1}(\delta, d_1, \dots, d_p, \overset{s}{\dots}, d_p)$.*
2. *if $t < q$ then $G_p^q H_p^t - G_p^t H_p^q \le 0$.*

Proof. Let us prove the first statement proceeding by induction on $p \in \mathbb{N}$. Lemma 5 solves the case $p = 1$. Next, fix $p > 1$, and assume that for all $s \in \mathbb{N}$, $\hat{\mu}_{p-1}(d_1 - \delta, \dots, d_{p-1} - \delta, \overset{s}{\dots}, d_{p-1} - \delta) \le \hat{\mu}_p(\delta, d_1, \dots, d_{p-1}, \overset{s}{\dots}, d_{p-1})$, or, equivalently, for all $s \in \mathbb{N}$, $\frac{G_{p-1}^{s+1}}{G_{p-1}^s} \le \frac{H_{p-1}^{s+1}}{H_{p-1}^s}$. We claim that

$$G_{p-1}^q H_{p-1}^t - G_{p-1}^t H_{p-1}^q \le 0, \text{ for all } t < q. \tag{12}$$

Indeed $\frac{G^q_{p-1}}{G^t_{p-1}} \leq \frac{H^q_{p-1}}{H^t_{p-1}}$ because of the induction hypothesis and the fact that $\frac{G^q_{p-1}}{G^t_{p-1}} = \frac{G^q_{p-1}}{G^{q-1}_{p-1}} \frac{G^{q-1}_{p-1}}{G^{q-2}_{p-1}} \cdots \frac{G^{t+1}_{p-1}}{G^t_{p-1}}$ and $\frac{H^q_{p-1}}{H^t_{p-1}} = \frac{H^q_{p-1}}{H^{q-1}_{p-1}} \frac{H^{q-1}_{p-1}}{H^{q-2}_{p-1}} \cdots \frac{H^{t+1}_{p-1}}{H^t_{p-1}}$. In order to establish the result for $p > 1$, we have to prove that for all $s \in \mathbb{N}$, $G^{s+1}_p H^s_p - G^s_p H^{s+1}_p \leq 0$. From Proposition 3, $G^{s+1}_p = \sum_{i=0}^{s+1} G^{i+1}_{p-1} Z_{s+1-i}$, $H^s_p = \sum_{i=0}^{s} H^{i+1}_{p-1} Z_{s-i}$, $G^s_p = \sum_{i=0}^{s} G^{i+1}_{p-1} Z_{s-i}$ and $H^{s+1}_p = \sum_{i=0}^{s+1} H^{i+1}_{p-1} Z_{s+1-i}$, where $Z_0 = 1$ and $Z_r = V_r(d_p - d_{p-1}, \ldots, d_p - d_{p-1})$, for all $r = 1, \ldots, s+1$. By Lemma 4, $G^{s+1}_p H^s_p - G^s_p H^{s+1}_p = \sum_{i=0}^{s}\sum_{j=i}^{s} \big(G^{s+2-i}_{p-1} H^{s+1-j}_{p-1} - G^{s+1-j}_{p-1} H^{s+2-i}_{p-1}\big)\big(Z_i Z_j - Z_{i-1} Z_{j+1}\big)$, where $Z_r = 0$ for all $r \neq 0, \ldots, s+1$ and $G^r_{p-1} = H^r_{p-1} = 0$ for all $r \neq 1, \ldots, s+2$. Applying Equation (12), the induction hypothesis, and Proposition 4, we obtain that indeed, $G^{s+1}_p H^s_p - G^s_p H^{s+1}_p \leq 0$.

As for the second statement, it is in fact a generalization of statement 1. Indeed, we have just proved that for all $s \in \mathbb{N}$, $G^{s+1}_p H^s_p - G^s_p H^{s+1}_p \leq 0$ or, equivalently, $\frac{G^{s+1}_p}{G^s_p} \leq \frac{H^{s+1}_p}{H^s_p}$. Now, given $t < q$, we have that $G^q_p H^t_p - G^t_p H^q_p \leq 0$ if and only if $\frac{G^q_p}{G^t_p} \leq \frac{H^q_p}{H^t_p}$. Again, this inequality follows directly from the hypothesis and the decompositions $\frac{G^q_p}{G^t_p} = \frac{G^q_p}{G^{q-1}_p} \frac{G^{q-1}_p}{G^{q-2}_p} \cdots \frac{G^{t+1}_p}{G^t_p}$ and $\frac{H^q_p}{H^t_p} = \frac{H^q_p}{H^{q-1}_p} \frac{H^{q-1}_p}{H^{q-2}_p} \cdots \frac{H^{t+1}_p}{H^t_p}$. □

Finally, we can state and prove the main result of this section.

Theorem 1. *For all $p, k \in \mathbb{N}$ such that $k \geq p$, and all $0 < \delta \leq d_1 \cdots \leq d_p \leq \cdots \leq d_k$, we have that*

$$\hat{\mu}_p(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta) \leq \hat{\mu}_{p+1}(\delta, d_1, \ldots, d_p, \ldots, d_k).$$

Proof. By definition,

$$\hat{\mu}_p(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta) = \frac{V_{k+1}(d_1 - \delta, \ldots, d_p - \delta, d_p - \delta, \ldots, d_k - \delta)}{V_k(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta)}$$
$$\hat{\mu}_{p+1}(\delta, d_1, \ldots, d_p, \ldots, d_k) = \frac{V_{k+2}(\delta, d_1, \ldots, d_p, d_p \ldots, d_k)}{V_{k+1}(\delta, d_1, \ldots, d_p, \ldots, d_k)}.$$

Therefore, $\hat{\mu}_p(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta) \leq \hat{\mu}_{p+1}(\delta, d_1, \ldots, d_p, \ldots, d_k)$ if and only if

$$\begin{aligned}\Delta = {} & V_{k+1}(d_1 - \delta, \ldots, d_p - \delta, d_p - \delta, \ldots, d_k - \delta) V_{k+1}(\delta, d_1, \ldots, d_p, \ldots, d_k) \\ & - V_k(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta) V_{k+2}(\delta, d_1, \ldots, d_p, d_p, \ldots, d_k) \leq 0. \end{aligned} \tag{13}$$

Now, applying Proposition 3, we decompose each of the four factors in the last inequality as sums involving volumes of the types G^s_p and H^s_p. Then,

$$V_k(d_1 - \delta, \ldots, d_k - \delta) = \sum_{i=0}^{k-p} G^{i+1}_p Z_{k-p-i}, \qquad V_{k+1}(d_1 - \delta, \ldots, d_p - \delta, d_p - \delta, \ldots, d_k - \delta) = \sum_{i=0}^{k-p} G^{i+2}_p Z_{k-p-i},$$
$$V_{k+1}(\delta, d_1, \ldots, d_k) = \sum_{i=0}^{k-p} H^{i+1}_p Z_{k-p-i}, \qquad V_{k+2}(\delta, d_1, \ldots, d_p, d_p, \ldots, d_k) = \sum_{i=0}^{k-p} H^{i+2}_p Z_{k-p-i},$$

where $Z_0 = 1$ and $Z_t = V_t(d_{k-t+1} - \delta, \ldots, d_k - \delta)$, $t = 1, \ldots, k-p$. Therefore, applying Lemma 4,

$$\Delta = \sum_{i=0}^{k-p}\sum_{j=i}^{k-p} \big(G^{k-p+2-i}_p H^{k-p+1-j}_p - G^{k-p+1-j}_p H^{k-p+2-i}_p\big)\big(Z_i Z_j - Z_{i-1} Z_{j+1}\big).$$

where $Z_r = 0$ for all $r \neq 0, \ldots, k-p$ and $G_p^r = H_p^r = 0$ for all $r \neq 1, \ldots, k-p+2$. Therefore, for $\Delta \leq 0$ it is sufficient to establish that $\Delta(q,t) \leq 0$ whenever $t < q$ and that $Z_t Z_q - Z_{t-1} Z_{q+1} \geq 0$ if $t \leq q$. These two properties were already proved in Propositions 5 and 4, respectively. □

As a particular case of Theorem 1 we deduce that the core-center satisfies higher-cost decreasing monotonicity.

Proposition 6. *The core-center satisfies higher-cost decreasing monotonicity.*

Proof. Let $c \in \mathcal{C}^N$ be an airport problem. Given $j \in \{2, \ldots, n-1\}$ and $r \in \{0, \ldots, j-2\}$, the inequality $\hat{\mu}_{j-r}(c_{r+1}-c_r, \ldots, c_{n-1}-c_r) \geq \hat{\mu}_{j-r-1}(c_{r+2}-c_{r+1}, \ldots, c_{n-1}-c_{r+1})$ follows by taking $k = n-r-2$, $p = j-r-1$ and $(\delta, d_1, \ldots, d_k) = (c_{r+1}-c_r, c_{r+2}-c_r, \ldots, c_{n-1}-c_r)$ in Theorem 1. Therefore, if $j \in \{2, \ldots, n-1\}$ then

$$\hat{\mu}_j(c_1, \ldots, c_{n-1}) \geq \hat{\mu}_{j-1}(c_2-c_1, \ldots, c_{n-1}-c_1) \geq \cdots \geq \hat{\mu}_1(c_j-c_{j-1}, \ldots, c_{n-1}-c_{j-1}).$$

Now, applying Proposition 1, if $i, j \in N \backslash \{n\}$, $i < j$, then $\mu_j(c)$ is decreasing with respect to c_i. □

5 Lower-cost increasing monotonicity

The lower-cost increasing monotonicity property states that if a single cost c_i increases, then the contributions requested of the agents with cost lower than c_i should not decrease.

Definition 2. *A rule ψ satisfies lower-cost increasing monotonicity if for each pair $c, c' \in \mathcal{C}^N$ and each $i \in N$ such that $c'_i \geq c_i$ and $c'_j = c_j$ for all $j \in N \backslash \{i\}$, then $\psi_j(c') \leq \psi_j(c)$ whenever $c_j \leq c_i$.*

In order to establish that the core-center satisfies lower-cost increasing monotonicity, we follow the scheme developed in Section 4 but with a significant difference: the decomposition of a given volume provided by Proposition 3 has to be changed (Proposition 7). We need some extra notations. Given $p, k, s, t, q \in \mathbb{N}$ such that $k \geq p$ and $0 < c_1 \leq \cdots \leq c_p \leq \cdots \leq c_k$, write $A_{k,s}^p = V_{k+s-1}(c_1, \ldots, c_p, \ldots, c_k, \overset{s}{\ldots}, c_k)$, $\hat{A}_{k,s}^p = V_{k+s}(c_1, \ldots, c_p, c_p \ldots, c_k, \overset{s}{\ldots}, c_k)$ and $\Delta_k^p(t,q) = \hat{A}_{k,t}^p A_{k,q}^p - A_{k,t}^p \hat{A}_{k,q}^p$. The superscript in $A_{k,s}^p$, though somehow unnecessary or ambiguous in cases like $\hat{A}_{p,s}^p = A_{p,s+1}^p$, is helpful to refer to a particular coordinate of the core-center. It is also worth noting that $\Delta_k^p(t,t) = 0$.

Proposition 7. *Given $p, k, s \in \mathbb{N}$ such that $k \geq p > 1$ and $0 < c_1 \leq \cdots \leq c_p \leq \cdots \leq c_k$, let $\delta(p,k) = p-1$ if $p = k$ and $\delta(p,k) = p$ if $p < k$. Then,*

$$A_{k,s}^p = \sum_{i=0}^{s} \frac{1}{i!} A_{k-1,s+1-i}^{\delta(p,k)} \, (c_k - c_{k-1})^i, \ \ \hat{A}_{k,s}^p = \sum_{i=0}^{s} \frac{1}{i!} \hat{A}_{k-1,s+1-i}^{\delta(p,k)} \, (c_k - c_{k-1})^i.$$

Proof. Let $p, k, s \in \mathbb{N}$ with $k > p > 1$. Then

$$A_{k,s}^p = V_{k+s-1}(c_1, \ldots, c_p, \ldots, c_k, \overset{s}{\ldots}, c_k) = \int_0^{c_1} \cdots \int_0^{c_{k-1}-\sum_{j=1}^{k-2} x_j} V_s\big(c_k - \sum_{j=1}^{k-1} x_j, \ldots, c_k - \sum_{j=1}^{k-1} x_j\big) dx_{k-1} \ldots dx_1$$

$$= \int_0^{c_1} \cdots \int_0^{c_{k-1}-\sum_{j=1}^{k-2} x_j} \frac{1}{s!}\big(c_k - \sum_{j=1}^{k-1} x_j\big)^s dx_{k-1} \ldots dx_1$$

where the last equality is obtained applying Lemma 1. Now, we expand the integrand by the binomial theorem, setting $X_k = c_k - c_{k-1}$ and $Y_{k-1} = c_{k-1} - \sum_{j=1}^{k-1} x_j$. Then,

$$\frac{1}{s!}\big(c_k - \sum_{j=1}^{k-1} x_j\big)^s = \frac{1}{s!}(X_k + Y_{k-1})^s = \frac{1}{s!} \sum_{i=0}^{s} \binom{s}{i} X_k^i Y_{k-1}^{s-i} = \sum_{i=0}^{s} \frac{1}{i!(s-i)!} X_k^i Y_{k-1}^{s-i}.$$

Therefore, $A^p_{k,s} = \sum_{i=0}^{s} \frac{1}{i!}\Big(\int_0^{c_1} \dots \int_0^{c_{k-1}-\sum_{j=1}^{k-2} x_j} \frac{1}{(s-i)!} Y_{k-1}^{s-i} dx_{k-1} \dots dx_1\Big) X_k^i$. Again, by Lemma 1, $\frac{1}{(s-i)!} Y_{k-1}^{s-i} = V_{s-i}(Y_{k-1}, \dots, Y_{k-1})$ and, consequently,

$$\int_0^{c_1} \dots \int_0^{c_{k-1}-\sum_{j=1}^{k-2} x_j} \frac{1}{(s-i)!} Y_{k-1}^{s-i} dx_{k-1} \dots dx_1 = \int_0^{c_1} \dots \int_0^{c_{k-1}-\sum_{j=1}^{k-2} x_j} V_{s-i}(Y_{k-1}, \dots, Y_{k-1}) dx_{k-1} \dots dx_1$$
$$= \int_0^{c_1} \dots \int_0^{c_{k-2}-\sum_{j=1}^{k-3} x_j} V_{s+1-i}\Big(c_{k-1} - \sum_{j=1}^{k-2} x_j, \dots, c_{k-1} - \sum_{j=1}^{k-2} x_j\Big) dx_{k-2} \dots dx_1 = A^p_{k-1,s+1-i}.$$

The above equality leads to $A^p_{k,s} = \sum_{i=0}^{s} \frac{1}{i!} A^p_{k-1,s+1-i}\ X_k^i$.

The case $k = p$ and the second part of the proof can be easily adapted from the previous one. □

Lemma 6. *Given $p, k, s \in \mathbb{N}$ such that $k \geq p > 1$ and $0 < c_1 \leq \dots \leq c_p \leq \dots \leq c_k$, let $X_k = c_k - c_{k-1}$. Then,*

$$\Delta^p_k(s, s+1) = \sum_{i=0}^{2s} \Big(\sum_{r=0}^{r_i} B(i,r) \Delta^{\delta(p,k)}_{k-1}(t(i,r), q(i,r)) \Big) X_k^i,$$

where, $r_i \in \mathbb{N}$ for all $i \in \{0, \dots, 2s\}$. Besides, for all $r \in \{0, \dots, r_i\}$, it holds that $B(i,r) \geq 0$ and $t(i,r) < q(i,r)$.

Proof. Using the decomposition of Proposition 7, one can derive that

$$\Delta^p_k(s, s+1) = \sum_{i=0}^{s} \Big(\sum_{r=0}^{i} \frac{1}{r!(i-r)!} \Delta^{\delta(p,k)}_{k-1}(t_1(i,r), q_1(i,r)) \Big) X_k^i$$
$$+ \sum_{i=s+1}^{2s+1} \Big(\sum_{r=0}^{2s+1-i} \frac{1}{(s+1-r)!(i-(s+1-r))!} \Delta^{\delta(p,k)}_{k-1}(t_2(i,r), q_2(i,r)) \Big) X_k^i,$$

where $t_1(i,r) = s+1-r$, $q_1(i,r) = s+2-(i-r)$, $t_2(i,r) = 2s+2-i-r$ and $q_2(i,r) = r+1$.

First, we examine the coefficients of the powers X_k^i, $i = 0, \dots, s$, in the sum above. Observe that the coefficient of X_k^0 that corresponds to $i = 0$, $r = 0$, is just $\Delta^{\delta(p,k)}_{k-1}(s+1, s+2)$. Next, fix $i \in \{1, \dots, s\}$ and introduce $r_i^* = \frac{i-1}{2}$. Clearly, if i is an odd number, $r_i^* \in \mathbb{N}$ and the term corresponding to the index $r = r_i^*$ is zero, because $t_1(i, r_i^*) = q_1(i, r_i^*)$. As a consequence, the coefficient of X_k^i has an odd number of addends, in fact, i when i is odd and $i+1$ when i is even. In any case, the term corresponding to the index $r = i$ (the last one) is $B(i,i)\Delta^{\delta(p,k)}_{k-1}(t(i,i), q(i,i))$ where $B(i,i) = \frac{1}{i!} \geq 0$, $t(i,i) = s+1-i$, and $q(i,i) = s+2$. Since $i \in \{0, \dots, s\}$, $t(i,i) < q(i,i)$. Therefore, we are left with an even number of terms.

Now, consider the terms corresponding to indices $r_1, r_2 \in \{0, \dots, i-1\}$ such that $r_1 < r_2$ and $r_1 + r_2 = i-1$. We have that $r_1 < r_i^* < r_2$, $t_1(i, r_1) = q_1(i, r_2)$, and $q_1(i, r_1) = t_1(i, r_2)$. Subsequently, $\Delta^{\delta(p,k)}_{k-1}(t_1(i,r_1), q_1(i,r_1)) = -\Delta^{\delta(p,k)}_{k-1}(t_1(i,r_2), q_1(i,r_2))$, so we can add up both terms and write

$$\frac{1}{r_1!(i-r_1)!} \Delta^{\delta(p,k)}_{k-1}(t_1(i,r_1), q_1(i,r_1)) + \frac{1}{r_2!(i-r_2)!} \Delta^{\delta(p,k)}_{k-1}(t_1(i,r_2), q_1(i,r_2))$$
$$= \Big(\frac{1}{r_2!(i-r_2)!} - \frac{1}{r_1!(i-r_1)!} \Big) \Delta^{\delta(p,k)}_{k-1}(t_1(i,r_2), q_1(i,r_2)).$$

Therefore each pair of indices r_1, r_2, with the properties listed above, produces a single term of the form $B(i,r)\Delta^{\delta(p,k)}_{k-1}(t(i,r), q(i,r))$ satisfying:

1. $B(i,r) \geq 0$.
Indeed, $\frac{1}{r_2!(i-r_2)!} - \frac{1}{r_1!(i-r_1)!} \geq 0$ if and only if $\frac{(i-r_1)!}{r_2!} \geq \frac{(i-r_2)!}{r_1!}$. But, $r_1 + r_2 = i - 1$ implies that $i - r_1 = r_2 + 1$, $i - r_2 = r_1 + 1$. Then, $\frac{(i-r_1)!}{r_2!} = r_2 + 1 > \frac{(i-r_2)!}{r_1!} = r_1 + 1$.

2. $t(i,r) < q(i,r)$.
Certainly, $t(i,r) = t_1(i,r_2) = s + 1 - r_2 < q(i,r) = q_1(i,r_2) = s + 2 - (i - r_2)$ if and only if $r_2 > \frac{i-1}{2} = r_i^*$.

A similar analysis can be done for the coefficients of the powers X_k^i, $i = s+1, \ldots, 2s+1$. □

Lemma 7. *For all* $p, s \in \mathbb{N}$, $\mu_p(c_1, \ldots, c_p, \overset{s}{\ldots}, c_p) \geq \mu_p(c_1, \ldots, c_p, \overset{s+1}{\ldots}, c_p)$.

Proof. We proceed by induction on p. The case $p = 1$, that is, $\mu_1(c_1, \overset{s}{\ldots}, c_1) \geq \mu_1(c_1, \overset{s+1}{\ldots}, c_1)$ for all $s \in \mathbb{N}$, is a simple consequence of the fact that $\mu_1(c_1, \overset{s}{\ldots}, c_1) = \frac{c_1}{s}$. Next, assume that the result holds for $p-1$, that is, for all $s \in \mathbb{N}$, $\mu_{p-1}(c_1, \ldots, c_{p-1}, \overset{s}{\ldots}, c_{p-1}) \geq \mu_{p-1}(c_1, \ldots, c_{p-1}, \overset{s+1}{\ldots}, c_{p-1})$. Then, it follows that $\Delta_{p-1}^{p-1}(s, s+1) \geq 0$ for all $s \in \mathbb{N}$, or equivalently, $\Delta_{p-1}^{p-1}(t,q) \geq 0$ whenever $t < q$. We have to prove that the result holds for p. But, again, that is equivalent to proving that $\Delta_p^p(s, s+1) \geq 0$, for all $s \in \mathbb{N}$, which is a direct consequence of Lemma 6 and the induction hypothesis. □

Lemma 8. *For all* $p, k, s \in \mathbb{N}$ *such that* $k \geq p$, $\mu_p(c_1, \ldots, c_p, \ldots, c_k, \overset{s}{\ldots}, c_k) \geq \mu_p(c_1, \ldots, c_p, \ldots, c_k, \overset{s+1}{\ldots}, c_k)$.

Proof. We proceed by induction on k. The case $k = p$ was proven in Lemma 7. Next, assume that the result holds for $k - 1 \geq p$, that is, $\mu_p(c_1, \ldots, c_p, \ldots, c_{k-1}, \overset{s}{\ldots}, c_{k-1}) \geq \mu_p(c_1, \ldots, c_p, \ldots, c_{k-1}, \overset{s+1}{\ldots}, c_{k-1})$, for all $s \in \mathbb{N}$. Then, $\Delta_{k-1}^p(s, s+1) \geq 0$ for all $s \in \mathbb{N}$, or equivalently, $\Delta_{k-1}^p(t,q) \geq 0$ whenever $t < q$. We have to prove that the result holds for $k > p$. But, again, that is equivalent to proving that $\Delta_k^p(s, s+1) \geq 0$, for all $s \in \mathbb{N}$. But this inequality follows immediately from Lemma 6 and the induction hypothesis. □

Theorem 2. *Given* $p, s \in \mathbb{N}$ *and costs* $0 < c_1 \leq \cdots \leq c_p \leq c_{p+1} \leq \cdots \leq c_{p+s}$, *we have that*

$$\mu_p(c_1, \ldots, c_p) \geq \mu_p(c_1, \ldots, c_p, c_{p+1}) \geq \cdots \geq \mu_p(c_1, \ldots, c_p, c_{p+1}, \ldots, c_{p+s}).$$

Proof. Observe that $\mu_p(c_1, \ldots, c_p) = (c_p - c_{p-1}) + \hat{\mu}_{p-1}(c_1, \ldots, c_{p-1})$ and $\mu_p(c_1, \ldots, c_p, c_{p+1}) = \hat{\mu}_p(c_1, \ldots, c_p)$. Then, $\mu_p(c_1, \ldots, c_p) \geq \mu_p(c_1, \ldots, c_p, c_{p+1})$ if and only if $\Delta_p = (c_p - c_{p-1}) A_{p-1,1}^{p-1} A_{p,1}^p + A_{p-1,2}^{p-1} A_{p,1}^p - A_{p,2}^p A_{p-1,1}^{p-1} \geq 0$. Note that Δ_p does not depend on the cost c_{p+1}, therefore, using Lemma 7, it is easy to see that the first inequality of the chain is satisfied.

Now, whenever $k > p$, $\mu_p(c_1, \ldots, c_k) \geq \mu_p(c_1, \ldots, c_k, c_{k+1})$ if and only if $\hat{\mu}_p(c_1, \ldots, c_{k-1}) \geq \hat{\mu}_p(c_1, \ldots, c_k)$. The last inequality is equivalent to $\hat{A}_{k-1,1}^p A_{k,1}^p - \hat{A}_{k,1}^p A_{k-1,1}^p \geq 0$. Consider the left-hand expression as a function of the cost c_k, that is, $f(c_k) = \hat{A}_{k-1,1}^p A_{k,1}^p - \hat{A}_{k,1}^p A_{k-1,1}^p$, $c_k \in [c_{k-1}, c_{k+1}]$. A straightforward computation shows that $f'(c_k) = 0$. Henceforth, f is constant in the interval $[c_{k-1}, c_{k+1}]$. Consequently, $f(c_k) \geq 0$ if and only if $f(c_{k-1}) \geq 0$. Since $f(c_{k-1}) = \Delta_{k-1}^p(1,2)$ then $f(c_{k-1}) \geq 0$ if and only if $\hat{\mu}_p(c_1, \ldots, c_p, \ldots, c_{k-1}) \geq \hat{\mu}_p(c_1, \ldots, c_p, \ldots, c_{k-1}, c_{k-1})$. Finally, the last inequality has already been established in Lemma 8. □

Immediately from Theorem 2, we deduce that the core-center satisfies lower-cost increasing monotonicity.

Proposition 8. *The core-center satisfies lower-cost increasing monotonicity.*

Proof. Let $c \in \mathcal{C}^N$ be an airport problem. Then the core-center satisfies lower-cost increasing monotonicity if and only if $\mu_j(c)$ is increasing with respect to c_i for all $j \leq i \leq n$. First, assume that $j \leq i < n$. According to Theorem 2, $\mu_j(c_1, \ldots, c_j) \geq \mu_j(c_1, \ldots, c_j, \ldots, c_i) \geq \mu_j(c_1, \ldots, c_n) = \mu_j(c)$, and lower-cost increasing monotonicity is now a direct consequence of Proposition 1. As for the case $j \leq i = n$, we already know that $\mu_j(c)$, $j = 1, \ldots, n-1$, is independent of c_n, and that $\frac{\partial \mu_n}{\partial c_n}(c) = 1$. □

6 Monotonicity properties

The monotonicity properties in the literature on airport problems focus on how changes in the cost parameters of an agent, or group of agents, impact the payoff assigned by a single-valued rule to that particular agent or group of agents. Of course, such cost variations also have an effect on the core of the airport game. But, how to measure it? We argue that the core-center, as the "average" stable payoff vectors, is a good indicator of the core monotonicity. In fact, the lower-cost increasing monotonicity and higher-cost decreasing monotonicity properties studied in the previous sections capture the general behavior of the core-center with respect to such changes, providing an insight on the corresponding core fluctuations. In this section, building on these two properties, we analyze whether or not the core-center satisfies the usual monotonicity properties listed in Section 2.

First, we state several implications relating some of these well known properties with higher-cost decreasing monotonicity and lower-cost increasing monotonicity.

Proposition 9. *If a rule satisfies lower-cost increasing monotonicity then it satisfies individual cost monotonicity. If a rule satisfies downstream cost monotonicity and lower-cost increasing monotonicity then it satisfies weak cost monotonicity.*

Proof. The first statement is trivial. Then, we just have to prove that, for each pair $c, c' \in \mathcal{C}^N$ such that $c' = c + c''$ for some $c'' \in \mathcal{C}^N$, then $\psi(c') \geq \psi(c)$. Consider the following airport problems:

Problem	Costs						
$c^0 = c$	c_1	c_2	$\dots$	c_i	$\dots$	c_n
c^1	$c_1 + c''_1$	$c_2 + c''_1$	$\dots$	$c_i + c''_1$	$\dots$	$c_n + c''_1$
$\dots$						
c^i	$c_1 + c''_1$	$c_2 + c''_2$	$\dots$	$c_i + c''_i$	$\dots$	$c_n + c''_i$
$\dots$						
$c^n = c'$	$c_1 + c''_1$	$c_2 + c''_2$	$\dots$	$c_i + c''_i$	$\dots$	$c_n + c''_n$

Now, noting that $c' = c^n$ and combining downstream cost monotonicity (DOWN) and lower-cost increasing monotonicity (LCIM) we have

$$\begin{array}{l}
\psi_1(c^n) \geq^{\text{LCIM}} \dots \geq^{\text{LCIM}} \psi_1(c^i) \geq^{\text{LCIM}} \dots \geq^{\text{LCIM}} \psi_1(c^1) \geq^{\text{DOWN}} \psi_1(c) \\
\psi_2(c^n) \geq^{\text{LCIM}} \dots \geq^{\text{LCIM}} \psi_2(c^i) \geq^{\text{LCIM}} \dots \geq^{\text{DOWN}} \psi_2(c^1) \geq^{\text{DOWN}} \psi_2(c) \\
\dots \\
\psi_i(c^n) \geq^{\text{LCIM}} \dots \geq^{\text{LCIM}} \psi_i(c^i) \geq^{\text{DOWN}} \dots \geq^{\text{DOWN}} \psi_i(c^1) \geq^{\text{DOWN}} \psi_i(c) \\
\dots \\
\psi_n(c^n) \geq^{\text{DOWN}} \dots \geq^{\text{DOWN}} \psi_n(c^i) \geq^{\text{DOWN}} \dots \geq^{\text{DOWN}} \psi_n(c^1) \geq^{\text{DOWN}} \psi_n(c).
\end{array}$$

Then, in fact, $\psi(c') \geq \psi(c)$. □

As an immediate consequence of Propositions 8 and 9 we have the following result.

Proposition 10. *The core-center satisfies individual cost monotonicity.*

Obviously, if a rule satisfies others-oriented cost monotonicity then it also satisfies higher-cost decreasing monotonicity. Nevertheless, the converse is not true.

Example 1. *Let $N = \{1, 2, 3\}$. Consider the pair of airport problems $c, c' \in \mathcal{C}^N$ where $c = (1, 2, 3)$ and $c' = (1, 3, 3)$. Then, $\mu_1(c) = \frac{4}{9} < \mu_1(c') = \frac{7}{15}$. Observe that an increase in the cost of player 2 results in a lower core-center payoff for player 1. Then, the core-center violates others-oriented cost monotonicity.*

The analysis of the downstream cost monotonicity property for the core-center follows a similar structure as that of Theorem 1. Hence we just provide an outline.

Theorem 3. *Given indices $i, j \in \mathbb{N}$, $j \geq i$, a value $\gamma > 0$ and costs $0 < c_1 \leq \cdots \leq c_k$, we have that*

$$\mu_j(c_1, c_2, \ldots, c_i + \gamma, \ldots, c_k + \gamma) \geq \mu_j(c_1, c_2, \ldots, c_k).$$

Proof. Let us examine some simple situations. If $i = j = k$, then $\mu_k(c_1, \ldots, c_{k-1}, c_k + \gamma) = \gamma + \mu_k(c_1, \ldots, c_k) \geq \mu_k(c_1, \ldots, c_k)$. If $i < k$ and $j = k$ then $\mu_k(c_1, \ldots, c_i+\gamma, \ldots, c_k+\gamma) = (c_k - c_{k-1}) + \hat{\mu}_{k-1}(c_1, \ldots, c_i+\gamma, \ldots, c_{k-1}+\gamma)$ and $\mu_k(c_1, \ldots, c_k) = (c_k - c_{k-1}) + \hat{\mu}_{k-1}(c_1, \ldots, c_{k-1})$. Therefore, $\mu_k(c_1, \ldots, c_i + \gamma, \ldots, c_k + \gamma) \geq \mu_k(c_1, \ldots, c_k)$ if and only if $\hat{\mu}_{k-1}(c_1, \ldots, c_i+\gamma, \ldots, c_{k-1}+\gamma) \geq \hat{\mu}_{k-1}(c_1, \ldots, c_{k-1})$. Obviously, if $i \leq j < k$ then $\mu_j(c_1, c_2, \ldots, c_i + \gamma, \ldots, c_k + \gamma) = \hat{\mu}_j(c_1, \ldots, c_i + \gamma, \ldots, c_{k-1} + \gamma)$ and $\mu_j(c_1, \ldots, c_k) = \hat{\mu}_j(c_1, \ldots, c_{k-1})$. It suffices to prove that for all $i \leq j \leq k$, $\hat{\mu}_j(c_1, \ldots, c_{i-1}, c_i + \gamma, \ldots, c_j + \gamma, \ldots, c_k + \gamma) \geq \hat{\mu}_j(c_1, \ldots, c_k)$. This is equivalent to

$$\begin{aligned}\Delta &= V_{k+1}(c_1, \ldots, c_{i-1}, c_i + \gamma, \ldots, c_j + \gamma, c_j + \gamma, \ldots, c_k + \gamma)V_k(c_1, \ldots, c_k) \\ &\quad - V_k(c_1, \ldots, c_{i-1}, c_i + \gamma, \ldots, c_j + \gamma, \ldots, c_k + \gamma)V_{k+1}(c_1, \ldots, c_j, c_j, \ldots, c_k) \geq 0.\end{aligned}$$

Denote $Z_0 = 1$ and

$$\begin{aligned}G_j^s &= V_{j+s-1}(c_1, \ldots, c_{i-1}, c_i + \gamma, \ldots, c_j + \gamma, \overset{s}{\ldots}, c_j + \gamma), \ s = 1, \ldots, k - j + 2 \\ H_j^s &= V_{j+s-1}(c_1, \ldots, c_j, \overset{s}{\ldots}, c_j), \ s = 1, \ldots, k - j + 2 \\ Z_t &= V_t(c_{k-t+1} - c_j, \ldots, c_k - c_j), \ t = 1, \ldots, k - j.\end{aligned}$$

Applying Proposition 3, $V_k(c_1, \ldots, c_k) = \sum_{r=0}^{k-j} H_j^{r+1} Z_{k-j-r}$, $V_{k+1}(c_1, \ldots, c_j, c_j, \ldots, c_k) = \sum_{r=0}^{k-j} H_j^{r+2} Z_{k-j-r}$, and

$$\begin{aligned}V_{k+1}(c_1, \ldots, c_i + \gamma, \ldots, c_j + \gamma, c_j + \gamma, \ldots, c_k + \gamma) &= \sum_{r=0}^{k-j} G_j^{r+2} Z_{k-j-r} \\ V_k(c_1, \ldots, c_i + \gamma, \ldots, c_j + \gamma, \ldots, c_k + \gamma) &= \sum_{r=0}^{k-j} G_j^{r+1} Z_{k-j-r}\end{aligned}$$

Therefore, applying Lemma 4,

$$\Delta = \sum_{r=0}^{k-j} \sum_{t=r}^{k-j} \left(G_j^{k-j+2-r} H_j^{k-j+1-t} - G_j^{k-j+1-t} H_j^{k-j+2-r}\right)\left(Z_r Z_t - Z_{r-1} Z_{t+1}\right),$$

where $Z_r = 0$ for all $r \neq 0, \ldots, k - j$ and $G_j^r = H_j^r = 0$ for all $r \neq 1, \ldots, k - j + 2$. Then, in order to prove that $\Delta \geq 0$ it is sufficient to establish that $\Delta(q, t) \geq 0$ whenever $t < q$ and that $Z_r Z_t - Z_{r-1} Z_{t+1} \geq 0$ if $r \leq t$. The first property can be established, with very few adjustments, as in Proposition 5, and the second holds by Proposition 4. □

Proposition 11. *The core-center satisfies downstream cost monotonicity.*

Proof. Let $c \in \mathcal{C}^N$ be an airport problem. Observe that downstream cost monotonicity can be rewritten as follows. If for each pair $c, c' \in \mathcal{C}^N$ and each $i \in N$, if for each $j \in N$ such that $c_j < c_i$, $c'_j = c_j$ and each $j \in N$ such that $c_j \geq c_i$, $c'_j = c_j + \gamma$ ($\gamma \geq 0$), then for each $j \in N$ such that $c_j \geq c_i$, $\psi_j(c') \geq \psi_j(c)$. Therefore the cost vectors c and c' can be written as $c = (c_1, c_2, \ldots, c_{i-1}, c_i, \ldots, c_n)$ and $c' = (c_1, c_2, \ldots, c_{i-1}, c_i + \gamma, \ldots, c_n + \gamma)$. Now, the result is a direct consequence of Theorem 3. □

By applying Propositions 8, 9 and 11 we derive the following result.

Proposition 12. *The core-center satisfies weak cost monotonicity.*

Example 2. *The core-center does not satisfy strong cost monotonicity. Indeed, let* $N = \{1,2,3\}$ *and* $c = (1,2,4) \in \mathcal{C}^N$*. Then* $\mu(c) = (\frac{4}{9}, \frac{7}{9}, \frac{25}{9})$*. Now, for the airport problem* $c' = (1,3,4) \in \mathcal{C}^N$*,* $\mu(c') = (\frac{7}{15}, \frac{19}{15}, \frac{34}{15})$*. Thus,* $c \leq c'$ *but* $\mu_3(c') < \mu_3(c)$*.*

Proposition 13. *The core-center satisfies population monotonicity.*

Proof. We prove the result for the case in which there is $k \in N$ such that $N = N' \cup \{k\}$. The general case follows from repeated application of that property. Thus, given $N' = N \backslash \{k\}$, we prove that $\mu_{N'}(c) \leq \mu(c_{N'})$. We distinguish three cases.

Case 1: $\boldsymbol{c_k = c_n}$. Then, for each $i \in N'$, $c_i \leq c_k = c_n$. But, by Theorem 2, $\mu_i(c_{N'}) = \mu_i(c_1, \ldots, c_{n-1}) \geq \mu_i(c_1, \ldots, c_{n-1}, c_n) = \mu_i(c)$, for each $i \in N'$.

Case 2: $\boldsymbol{c_k = c_1}$. Then $c_i \geq c_k = c_1$ for all $i \in N'$. Now, for each $\varepsilon \geq 0$, let $c^\varepsilon = (\varepsilon, c_2, \ldots, c_n)$. Clearly, $\mu(c_{N'}) = \mu_{N'}(c^0)$. By higher-cost decreasing monotonicity, for each $\varepsilon \in (0, c_1]$, $\mu_{N'}(c) \leq \mu_{N'}(c^\varepsilon)$ and, by continuity, $\mu_{N'}(c) \leq \mu_{N'}(c^0)$. Therefore, $\mu_{N'}(c) \leq \mu_{N'}(c^0) = \mu(c_{N'})$.

Case 3: $\boldsymbol{c_1 < c_k < c_n}$. Let $i \in N'$. We distinguish two subcases.

$\boldsymbol{c_i > c_k}$: Consider the airport problems $c^\varepsilon \in \mathcal{C}^N$, with $c^\varepsilon = (\varepsilon, c_1, \ldots, c_{k-1}, c_{k+1}, c_{k+2}, \ldots, c_n)$ and $\varepsilon \in (0, c_1]$. As in Case 2, higher-cost decreasing monotonicity and continuity ensure that $\mu_i(c_{N'}) = \mu_i(c^0) \geq \mu_i(c^\varepsilon)$. Combining this with a repeated application of higher-cost decreasing monotonicity, we have

$$\begin{aligned}\mu_i(c_{N'}) = \mu_i(c^0) &\geq \mu_i(c_1, c_1, c_2, \ldots, c_{k-1}, c_{k+1}, c_{k+2}, \ldots, c_n) \geq \mu_i(c_1, c_2, c_2, \ldots, c_{k-1}, c_{k+1}, c_{k+2}, \ldots, c_n) \\ &\geq \cdots \geq \mu_i(c_1, c_2, c_3, \ldots, c_{k-1}, c_k, c_{k+1}, \ldots, c_n) = \mu_i(c).\end{aligned}$$

$\boldsymbol{c_i \leq c_k}$: If $c_i = c_k$, we assume, without loss of generality, that $i < k$. Now, applying lower-cost increasing monotonicity repeatedly, $\mu_i(c_{N'}) = \mu_i(c_1, \ldots, c_{k-1}, c_{k+1}, c_{k+2}, \ldots, c_n) \geq \mu_i(c_1, \ldots, c_{k-1}, c_k, c_{k+1}, \ldots, c_{n-1})$. Now, by Case 1, we also have that $\mu_i(c_1, \ldots, c_{k-1}, c_k, c_{k+1}, \ldots, c_{n-1}) \geq \mu_i(c_1, \ldots, c_{k-1}, c_k, c_{k+1}, \ldots, c_{n-1}, c_n) = \mu_i(c)$. Combining the inequalities in both equations we get that $\mu_i(c_{N'}) \geq \mu_i(c)$. □

Example 3. *The core-center does not satisfy marginalism. Consider the airport problems with player set* $N = \{1,2,3\}$ *and costs* $c = (1,2,3)$ *and* $c' = (1,3,4)$ *that satisfy the hypothesis of downstream cost monotonicity (with* $i = 2$*). Their respective core-centers are* $\mu(c) = (\frac{4}{9}, \frac{7}{9}, \frac{16}{9})$ *and* $\mu(c') = (\frac{7}{15}, \frac{19}{15}, \frac{34}{15})$ *so, in particular,* $\mu_1(c) \neq \mu_1(c')$*.*

Remark 2. *The core-center is an intuitive but quite complex solution concept defined for the class of balanced games. What we show up in this paper is a well established model, the class of airport problems, for which the core-center has good monotonicity properties. The task of proving the results of the paper requires a deep analysis of the geometric structure of the core.*

Besides, two new monotonicity properties, higher-cost decreasing monotonicity and lower-cost increasing monotonicity, are the key to check if the core-center satisfies the usual monotonicity properties. The relation of these two properties with the usual properties for airport problems and their implications for the ranking of rules has been studied in Mirás-Calvo et al. (2014).

As a summary, of the monotonicity properties listed in Section 2, the core-center violates others-oriented cost monotonicity, strong cost monotonicity, and marginalism, but it satisfies individual cost monotonicity, downstream cost monotonicity, weak cost monotonicity, and population monotonicity.

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References

Arin, J. (2013). Monotonic core solutions: Beyond Young's theorem. *International Journal of Game Theory*, 42:325–337.

Calleja, P., Rafels, C., and Tijs, S. (2009). The aggregate-monotonic core. *Games and Economic Behavior*, 66:742–748.

Getán, J., Montes, J., and Rafels, C. (2009). A note on the monotonic core. *International Game Theory Review*, 11:229–235.

Gillies, D. B. (1953). *Some Theorems on n-Person Games*. PhD thesis, Princeton.

González-Díaz, J., Mirás-Calvo, M. A., Quinteiro-Sandomingo, C., and Sánchez-Rodríguez, E. (2014). On the core of an airport game and the properties of its center. *Reports in Statistics and Operations Research*, 14-01:1–32. University of Santiago de Compostela.

González-Díaz, J. and Sánchez-Rodríguez, E. (2007). A natural selection from the core of a TU game: the core-center. *International Journal of Game Theory*, 36:27–46.

Housman, D. and Clark, L. (1998). Core and monotonic allocation methods. *International Journal of Game Theory*, 27:611–616.

Littlechild, S. C. and Owen, G. (1973). A simple expression for the Shapley value in a special case. *Management Science*, 20:370–372.

Megiddo, N. (1974). On the nonmonotonicity of the bargaining set, the kernel and the nucleolus of a game. *Siam Journal on Applied Mathematics*, 27:355–358.

Mirás-Calvo, M. A., Quinteiro-Sandomingo, C., and Sánchez-Rodríguez, E. (2014). Monotonicity implications to the ranking of rules for airport problems. Mimeo.

Thomson, W. (2007). Cost allocation and airport problems. Rochester Center for Economic Research Working Paper.

Young, H. (1985). Monotonic solutions of cooperatives games. *International Journal of Game Theory*, 14:65–72.