Existence of solution to a model for gas transportation networks on non-flat topography

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Abstract

In this paper we prove the existence of solution to a mathematical model for gas transportation networks on non-flat topography. Firstly, the network topology is represented by a directed graph and then a nonlinear system of numerical equations is introduced whose unknowns are the pressures at the nodes and the mass flow rates at the edges of the graph. This system is written in a compact vector form in terms of the vector of the square pressures at the nodes and then an existence result is proved under some simplifying assumptions. The proof is done in two steps: the first one uses convex analysis tools and the second one the Brouwer fixed-point theorem.

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1 Introduction

Due to their nature, gas transmission networks occupy vast extensions of land which can be measured in thousands of kilometers. They are generally managed from a control center by the Technical System Manager (TSM) based on the data they receive from the different elements that form the network, namely, compression stations, pressure control valves, flow control valves, closing valves, regasification plants, international connections, underground storages and deposit fields. A gas transmission network is defined by its topology and its elements. More specifically, the gas pipeline connections and geometrical properties as length, diameter, roughness, and geographic coordinates of pipes and nodes.

Mathematical modelling of gas flow in pipelines is an important subject in planning and operating gas transportation networks (see reference books, [1], [2]). Some

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recent papers have been devoted to the transient case (see [3], [4]). Usually, they assume isothermal or isentropic flow (see [5]) but in real networks neither temperature nor entropy remain constant because, first, there is heat exchange with the environment (see [6],[7]) and second there is dissipation in the boundary layer near the wall of the pipelines due to viscous friction. These features complicate the model because they lead to two respective source terms in the physical balance laws.

However, most papers and computer programs on the subject deal with the case of steady-state because, based on simplified steady state models, network optimization problems can be stated and solved even for large networks. Currently, in most networks the first aim is to meet all consumer demands, the so-called *security of supply*. However, compression stations and regasification plants use themselves the gas as an internal power source, thus leading to operation costs. In order to minimize these costs it is necessary to pay particular attention to the way in which the network is managed. Mathematical optimization theory is an important tool to handle this problem (see [8], [9], [10], [11], [12]). Nevertheless, the analysis of these optimization problems is beyond the scope of this paper which is rather focused in the network simulation problem when the flow inputs and outputs are given and the operating parameters of compressors (compression ratio or pressure jump) are prescribed; more specifically, we deal with the existence of solution of a particular but frequently used network mathematical model.

Despite their practical importance, to the best of the authors' knowledge, the proof of the existence of a solution to these simplified models has been only done for the case of flat topography (see [13, Corollary 2]) and strictly speaking for a network involving only pipes, neither compressors nor valves are considered. Thus, the nontrivial extension of this result to the full more general model with non-flat topography remains an open problem and it is the main goal of the present research. It is worth emphasizing that the fact that nodes can have different heights has an extremely important influence on the behaviour of gas networks. For instance, an upwinding node may have a lower pressure than a downwinding one if the level of the former is higher than the level of the latter. Of course this would never occur if both are at the same level, unless the gas in the pipe between them does not move.

The paper is organized as follows: in Sections 2 and 3 the mathematical model is established (further details are given in AppendixA). Then, in Section 4 the existence of a solution to this model is proved as well as uniqueness in the case of flat topography. At the end, two appendices deal, respectively, with obtaining the model equations from the conservation principles of continuum thermomechanics and with some elements on graph theory.

2 Mathematical model: data and unknowns

The goal of this section is to introduce a mathematical model for steady-state gas flow in a gas transportation network which will be subsequently analyzed in Section 4. A gas transportation network consists of different elements and devices such as entry points, exit points, gas pipelines with different sizes, compression stations, flow control valves (FCV), closing valves and pressure control valves (PCV). Its topology can be represented by a directed graph corresponding to the following choices:

- The *nodes* represent the gas supply points, the gas consumption points, the underground storages, the suction or discharge points in a compression station, the interconnection points among pipelines, and the points where the latter change diameter or some other property.

- The *edges* represent the pipelines, the compressors (each compressor links the suction node and the discharge node by the ratio of their increasing pressures), the flow control valves (FCV) (where the mass flow rate is imposed), the closed closing valves (where the mass flow is zero), the bypasses or open closing valves, the pressure control valves (PCV) (which link two nodes by the ratio of their decreasing pressures).

The total number of nodes is denoted by n and the total number of edges by e. Concerning the flow, the magnitudes involved in the model are:

1. The *pressure* at the nodes: $\{p_i : \; i = 1, \ldots, n\}$. We denote by $\mathbf{p}$ the column vector of n components: $\mathbf{p} = (p_1, \ldots, p_n)^t$.
2. The *mass flow rate* exchanged with the outside of the network at the nodes: $\{c_i : \; i = 1, \ldots, n\}$. We denote by $\mathbf{c}$ the column vector of n components: $\mathbf{c} = (c_1, \ldots, c_n)^t$.
3. The *mass flow rate* at the edges: $\{q_j : \; j = 1, \ldots, e\}$. We denote by $\mathbf{q}$ the column vector of e components: $\mathbf{q} = (q_1, \ldots, q_e)^t$.

Remark 2.1. *It is possible to impose the values of some of the above magnitudes. Thus, each of the above vectors will be divided into two parts: one corresponding to imposed values (data of the model) and another one corresponding to values that have to be computed (unknowns of the model). Moreover, the unknowns at the nodes have to be alternatively chosen as either the mass flow rate exchanged with the outside (if the pressure is imposed), or the pressure (if the mass flow rate exchanged with the outside of the network is imposed).*

In order to analyze and solve the model it is convenient to introduce the *square pressure* at the nodes: $u_i, \; i = 1, \ldots, n$. Vector $\mathbf{u}$ will denote the column vector of n components, $\mathbf{u} = (u_1, \ldots, u_n)^t = (p_1^2, \ldots, p_n^2)^t$.

2.1 Data and unknowns of the model

Firstly, let us define the dimension of the different kind of nodes and edges:

- n_p: number of nodes where the pressure is imposed,
- e_r: number of edges corresponding to compressors or pressure control valves,
- e_t: number of flow control valves,

- e_c: number of closed closing valves,
- $e_f = e - e_t - e_c - e_r$: number of edges which are neither flow control valves, nor closed closing valves, nor compressors, nor pressure control valves. We refer to these edges as *free edges* and, for the sake of exposition, they will be numbered first.

Then the *data* of the model are the following:

- $\boldsymbol{\alpha}^R$: vector of differences of square pressures between the two nodes of edges associated with compressors or pressure control valves (e_r components),
- $\mathbf{p}^U$: vector of imposed pressures (n_p components),
- $\mathbf{u}^U$: vector of imposed square pressures (n_p components),
- $\mathbf{c}^D$: vector of imposed mass flow rates exchanged with the outside of the network ($n - n_p$ components),
- $\mathbf{q}^V$: vector of imposed mass flow rates (e_v components, with $e_v = e_t + e_c$).

Consequently, the *unknowns* of the model are the following:

- $\mathbf{p}^D$: vector of pressures at nodes where pressure is not imposed ($n - n_p$ components),
- $\mathbf{u}^D$: vector of square pressures at nodes where square pressure is not imposed ($n - n_p$ components),
- $\mathbf{q}^R$: vector of mass flow rates along the edges associated with compressors or pressure control valves (e_r components),
- $\mathbf{q}^F$: vector of mass flow rates along the free edges (e_f components),
- $\mathbf{c}^U$: vector of mass flow rates exchanged with the outside of the network at nodes where pressure is imposed (n_p components),

To clarify notations, let us summarize the meaning of superscripts U, D, V, R and F:

- U identifies vectors whose components are associated with nodes where the pressures are imposed,
- D identifies vectors whose components are associated with nodes where the mass flow rates exchanged with the outside of the network are imposed,
- V identifies vectors whose components are associated with edges where the mass flow rates are imposed,
- R identifies vectors whose components are associated with edges corresponding to compressors or pressure control valves,
- F identifies vectors whose components are associated with the free edges.

Associated with the above vectors it is convenient to introduce the following matrices:

- Matrix $\mathcal{U}$, of order $n_p \times n$, extracts, from a vector of n components, the sub-vector whose components correspond to the nodes where pressure is imposed (and then the mass flow rate exchanged with the outside cannot *not* imposed)

- Matrix $\mathcal{D}$, of order $(n - n_p) \times n$, extracts, from a vector of n components, the sub-vector whose components correspond to the nodes where the mass flow rate exchanged with the outside is imposed (and then the pressure cannot *not* imposed).
- Matrix $\mathcal{V}$, of order $e_v \times e$, extracts, from a vector of e components, the sub-vector of those corresponding to the edges where the mass flow rate is imposed, namely, edges with flow control valves (FCV) or closing valves.
- Matrix $\mathcal{R}$, of order $e_r \times e$, extracts, from a vector of e components, the sub-vector of those corresponding to the edges associated with compressors or pressure control valves (PCV).
- Matrix $\mathcal{F}$, of order $e_f \times e$, extracts, from a vector of e components, the sub-vector of those associated with free edges, i.e., not corresponding to either FCVs, or closing valves, or compressors, or PCVs).

According to these notations, vectors $\mathbf{p}$, $\mathbf{u}$, $\mathbf{c}$ and $\mathbf{q}$ can be written as follows:

$$\begin{aligned}
\mathbf{p} &= \mathcal{D}^t\mathbf{p}^D + \mathcal{U}^t\mathbf{p}^U, \\
\mathbf{u} &= \mathcal{D}^t\mathbf{u}^D + \mathcal{U}^t\mathbf{u}^U, \\
\mathbf{c} &= \mathcal{U}^t\mathbf{c}^U + \mathcal{D}^t\mathbf{c}^D, \\
\mathbf{q} &= \mathcal{F}^t\mathbf{q}^F + \mathcal{R}^t\mathbf{q}^R + \mathcal{V}^t\mathbf{q}^V,
\end{aligned}$$

where the first terms on the right-hand sides represent the unknowns and the last ones the data. Related to the above decomposition of vector $\mathbf{q}$, a block splitting of the incidence matrix of the graph, $\mathcal{A}$, (see Appendix B) arises:

$$\begin{aligned}
\mathcal{A}\mathbf{q} = \mathcal{A}\mathcal{F}^t\mathbf{q}^F + \mathcal{A}\mathcal{R}^t\mathbf{q}^R + \mathcal{A}\mathcal{V}^t\mathbf{q}^V &= \mathcal{A}_F\mathbf{q}^F + \mathcal{A}_R\mathbf{q}^R + \mathcal{A}_V\mathbf{q}^V \\
&= \begin{pmatrix} \mathcal{A}_F & \mathcal{A}_R & \mathcal{A}_V \end{pmatrix} \begin{pmatrix} \mathbf{q}^F \\ \mathbf{q}^R \\ \mathbf{q}^V \end{pmatrix},
\end{aligned}$$

where $\mathcal{A}_F = \mathcal{A}\mathcal{F}^t$, $\mathcal{A}_R = \mathcal{A}\mathcal{R}^t$ and $\mathcal{A}_V = \mathcal{A}\mathcal{V}^t$. Thus, $\mathcal{A}_F \in M_{n\times e_f}$, $\mathcal{A}_R \in M_{n\times e_r}$ and $\mathcal{A}_V \in M_{n\times e_v}$ are the incidence matrices of the subgraphs including the free edges, those associated with compressors or PCVs, and those associated with FCVs or closed closing valves, respectively.

3 Mathematical model: equations

The equations of the model are mathematical expressions of mass conservation at nodes and head loss along edges.

3.1 Mass conservation

It is also known as Kirchhoff's first law of the network because of its analogy with this law for electric circuits. It establishes that, at any node, the sum of the ingoing mass flow rates must be equal to the sum of outgoing mass flow rates. Thanks to the incidence matrix of the graph representing the network, $\mathcal{A}$, it can be written in a compact way as

$$\mathcal{A}\mathbf{q} = \mathbf{c}. \tag{1}$$

An important property of matrix $\mathcal{A}$ is the following. Let $\mathbf{e}$ be the vector of $\mathbb{R}^n$ whose components are all equal to 1. From the definition of $\mathcal{A}$ it is straightforward to check that $\mathcal{A}^t\mathbf{e} = \mathbf{0}$. Then, scalar multiplication of equation (1) by $\mathbf{e}$ leads to

$$\mathbf{c} \cdot \mathbf{e} = \sum_{i=1}^{n} c_i = \mathcal{A}\mathbf{q} \cdot \mathbf{e} = \mathbf{q} \cdot \mathcal{A}^t\mathbf{e} = 0, \tag{2}$$

which is an obvious necessary condition for the existence of a solution to the network model: since the network is in steady state, the algebraic sum of the mass flow rates exchanged with the outside of the network has to be null.

The above property implies that the maximum number of independent equations in the linear system (1) is $n-1$. Thus, even if all components of vector $\mathbf{c}$ are known, in general we need other equations to uniquely compute the flows in the network. These additional equations will be written in the next section and come from the linear momentum conservation principle. Meanwhile, let us analyze the set of solutions of the mass conservation equation (1) assuming that $\mathbf{c}$ is given satisfying (2). For this purpose, let us denote by $\mathbf{q}^*$ a particular solution orthogonal to $\ker(\mathcal{A})$. Then the set of solutions is the linear manifold $\mathbf{q}^* + \ker(\mathcal{A})$. Let us take any $\mathbf{w} \in \ker(\mathcal{A})$. This means that $\mathcal{A}\mathbf{w} = \mathbf{0}$. If the only physical constraint were mass conservation, the flow corresponding to vector $\mathbf{w}$ could be considered as superfluous because it does not help to transport gas from emission to consumption points. However, superfluous flows are often needed to meet the linear momentum conservation equations to be given below. In other words, it is unlikely that the vector of mass flow rates in a real network be orthogonal to the vector space $\ker(\mathcal{A})$.

The flow vectors belonging to the kernel of $\mathcal{A}$ are called *cycling flows*. The orthogonal projection of the actual vector of mass flow rates in a network onto the space of cycling flows will be called the *superfluous flows* vector. We want to emphasize once again that the latter are often needed in order to comply with the momentum conservation principle.

For some particular calculations it can be necessary to "eliminate" the superfluous flow vector. This can be done by making the projection of the mass flow rate vector onto the orthogonal space to $\ker(\mathcal{A})$. A basis of this kernel can be obtained from the so-called *cycle matrix* which, in its turn, can be obtained by means of "graph algorithms" like the *depth-first search* (DFS) or "algebraic methods" based on the *singular-value decomposition* (SVD) of matrix $\mathcal{A}$.

Moreover, according to the vector decompositions given in Section 2, the mass conservation equation (1) can be rewritten in the form

$$\mathcal{A}_F\mathbf{q}^F + \mathcal{A}_R\mathbf{q}^R - \mathcal{U}^t\mathbf{c}^U = \mathcal{D}^t\mathbf{c}^D - \mathcal{A}_V\mathbf{q}^V. \tag{3}$$

Let us left-multiply this equality by matrix $\mathcal{D}$. We get (notice that $\mathcal{D}\mathcal{U}^t = 0$ and $\mathcal{D}\mathcal{D}^t = \mathcal{I}$),

$$\mathcal{D}\mathcal{A}_F\mathbf{q}^F + \mathcal{D}\mathcal{A}_R\mathbf{q}^R = \mathbf{g}, \tag{4}$$

with

$$\mathbf{g} := \mathbf{c}^D - \mathcal{D}\mathcal{A}_V\mathbf{q}^V. \tag{5}$$

3.2 Momentum conservation

It states that there is a pressure drop along pipelines due to the viscous stress arising from friction with their walls which can be computed with the function introduced in (A.14). This function can be rewritten as

$$G_j\left(p_{mj},\theta_{mj},q_j\right) = r_j\left(p_{mj},\theta_{mj}\right)\mu_j\left(q_j\right), \tag{6}$$

where

$$r_j\left(p_{mj},\theta_{mj}\right) := \frac{16L_j\,R}{\pi^2 D_j^5}\theta_{mj}Z\left(p_{mj},\theta_{mj}\right). \tag{7}$$

Let us recall that p_{mj} and θ_{mj} denote, respectively, average pressure and temperature along the j-th edge (see AppendixA).

Let us recall that the free edges are numbered first. We define the "diagonal" mapping $\mathbf{G}_F : \mathbb{R}^{e_f} \to \mathbb{R}^{e_f}$ by

$$\mathbf{G}_F\left(\mathbf{p}_m,\boldsymbol{\theta}_m,\mathbf{q}^F\right)_j = G_j\left(p_{mj},\theta_{mj},q_j^F\right),\ j = 1,\cdots,e_f$$

and the vector $\mathbf{b}^F \in \mathbb{R}^{e_f}$ by

$$b_j^F = \frac{2g}{R\theta_{mj}}\frac{u_{mj}}{Z\left(p_{mj},\theta_{mj}\right)}\left(H_{\mathcal{M}_{2,j}} - H_{\mathcal{M}_{1,j}}\right), \tag{8}$$

where H_i denotes the height of the i-th node, u_{mj} is the average value of u along the j-th edge given by

$$u_{mj} = \frac{u_{\mathcal{M}_{1,j}} + u_{\mathcal{M}_{2,j}}}{2},$$

and $\mathcal{M}_{1,j}$ and $\mathcal{M}_{2,j}$ are the two nodes of the j-th edge. We have,

$$\mathcal{A}_F^t\mathbf{u} - \mathbf{G}_F\left(\mathbf{p}_m,\boldsymbol{\theta}_m,\mathbf{q}^F\right) = \mathbf{b}^F(\mathbf{u}), \tag{9}$$

$$\mathcal{A}_R^t\mathbf{u} = \boldsymbol{\alpha}^R, \tag{10}$$

and, since $\mathbf{u} = \mathcal{U}^t\mathbf{u}^U + \mathcal{D}^t\mathbf{u}^D$, the first equation can also be written as

$$\mathcal{A}_F^t\mathcal{D}^t\mathbf{u}^D - \mathbf{G}_F\left(\mathbf{p}_m,\boldsymbol{\theta}_m,\mathbf{q}^F\right) = \mathbf{f}, \tag{11}$$

with

$$\mathbf{f} := \mathbf{b}^F(\mathbf{u}) - \mathcal{A}_F^t \mathcal{U}^t \mathbf{u}^U \tag{12}$$

and the second one as

$$\mathcal{A}_R^t \mathcal{D}^t \mathbf{u}^D = \mathbf{k}, \tag{13}$$

with

$$\mathbf{k} := \boldsymbol{\alpha}^R - \mathcal{A}_R^t \mathcal{U}^t \mathbf{u}^U. \tag{14}$$

Let us summarize the model of the gas transportation network:

Given,

- $\mathbf{u}^U$*: the vector of imposed square pressures,*
- $\mathbf{q}^V$*: the vector of mass flow rates at the edges with flow control valves,*
- $\boldsymbol{\alpha}^R$*: the vector of differences of square pressures between the two nodes of edges associated with compressors or pressure control valves,*
- $\mathbf{c}^D$*: the vector of the mass flow rates exchanged with the outside of the network, at nodes where pressure is not imposed,*

find vectors $\mathbf{u}^D$, $\mathbf{q}^F$, $\mathbf{q}^R$ *and* $\mathbf{c}^U$ *such that*

$$\mathcal{D}\mathcal{A}_F \mathbf{q}^F + \mathcal{D}\mathcal{A}_R \mathbf{q}^R = \mathbf{g}, \tag{15}$$

$$\mathcal{A}_F^t \mathcal{D}^t \mathbf{u}^D - \mathbf{G}_F\left(\mathbf{q}^F\right) = \mathbf{f}, \tag{16}$$

$$\mathcal{A}_R^t \mathcal{D}^t \mathbf{u}^D = \mathbf{k}, \tag{17}$$

$$\mathcal{A}_F \mathbf{q}^F + \mathcal{A}_R \mathbf{q}^R - \mathcal{U}^t \mathbf{c}^U = \mathcal{D}^t \mathbf{c}^D - \mathcal{A}_V \mathbf{q}^V, \tag{18}$$

with $\mathbf{g}$*,* $\mathbf{f}$ *and* $\mathbf{k}$ *given by (5), (12) and (14), respectively.*

Remark 3.1. *Notice that the unknowns of the model are* $\mathbf{u}^D$ *(*$n - n_p$ *numbers),* $\mathbf{q}^F$ *(*e_f *numbers),* $\mathbf{q}^R$ *(*e_r *numbers), and* $\mathbf{c}^U$ *(*n_p *numbers) so that the total number of unknowns is* $n - n_p + e_f + e_r + n_p = n + e_f + e_r$*, which is equal to the number of equations:* $n - n_p + e_f + e_r + n_p = n + e_f + e_r$*.*

Remark 3.2. *Let us notice that if we can solve equations (15), (16) and (17) for* $\mathbf{u}^D$*,* $\mathbf{q}^F$ *and* $\mathbf{q}^R$*, then (18) allows us to compute* $\mathbf{c}^U$ *by*

$$\mathbf{c}^U = \mathcal{U}\mathcal{A}_F \mathbf{q}^F + \mathcal{U}\mathcal{A}_R \mathbf{q}^R + \mathcal{U}\mathcal{A}_V \mathbf{q}^V,$$

in a second step. This is because $\mathcal{U}\mathcal{U}^t = \mathcal{I}$ *and* $\mathcal{U}\mathcal{D}^t = 0$*.*

Let us notice that unknown vector $\mathbf{u}^D$ appears in the expression of the right-hand side $\mathbf{f}$ of equation (16), namely, in vector $\mathbf{b}^F(\mathbf{u})$. Thus, it is important to rewrite this equation by putting this term on the left-hand side. For this purpose, let us introduce the following notation:

$$w_j = \frac{g}{R\theta_{mj} Z(p_m, \theta_{mj})} \left(H_{\mathcal{M}_{2j}} - H_{\mathcal{M}_{1j}} \right), \tag{19}$$

for $1 \leq j \leq e_f$, and $\mathcal{W}$ denotes the $e_f \times e_f$ diagonal matrix

$$\mathcal{W}_{lj} = w_j \delta_{lj}, \quad 1 \leq l, j \leq e_f.$$

Let the $n \times e$ matrix Λ be defined, for $1 \leq i \leq n, \ 1 \leq j \leq e$, by

$$\Lambda_{ij} = \delta_{i\,\mathcal{M}_{1j}} + \delta_{i\,\mathcal{M}_{2j}}$$

where δ is the Kronecker's delta. Then vector $\mathbf{b}^F(\mathbf{u})$ can be written as

$$\mathbf{b}^F(\mathbf{u}) = \mathcal{W}\mathcal{F}\Lambda^t \mathbf{u} = \mathcal{W}\mathcal{F}\Lambda^t \big(\mathcal{D}^t \mathbf{u}^D + \mathcal{U}^t \mathbf{u}^U\big)$$

and (16) becomes

$$\big(\mathcal{A}_F^t - \mathcal{W}\mathcal{F}\Lambda^t\big)\mathcal{D}^t \mathbf{u}^D - \mathbf{G}_F\left(\mathbf{q}^F\right) = \mathbf{h}, \tag{20}$$

with

$$\mathbf{h} := \big(\mathcal{W}\mathcal{F}\Lambda^t - \mathcal{A}_F^t\big)\mathcal{U}^t \mathbf{u}^U.$$

Now, from (20) we can obtain $\mathbf{q}^F$ as

$$\mathbf{q}^F = \mathbf{G}_F^{-1}\Big(\big(\mathcal{A}_F^t - \mathcal{W}\mathcal{F}\Lambda^t\big)\mathcal{D}^t \mathbf{u}^D - \mathbf{h}\Big),$$

and replacing this expression in (15) it can be written in terms of $\mathbf{u}^D$ and $\mathbf{q}^R$, namely,

$$\mathcal{D}\mathcal{A}_F \mathbf{G}_F^{-1}\Big(\big(\mathcal{A}_F^t - \mathcal{W}\mathcal{F}\Lambda^t\big)\mathcal{D}^t \mathbf{u}^D - \mathbf{h}\Big) + \mathcal{D}\mathcal{A}_R \mathbf{q}^R = \mathbf{g}. \tag{21}$$

In order to prove the existence of a solution to (17) and (21) it is convenient to subtract the term

$$\mathcal{D}\Lambda\mathcal{F}^t\mathcal{W}\mathbf{G}_F^{-1}\Big(\big(\mathcal{A}_F^t - \mathcal{W}\mathcal{F}\Lambda^t\big)\mathcal{D}^t \mathbf{u}^D - \mathbf{h}\Big),$$

to both sides of (21). We get

$$\begin{aligned}\mathcal{D}\big(\mathcal{A}_F - \Lambda\mathcal{F}^t\mathcal{W}\big)\mathbf{G}_F^{-1}\Big(\big(\mathcal{A}_F^t - \mathcal{W}\mathcal{F}\Lambda^t\big)\mathcal{D}^t \mathbf{u}^D - \mathbf{h}\Big) + \mathcal{D}\mathcal{A}_R \mathbf{q}^R \\ = \mathbf{g} - \mathcal{D}\Lambda\mathcal{F}^t\mathcal{W}\mathbf{G}_F^{-1}\Big(\big(\mathcal{A}_F^t - \mathcal{W}\mathcal{F}\Lambda^t\big)\mathcal{D}^t \mathbf{u}^D - \mathbf{h}\Big).\end{aligned} \tag{22}$$

Finally, by introducing the $(n - n_p) \times e_f$ matrix

$$\mathcal{B} := \mathcal{D}\big(\mathcal{A}_F - \Lambda\mathcal{F}^t\mathcal{W}\big),$$

this equation can be rewritten as

$$\mathcal{B}\mathbf{G}_F^{-1}\Big(\mathcal{B}^t \mathbf{u}^D - \mathbf{h}\Big) + \mathcal{D}\mathcal{A}_R \mathbf{q}^R = \mathbf{g} - \mathcal{D}\Lambda\mathcal{F}^t\mathcal{W}\mathbf{G}_F^{-1}\Big(\mathcal{B}^t \mathbf{u}^D - \mathbf{h}\Big). \tag{23}$$

Remark 3.3. *As mentioned before, matrix $\mathcal{W}$ depends on the solution through the compressibility factor Z because this parameter is a function of pressure which, in its turn, is the square root of u. The same is true for mapping $\mathbf{G}_F$.*

4 Existence of solution

In order to simplify the analysis, the existence theorem below will be proved for an approximate model. It is obtained by freezing, at each edge, the friction factor λ present in $\mu(q) = \lambda(q)|q|q$ (see (6)) and the compressibility factor Z (see AppendixA) to constant values. These assumptions have also been done in [13] where, in addition, the topography is assumed to be flat, i.e., the term (8) involving the difference of heights at the nodes is not considered. We want to emphasize that the presence of this term prevents us from using the same method as the one employed in that paper, in order to prove the existence of a solution to the gas network model. In particular, the system of equations (7) in [13] is no longer true in our case.

The above simplifying assumptions mean that the pressure loss function $G_j(p_{mj}, \theta_{mj}, q_j)$ is replaced by

$$\tilde{G}_j\,(q) := \tilde{r}_j \tilde{\lambda}_j |q| q, \tag{24}$$

with

$$\tilde{r}_j := \frac{16 L_j\, R}{\pi^2 D_j^5} \theta_{mj} \tilde{Z}_j,$$

where $\tilde{\lambda}_j$ and $\tilde{Z}_j,\ j = 1, \cdots, e_f$ are constant values given for each edge of the network, and that numbers w_j are approximated by

$$\tilde{w}_j = \frac{g}{R\theta_{mj}\tilde{Z}_j} \left(H_{m(2,j)} - H_{m(1,j)} \right),$$

for $1 \leq j \leq e_f$. Thus, we replace the diagonal matrix $\mathcal{W}$ by $\tilde{\mathcal{W}}$ with $\tilde{\mathcal{W}}_{ij} := \tilde{w}_j \delta_{ij}$, $1 \leq i, j \leq e$. For the sake of simplicity in notation it is convenient to introduce matrix $\tilde{\mathcal{B}}$ by

$$\tilde{\mathcal{B}} := \mathcal{D}\big(\mathcal{A}_F - \Lambda \mathcal{F}^t \tilde{\mathcal{W}}\big).$$

The goal of this section is to prove the existence of a solution to the problem

$$\tilde{\mathcal{B}} \tilde{\mathbf{G}}_F^{-1} \Big(\tilde{\mathcal{B}}^t \mathbf{u}^D - \mathbf{h} \Big) + \mathcal{D}\mathcal{A}_R \mathbf{q}^R = \mathbf{g} - \mathcal{D}\Lambda\mathcal{F}^t \tilde{\mathcal{W}} \tilde{\mathbf{G}}_F^{-1} \Big(\tilde{\mathcal{B}}^t \mathbf{u}^D - \mathbf{h} \Big), \tag{25}$$

$$\mathcal{A}_R^t \mathcal{D}^t \mathbf{u}^D = \mathbf{k} \tag{26}$$

where

$$\big(\tilde{\mathbf{G}}_F(\mathbf{q}^F) \big)_j = \tilde{G}_j(q_j).$$

It will be achieved in two steps:

1. In the first step, we freeze the value of $\mathbf{u}^D$ on the right-hand side of (25) to a given vector $\mathbf{u}^{D*}$ and consider the one on the left-hand side as unknown, i.e., we look for $\mathbf{u}^D$ satisfying the system of equations

$$\tilde{\mathcal{B}} \tilde{\mathbf{G}}_F^{-1} \Big(\tilde{\mathcal{B}}^t \mathbf{u}^D - \mathbf{h} \Big) + \mathcal{D}\mathcal{A}_R \mathbf{q}^R = \mathbf{g} - \mathcal{D}\Lambda\mathcal{F}^t \tilde{\mathcal{W}} \mathbf{q}^{F*}, \tag{27}$$

$$\mathcal{A}_R^t \mathcal{D}^t \mathbf{u}^D = \mathbf{k}, \tag{28}$$

where

$$\mathbf{q}^{F*} := \tilde{\mathbf{G}}_F^{-1}\Big(\tilde{\mathcal{B}}^t\mathbf{u}^{D*} - \mathbf{h}\Big).$$

Then we prove the existence of a unique solution to this problem.

2. In the second step, we prove the existence of a solution to (25) by showing that the mapping

$$\mathbf{q}^{F*} \longrightarrow \tilde{\mathbf{F}}(\mathbf{q}^{F*}) := \tilde{\mathbf{G}}_F^{-1}\Big(\tilde{\mathcal{B}}^t\mathbf{u}^{D} - \mathbf{h}\Big),$$

being $\mathbf{u}^D$ the solution of the system given by (27), (28), has a fixed point.

4.1 First step: existence and uniqueness of solution to the system given by (27) and (28)

Let $\tilde{\chi}_j : \mathbb{R} \to \mathbb{R}$ be the primitive function of $\tilde{G}_j$ satisfying $\tilde{\chi}_j(0) = 0$. Since $\tilde{G}_j$ is strictly monotone, then $\tilde{\chi}_j$ is strictly convex. In fact, it is easy to see that

$$\tilde{\chi}_j(q) = \frac{1}{3}\tilde{r}_j\tilde{\lambda}_j|q|^3. \tag{29}$$

Moreover, from convex analysis it is well known that $\tilde{G}_j^{-1}$ is the derivative of the conjugate function of $\tilde{\chi}_j$, to be denoted by $\tilde{\chi}_j^*$, which is defined by (see, for instance [14]),

$$\tilde{\chi}_j^*(y) = \sup_{q\in\mathbb{R}}\left(yq - \tilde{\chi}_j(q)\right).$$

That is,

$$\tilde{\chi}_j^{*\,\prime}(y) = \tilde{G}_j^{-1}(y). \tag{30}$$

From the definition of $\tilde{\chi}_j^*$ and (29) we easily deduce that

$$\tilde{\chi}_j^*(y) = \frac{2}{3}\frac{|y|^{3/2}}{(\tilde{r}_j\tilde{\lambda}_j)^{1/2}},$$

so the proof of the next result is straightforward.

Lemma 4.1. *The following coerciveness property for function $\tilde{\chi}_j^*$ holds:*

$$\lim_{|y|\to\infty}\frac{\tilde{\chi}_j^*(y)}{|y|} = \infty. \tag{31}$$

Let $\tilde{\chi} : \mathbb{R}^{e_f} \to \mathbb{R}$ be defined by

$$\tilde{\chi}\left(\mathbf{q}^F\right) = \sum_{j=1}^{e_f}\tilde{\chi}_j\left(q_j^F\right).$$

Then the conjugate function of $\tilde{\chi}$ is given by

$$\tilde{\chi}^* (\mathbf{y}) = \sum_{j=1}^{e_f} \tilde{\chi}_j^* (y_j) \, .$$

We will prove that (27) and (28) are the necessary and sufficient optimality conditions for a particular linearly constrained convex minimization problem. For this purpose, let us introduce $\phi : \mathbb{R}^{n-n_p} \to \mathbb{R}$ to be the function defined by

$$\phi \left(\mathbf{u}^D\right) := \tilde{\chi}^* \left(\tilde{\mathcal{B}}^t \mathbf{u}^D - \mathbf{h}\right) - \mathbf{g}^* \cdot \mathbf{u}^D, \tag{32}$$

where

$$\mathbf{g}^* := \mathbf{g} - \mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}\tilde{\mathbf{G}}_F^{-1}\big(\tilde{\mathcal{B}}^t\mathbf{u}^{D*} - \mathbf{h}\big).$$

By using the chain rule it is easy to see that

$$\mathbf{grad}\phi \left(\mathbf{u^D}\right) = \tilde{\mathcal{B}}\tilde{\mathbf{G}}_F^{-1}\left(\tilde{\mathcal{B}}^t\mathbf{u}^D - \mathbf{h}\right) - \mathbf{g}^*,$$

We consider the constrained optimization problem

$$\min\{\phi(\mathbf{u}^D) : \, \mathbf{u}^D \in \mathbb{R}^{n-n_p}, \; \mathcal{A}_R^t\mathcal{D}^t\mathbf{u}^D = \mathbf{k}\}. \tag{33}$$

We notice that (27) and (28) are the Karush-Kuhn-Tucker necessary (and actually sufficient) optimality conditions of the above minimization problem. In particular, vector $\mathbf{q}^R$ is the vector Lagrange multiplier associated with equality constraint (28).

The following lemmas will be used to prove the existence and uniqueness of solution to (33).

Lemma 4.2. *Let us assume $n_p < n$ and that there is a node in each connected component of the free-edges graph where the pressure is imposed. Then the linear mapping $\mathcal{A}_F^t\mathcal{D}^t$ is injective. Hence it has a left-inverse and consequently there exists a positive constant C such that*

$$\|\mathbf{w}^D\| \leq C\|\mathcal{A}_F^t\mathcal{D}^t\mathbf{w}^D\| \quad \forall\mathbf{u}^D \in \mathbb{R}^{n-n_p}. \tag{34}$$

Proof: Firstly, let us notice that $\mathcal{A}_F^t\mathcal{D}^t$ is an $e_f \times (n - n_p)$ matrix. From graph theory (see Appendix B), $m := \dim\ker(\mathcal{A}_F^t)$ is equal to the number of connected components of the free-edges graph. Let $\{\mathbf{a}^1, \ldots, \mathbf{a}^m\} \subset \mathbb{R}^n$ be the basis of $\ker(\mathcal{A}_F^t)$ where $\mathbf{a}^i$ is the vector whose components corresponding to nodes in the i-th connected component are ones and the rest of them are null. We notice that vectors $\mathbf{a}^i, \; i = 1, \cdots, m$, cannot belong to $\mathrm{im}(\mathcal{D}^t)$ which is the vector space spanned by the rows of $\mathcal{D}$. Indeed, in each row of $\mathcal{D}$ there is exactly a one and zeros. Moreover, by assumption, the pressure is imposed in at least one of the nodes of the i-th connected component of the free-edges graph and hence there is not any row in $\mathcal{D}$ having the one in the column corresponding to this node. Therefore, for $i = 1, \cdots, m$, $\mathbf{a}_i$ cannot be written as a linear combination of the rows of $\mathcal{D}$ which implies that

$$\ker\left(\mathcal{A}_F^t\right) \cap \operatorname{im}\left(\mathcal{D}^t\right) = \{\mathbf{0}\}.$$

Hence $\mathcal{A}_F^t \mathcal{D}^t$ is injective and has a left-inverse given by $(\mathcal{D}\mathcal{A}_F\mathcal{A}_F^t\mathcal{D}^t)^{-1}\mathcal{D}\mathcal{A}_F$. Taking norms in the equality

$$\mathbf{w}^D = (\mathcal{D}\mathcal{A}_F\mathcal{A}_F^t\mathcal{D}^t)^{-1}\mathcal{D}\mathcal{A}_F\mathcal{A}_F^t\mathcal{D}^t\mathbf{w}^D \qquad \forall \mathbf{w}^D \in \mathbb{R}^{n-n_p},$$

we easily get (34).

Remark 4.1. *Under the assumption in the previous lemma, since $\mathcal{A}_F^t\mathcal{D}^t$ is injective then $n - n_p \leq e_f$.*

Remark 4.2. *It is not difficult to prove that, in absence of topography terms given in (8), the above result and the properties of the pressure-loss function allow us to prove the existence and uniqueness of a solution to the gas network model by reformulating it as a linearly constrained minimization problem for a strictly convex and coercive function. However, if topography is non-flat this issue is more difficult as we will see below.*

In order to prove the existence theorem below we need a similar property as (34) but for matrix $\tilde{\mathcal{B}}^t$ instead of $\mathcal{A}_F^t\mathcal{D}^t$. This property will be taken as an assumption.

Lemma 4.3. *Under the assumption,*

$$\tilde{\mathcal{B}}^t = (\mathcal{A}_F^t - \tilde{\mathcal{W}}\mathcal{F}\Lambda^t)\mathcal{D}^t \text{ is injective,} \tag{35}$$

the following properties are satisfied by function ϕ:

1. *ϕ is differentiable,*
2. *ϕ is strictly convex,*
3. $\lim_{\|\mathbf{w}^D\|\to\infty} \phi\left(\mathbf{w}^D\right) = \infty$.

Proof:

1. It is a consequence of the fact that $\tilde{\chi}^*$ is differentiable at any $x \in \mathbb{R}$.
2. Let $\mathbf{u}^D, \mathbf{w}^D \in \mathbb{R}^{n-n_p}$, $\mathbf{u}^D \neq \mathbf{w}^D$. Then $\tilde{\mathcal{B}}^t\mathbf{u}^D \neq \tilde{\mathcal{B}}^t\mathbf{w}^D$ because $\tilde{\mathcal{B}}^t$ is injective. Since $\tilde{\chi}^*$ is strictly convex, for $\lambda \in (0,1)$ we have

$$\begin{aligned}
& \phi\left(\lambda\mathbf{u}^D + (1-\lambda)\,\mathbf{w}^D\right) \\
= \; & \tilde{\chi}^*\left(\lambda\tilde{\mathcal{B}}^t\mathbf{u}^D + (1-\lambda)\,\tilde{\mathcal{B}}^t\mathbf{w}^D\right) - \mathbf{g}^*\cdot\left(\lambda\mathbf{u}^D + (1-\lambda)\,\mathbf{w}^D\right) \\
< \; & \lambda\left[\tilde{\chi}^*\left(\tilde{\mathcal{B}}^t\mathbf{u}^D\right) - \mathbf{g}^*\cdot\mathbf{u}^D\right] + (1-\lambda)\left[\tilde{\chi}^*\left(\tilde{\mathcal{B}}^t\mathbf{w}^D\right) - \mathbf{g}^*\cdot\mathbf{w}^D\right] \\
= \; & \lambda\phi\left(\mathbf{u}^D\right) + (1-\lambda)\,\phi\left(\mathbf{w}^D\right).
\end{aligned}$$

3. Firstly, from Lemma 4.1 we deduce

$$\lim_{\|x\|\to\infty} \frac{\tilde{\chi}^*(x)}{\|x\|} = \infty\,. \tag{36}$$

Moreover,

$$\begin{aligned} \phi\left(\mathbf{u}^D\right) &= \tilde{\chi}^*\left(\tilde{\mathcal{B}}^t\mathbf{u}^D\right) - \mathbf{g}^*\cdot\mathbf{u}^D \\ \geq \quad \tilde{\chi}^*\left(\tilde{\mathcal{B}}^t\mathbf{u}^D\right) - \|\mathbf{g}^*\|\|\mathbf{u}^D\| &= \left(\frac{\tilde{\chi}^*\left(\tilde{\mathcal{B}}^t\mathbf{u}^D\right)}{\|\mathbf{u}^D\|} - \|\mathbf{g}^*\|\right)\|\mathbf{u}^D\| \\ \geq \left(\frac{\tilde{\chi}^*\left(\tilde{\mathcal{B}}^t\mathbf{u}^D\right)}{C\|\tilde{\mathcal{B}}^t\mathbf{u}^D\|} - \|\mathbf{g}^*\|\right)\|\mathbf{u}^D\|. \end{aligned} \tag{37}$$

From assumption (35) we get

$$\lim_{\|\mathbf{u}^D\|\to\infty} \|\tilde{\mathcal{B}}^t\mathbf{u}^D\| = \infty\,,$$

and then, (36) yields

$$\lim_{\|\mathbf{u}^D\|\to\infty} \frac{\tilde{\chi}^*\left(\tilde{\mathcal{B}}^t\mathbf{u}^D\right)}{\|\tilde{\mathcal{B}}^t\mathbf{u}^D\|} = \infty\,.$$

This result and (37) allow us to conclude the proof.

The next result shows that (35) is indeed a crucial assumption to ensure that the minimization problem (33) has a unique solution. We only need to complement (35) with any condition that guarantees that the set of constraints given by equation (28) has some solution.

Theorem 4.1. *Let vectors* $\mathbf{H} \in \mathbb{R}^n$, $\mathbf{u}^U \in \mathbb{R}^{n_p}$, $\mathbf{q}^V \in \mathbb{R}^{e_v}$, $\mathbf{c}^D \in \mathbb{R}^{n-n_p}$ *and* $\boldsymbol{\alpha}^R \in \mathbb{R}^{e_r}$ *be given. Under assumption (35) and one of the following:*

- *There are neither compressors nor PCVs in the network,*
- *The set* $\{\mathbf{u}^D \in \mathbb{R}^{n-n_p} : \mathcal{A}_R^t\mathcal{D}^t\mathbf{u}^D = \boldsymbol{\alpha}^R - \mathcal{A}_R^t\mathcal{U}^t\mathbf{u}^U\}$ *is nonempty,*

the minimization problem (33) has a unique solution $\mathbf{u}^D$.

Proof of the Theorem: It is a straightforward consequence of Lemma 4.3 and standard results in convex optimization theory (see for instance [15]).

Corollary 4.1. *Under the assumptions of Theorem 4.1 there exists a solution to the system given by (27) and (28). Moreover, if* $\mathbf{u}_i^D$, $\mathbf{q}_i^F$, $i = 1, 2$ *are two solutions, then*

$$\mathbf{u}_1^D = \mathbf{u}_2^D \quad \textit{and} \; \mathbf{q}_1^R - \mathbf{q}_2^R \in \ker(\mathcal{D}\mathcal{A}_R).$$

Proof: It can be immediately deduced from [14, Cor. 28.2.2] that allows us to write the Karush-Kuhn-Tucker optimality conditions of the above minimization problem, namely, (27) and (28).

Remark 4.3. *Notice that the Lagrange multiplier* $\mathbf{q}^R$ *is the vector of mass flow rates along edges corresponding to compressors or pressure control valves. If* $\ker(\mathcal{D}\mathcal{A}_R) \neq \{\mathbf{0}\}$*, there must exist cycles such that all their edges are either compressors or pressure control valves. Since this is quite unusual, in most practical cases all the mass flow rates are also unique.*

Now, from Theorem 4.1 we can easily deduce the existence result proved in [13]:

Corollary 4.2. *Let us assume that the network topography is flat and there are neither compressors nor valves. Then* $\tilde{\mathcal{B}} = \mathcal{D}\mathcal{A}_F$ *and hence, under the assumption of Lemma 4.2, there exists a unique solution to the gas network model.*

4.2 Second step: fixed-point method

The following technical lemmas are important tools for the proofs below.

Lemma 4.4. *We have*

$$(|x|x - |y|y)(x - y) \geq \frac{1}{2}|x - y|^3 \quad \forall x, y \in \mathbb{R} \tag{38}$$

Proof: Firstly, let us notice that if $y = 0$ then (38) is trivially true. Otherwise, since the expressions in the above inequality are homogeneous of degree three, then dividing (38) by $|y|^3$ we deduce that it is equivalent to

$$(|\frac{x}{y}|\frac{x}{y} - 1)(\frac{x}{y} - 1) \geq \frac{1}{2}|\frac{x}{y} - 1|^3 \quad \forall x, y \in \mathbb{R}$$

and also to

$$(|z|z - 1)(z - 1) \geq \frac{1}{2}|z - 1|^3 \quad \forall z \in \mathbb{R}.$$

Since this inequality trivially holds for $z = 1$ it is enough to prove that

$$\inf_{1 \neq z \in \mathbb{R}} \frac{(|z|z - 1)(z - 1)}{|z - 1|^3} \geq \frac{1}{2},$$

which is straightforward. Let us remark that the lower bound cannot be improved. Indeed, for $z = -1$ we have

$$\frac{(|z|z - 1)(z - 1)}{|z - 1|^3} = \frac{(|-1|(-1) - 1)(-1 - 1)}{|-1 - 1|^3} = \frac{1}{2}.$$

Lemma 4.5. *We have*

$$\Big| |x|x - |y|y \Big| \leq |x-y|(|x|+|y|) \quad \forall x, y \in \mathbb{R}. \tag{39}$$

Proof: Similarly, if $y = 0$ (39) is trivially true. Otherwise, since the expressions on the two members are homogeneous of degree two, then dividing by y^2 we deduce that (39) holds if and only if

$$(|z|z-1)(z-1) \leq |z-1|(|z|+1) \quad \forall z \in \mathbb{R}.$$

Since this inequality is trivially true for $z = 1$, it is enough to prove the following one,

$$\sup_{1 \neq z \in \mathbb{R}} \frac{(|z|z-1)(z-1)}{|z-1|(|z|+1)} \leq 1,$$

which easily holds true. Moreover, we notice that the upper bound cannot be improved because the value of the left-hand side is exactly 1 for $z = 0$.

Corollary 4.3. *Function $x \in \mathbb{R} \to |x|x \in \mathbb{R}$ is Lipschitz-continuous on bounded sets.*

Lemma 4.6. *We have*

$$(\tilde{G}_j(x) - \tilde{G}_j(y))(x-y) \geq \frac{1}{2}\tilde{r}_j\tilde{\lambda}_j|x-y|^3 \quad \forall x, y \in \mathbb{R} \tag{40}$$

Proof: It follows from (24) and estimate (38).

Lemma 4.7. *We have*

$$|\tilde{G}_j(x) - \tilde{G}_j(y)| \leq \tilde{r}_j\tilde{\lambda}_j|x-y|(|x|+|y|) \quad \forall x, y \in \mathbb{R}. \tag{41}$$

Proof:It follows from (24) and estimate (39).

Let us introduce the weighted norm

$$\|\mathbf{q}\| = \Big(\sum_{j=1}^{e_f} \tilde{r}_j\tilde{\lambda}_j|q_j|^3\Big)^{1/3}.$$

Its dual norm is

$$\|\mathbf{q}\|' = \Big(\sum_{j=1}^{e_f} (\tilde{r}_j\tilde{\lambda}_j)^{-1/2}|q_j|^{3/2}\Big)^{2/3}.$$

Indeed, let us prove that

$$|\mathbf{a}\cdot\mathbf{b}| \leq \min\{\|\mathbf{a}\|\|\mathbf{b}\|',\ \|\mathbf{a}\|'\|\mathbf{b}\|\} \quad \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^{e_f}.$$

Since 3 and 3/2 are conjugate numbers, by using Hölder's inequality we get

$$|\mathbf{a}\cdot\mathbf{b}| = \sum_{j=1}^{e_f} a_j b_j = \sum_{j=1}^{e_f} (\tilde{r}_j\tilde{\lambda}_j)^{1/3} a_j\, (\tilde{r}_j\tilde{\lambda}_j)^{-1/3} b_j$$

$$\leq \Big(\sum_{j=1}^{e_f} \tilde{r}_j\tilde{\lambda}_j a_j^3\Big)^{1/3} \Big(\sum_{j=1}^{e_f} (\tilde{r}_j\tilde{\lambda}_j)^{-1/2} b_j^{3/2}\Big)^{2/3} \leq \|\mathbf{a}\|\|\mathbf{b}\|'.$$

Lemma 4.8. *We have*

$$\Big(\tilde{\mathbf{G}}_F\left(\mathbf{q}_1^F\right) - \tilde{\mathbf{G}}_F\left(\mathbf{q}_2^F\right)\Big)\cdot\left(\mathbf{q}_1^F - \mathbf{q}_2^F\right) \geq \frac{1}{2}\|\mathbf{q}_1^F - \mathbf{q}_2^F\|^3 \;\forall \mathbf{q}_i^F \in \mathbb{R}^{e_f},\; i = 1,2. \tag{42}$$

Proof: From (40), we get

$$\Big(\tilde{\mathbf{G}}_F\left(\mathbf{q}_1^F\right) - \tilde{\mathbf{G}}_F\left(\mathbf{q}_2^F\right)\Big)\cdot\left(\mathbf{q}_1^F - \mathbf{q}_2^F\right) = \sum_{j=1}^{e_f}\Big(\tilde{G}_j\left(q_{1j}\right) - \tilde{G}_j\left(q_{2j}\right)\Big)\cdot\left(q_{1j} - q_{2j}\right)$$

$$\geq \frac{1}{2}\sum_{j=1}^{e_f}\tilde{r}_j\tilde{\lambda}_j|q_{1j} - q_{2j}|^3 = \frac{1}{2}\|\mathbf{q}_1^F - \mathbf{q}_2^F\|^3$$

Corollary 4.4. *We have*

$$\Big((\tilde{\mathbf{G}}_F)^{-1}\left(\mathbf{z}_1^F\right) - (\tilde{\mathbf{G}}_F)^{-1}\left(\mathbf{z}_2^F\right)\Big)\cdot\left(\mathbf{z}_1^F - \mathbf{z}_2^F\right)$$
$$\geq \frac{1}{2}\|(\tilde{\mathbf{G}}_F)^{-1}\left(\mathbf{z}_1^F\right) - (\tilde{\mathbf{G}}_F)^{-1}\left(\mathbf{z}_2^F\right)\|^3 \quad \forall \mathbf{z}_i^F \in \mathbb{R}^{e_f},\; i = 1,2. \tag{43}$$

Lemma 4.9. *We have*

$$\|\tilde{\mathbf{G}}_F\left(\mathbf{q}_1^F\right) - \tilde{\mathbf{G}}_F\left(\mathbf{q}_2^F\right)\|' \leq \|\mathbf{q}_1^F - \mathbf{q}_2^F\|\left(\|\mathbf{q}_1^F\| + \|\mathbf{q}_2^F\|\right)$$
$$\forall \mathbf{q}_i^F \in \mathbb{R}^{e_f},\; i = 1,2. \tag{44}$$

Proof. From (41) we get, by using (41) and Cauchy-Schwartz inequality,

$$\|\tilde{\mathbf{G}}_F\left(\mathbf{q}_1^F\right) - \tilde{\mathbf{G}}_F\left(\mathbf{q}_2^F\right)\|'^{3/2} = \sum_{j=1}^{e_f}(\tilde{r}_j\tilde{\lambda}_j)^{-1/2}\big(\tilde{G}_j(q_{1j}) - \tilde{G}_j(q_{2j})\big)^{3/2}$$

$$\leq \sum_{j=1}^{e_f}\tilde{r}_j\tilde{\lambda}_j\big(q_{1j} - q_{2j}\big)^{3/2}\big(|q_{1j}| + |q_{2j}|\big)^{3/2}$$

$$\leq \Big(\sum_{j=1}^{e_f}\big((\tilde{r}_j\tilde{\lambda}_j)^{1/2}\big(q_{1j} - q_{2j}\big)^{3/2}\big)^2\Big)^{1/2}\Big(\sum_{j=1}^{e_f}\big((\tilde{r}_j\tilde{\lambda}_j)^{1/2}\big(|q_{1j}| + |q_{2j}|\big)^{3/2}\big)^2\Big)^{1/2}$$

from which it follows that

$$\|\tilde{\mathbf{G}}_F\left(\mathbf{q}_1^F\right)-\tilde{\mathbf{G}}_F\left(\mathbf{q}_2^F\right)\|' \leq \Big(\sum_{j=1}^{e_f}(\tilde{r}_j\tilde{\lambda}_j)\big(q_{1j}-q_{2j}\big)^3\Big)^{1/3} \times \cdots$$

$$\Big(\sum_{j=1}^{e_f}\big((\tilde{r}_j\tilde{\lambda}_j)\big(|q_{1j}|+|q_{2j}|\big)^3\big)\Big)^{1/3} \leq \|\mathbf{q}_1^F-\mathbf{q}_2^F\|\left(\|\mathbf{q}_1^F\|+\|\mathbf{q}_2^F\|\right).$$

Lemma 4.10. *There exists a positive constant C such that*

$$\|\tilde{\mathbf{F}}(\mathbf{q}_1^{F*})-\tilde{\mathbf{F}}(\mathbf{q}_2^{F*})\|^2 \leq C\|\mathbf{q}_1^{F*}-\mathbf{q}^{F*}\|\big(\|\tilde{\mathbf{F}}(\mathbf{q}_1^{F*})\|+\|\tilde{\mathbf{F}}(\mathbf{q}_2^{F*})\|\big). \tag{45}$$

Proof: Let $\mathbf{q}_i^F=\tilde{\mathbf{F}}(\mathbf{q}_i^{F*}),\ i=1,2$. Then, $\mathbf{q}_i^F=\tilde{\mathbf{G}}_F^{-1}\Big(\tilde{\mathcal{B}}^t\mathbf{u}_i^D-\mathbf{h}\Big)$, with

$$\tilde{\mathcal{B}}(\tilde{\mathbf{G}}_F)^{-1}(\tilde{\mathcal{B}}^t\mathbf{u}_i^D-\mathbf{h})+\mathcal{D}\mathcal{A}_R\mathbf{q}_i^R=\mathbf{g}-\mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}\mathbf{q}_i^{F*}, \tag{46}$$

$$\mathcal{A}_R^t\mathcal{D}^t\mathbf{u}_i^D=\mathbf{k}, \tag{47}$$

for $i=1,2$.

By subtracting equalities (46) for $i=1,2$ and making the scalar product by $\mathbf{u}_1^D-\mathbf{u}_2^D$ we get

$$\begin{aligned}&\Big((\tilde{\mathbf{G}}_F)^{-1}(\tilde{\mathcal{B}}^t\mathbf{u}_1^D-\mathbf{h})-(\tilde{\mathbf{G}}_F)^{-1}(\tilde{\mathcal{B}}^t\mathbf{u}_2^D-\mathbf{h})\Big)\cdot\Big(\tilde{\mathcal{B}}^t\mathbf{u}_1^D-\tilde{\mathcal{B}}^t\mathbf{u}_2^D\Big)\\ &+(\mathbf{q}_1^R-\mathbf{q}_2^R)\cdot\mathcal{A}_R^t\mathcal{D}^t(\mathbf{u}_1^D-\mathbf{u}_2^D)=-\mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}\Big(\mathbf{q}_1^{F*}-\mathbf{q}_2^{F*}\Big)\cdot(\mathbf{u}_1^D-\mathbf{u}_2^D).\end{aligned}$$

Let us denote by $(\tilde{\mathcal{B}}^t)^{-1}$ the left-inverse of the injective mapping $\tilde{\mathcal{B}}^t$. By using the above equality with (43), (44) and (47), we get

$$\begin{aligned}\frac{1}{2}\|\tilde{\mathbf{F}}(\mathbf{q}_1^{F*})-\tilde{\mathbf{F}}(\mathbf{q}_2^{F*})\|^3&=\frac{1}{2}\|(\tilde{\mathbf{G}}_F)^{-1}\Big(\tilde{\mathcal{B}}^t\mathbf{u}_1^D-\mathbf{h}\Big)-(\tilde{\mathbf{G}}_F)^{-1}\Big(\tilde{\mathcal{B}}^t\mathbf{u}_2^D-\mathbf{h}\Big)\|^3\\ &\leq\Big((\tilde{\mathbf{G}}_F)^{-1}(\tilde{\mathcal{B}}^t\mathbf{u}_1^D-\mathbf{h})-(\tilde{\mathbf{G}}_F)^{-1}(\tilde{\mathcal{B}}^t\mathbf{u}_2^D-\mathbf{h})\Big)\cdot\Big(\tilde{\mathcal{B}}^t\mathbf{u}_1^D-\tilde{\mathcal{B}}^t\mathbf{u}_2^D\Big)\\ &=-\big((\tilde{\mathcal{B}}^t)^{-1}\big)^t\mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}(\mathbf{q}_1^{F*}-\mathbf{q}_2^{F*})\cdot(\tilde{\mathcal{B}}^t\mathbf{u}_1^D-\tilde{\mathcal{B}}^t\mathbf{u}_2^D)\\ &\leq\|\big((\tilde{\mathcal{B}}^t)^{-1}\big)^t\mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}\|\|\mathbf{q}_1^{F*}-\mathbf{q}_2^{F*}\|\|\tilde{\mathcal{B}}^t\mathbf{u}_1^D-\tilde{\mathcal{B}}^t\mathbf{u}_2^D\|'\\ &=\|\big((\tilde{\mathcal{B}}^t)^{-1}\big)^t\mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}\|\|\mathbf{q}_1^{F*}-\mathbf{q}_2^{F*}\|\|\tilde{\mathbf{G}}_F\big((\tilde{\mathbf{G}}_F)^{-1}\big(\tilde{\mathcal{B}}^t\mathbf{u}_1^D-\mathbf{h}\big)\big)\\ &\qquad-\tilde{\mathbf{G}}_F\big((\tilde{\mathbf{G}}_F)^{-1}\big(\tilde{\mathcal{B}}^t\mathbf{u}_2^D-\mathbf{h}\big)\big)\|'\\ &=\|\big((\tilde{\mathcal{B}}^t)^{-1}\big)^t\mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}\|\|\mathbf{q}_1^{F*}-\mathbf{q}_2^{F*}\|\|\tilde{\mathbf{G}}_F\big(\tilde{\mathbf{F}}(\mathbf{q}_1^{F*})\big)-\tilde{\mathbf{G}}_F\big(\tilde{\mathbf{F}}(\mathbf{q}_2^{F*})\big)\|'\\ &\leq\|\big((\tilde{\mathcal{B}}^t)^{-1}\big)^t\mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}\|\|\mathbf{q}_1^{F*}-\mathbf{q}_2^{F*}\|\|\tilde{\mathbf{F}}(\mathbf{q}_1^{F*})-\tilde{\mathbf{F}}(\mathbf{q}_2^{F*})\|\Big(\|\tilde{\mathbf{F}}(\mathbf{q}_1^{F*})\|+\|\tilde{\mathbf{F}}(\mathbf{q}_2^{F*})\|\Big),\end{aligned}$$

where

$$\|\big((\tilde{\mathcal{B}}^t)^{-1}\big)^t\mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}\|=\max\{\|\big((\tilde{\mathcal{B}}^t)^{-1}\big)^t\mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}\mathbf{q}^F\|:\|\mathbf{q}^F\|=1\}$$

If $\|\tilde{\mathbf{F}}(\mathbf{q}_1^{F*}) - \tilde{\mathbf{F}}(\mathbf{q}_2^{F*})\| = 0$ then (45) is trivially true. Otherwise, it is deduced for

$$C := 2\|\big((\tilde{\mathcal{B}}^t)^{-1}\big)^t \mathcal{D}\Lambda\mathcal{F}^t\tilde{\mathcal{W}}\|$$

by dividing the above inequality by $\|\tilde{\mathbf{F}}(\mathbf{q}_1^{F*}) - \tilde{\mathbf{F}}(\mathbf{q}_2^{F*})\|$.

Lemma 4.11. *Let us denote* $\tilde{\mathbf{F}}_{\mathbf{0}} := \tilde{\mathbf{F}}(\mathbf{0})$*. Then the following estimate holds:*

$$\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| \leq \max\{\|\tilde{\mathbf{F}}_{\mathbf{0}}\|, \frac{1}{4}\Big(C^{1/2}\|\mathbf{q}^{F*}\|^{1/2} + \sqrt{C\|\mathbf{q}^{F*}\| + 8\|\tilde{\mathbf{F}}_{\mathbf{0}}\|}\Big)^2 - \|\tilde{\mathbf{F}}_{\mathbf{0}}\|\} \quad \forall \mathbf{q}^{F*} \in \mathbb{R}^{e_f}. \tag{48}$$

Proof: If $\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| < \|\tilde{\mathbf{F}}_{\mathbf{0}}\|$ the results trivially follows. Otherwise, let us suppose that $\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| \geq \|\tilde{\mathbf{F}}_{\mathbf{0}}\|$. By taking $\mathbf{q}_1^{F*} = \mathbf{q}^{F*}$ and $\mathbf{q}_2^{F*} = \mathbf{0}$ in (45) we get

$$\|\tilde{\mathbf{F}}(\mathbf{q}^{F*}) - \tilde{\mathbf{F}}_{\mathbf{0}}\|^2 \leq C\|\mathbf{q}^{F*}\|\Big(\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| + \|\tilde{\mathbf{F}}_{\mathbf{0}}\|\Big). \tag{49}$$

Then

$$\big(\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| - \|\tilde{\mathbf{F}}_{\mathbf{0}}\|\big)^2 \leq \|\tilde{\mathbf{F}}(\mathbf{q}^{F*}) - \tilde{\mathbf{F}}_{\mathbf{0}}\|^2 \leq C\|\mathbf{q}^{F*}\|\Big(\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| + \|\tilde{\mathbf{F}}_{\mathbf{0}}\|\Big)$$

and also

$$\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| \leq C^{1/2}\|\mathbf{q}^{F*}\|^{1/2}\Big(\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| + \|\tilde{\mathbf{F}}_{\mathbf{0}}\|\Big)^{1/2} + \|\tilde{\mathbf{F}}_{\mathbf{0}}\|.$$

By adding $\|\tilde{\mathbf{F}}_{\mathbf{0}}\|$ to both sides of this inequality we get

$$\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| + \|\tilde{\mathbf{F}}_{\mathbf{0}}\| \leq C^{1/2}\|\mathbf{q}^{F*}\|^{1/2}\Big(\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| + \|\tilde{\mathbf{F}}_{\mathbf{0}}\|\Big)^{1/2} + 2\|\tilde{\mathbf{F}}_{\mathbf{0}}\|.$$

Let us denote $x := \Big(\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| + \|\tilde{\mathbf{F}}_{\mathbf{0}}\|\Big)^{1/2}$. Then the above inequality can be rewritten as follows:

$$x^2 - C^{1/2}\|\mathbf{q}^{F*}\|^{1/2}x - 2\|\tilde{\mathbf{F}}_{\mathbf{0}}\| \leq 0,$$

which implies

$$x \leq \frac{1}{2}\Big(C^{1/2}\|\mathbf{q}^{F*}\|^{1/2} + \sqrt{C\|\mathbf{q}^{F*}\| + 8\|\tilde{\mathbf{F}}_{\mathbf{0}}\|}\Big),$$

so finally

$$\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| \leq \frac{1}{4}\Big(C^{1/2}\|\mathbf{q}^{F*}\|^{1/2} + \sqrt{C\|\mathbf{q}^{F*}\| + 8\|\tilde{\mathbf{F}}_{\mathbf{0}}\|}\Big)^2 - \|\tilde{\mathbf{F}}_{\mathbf{0}}\|.$$

Corollary 4.5. *Mapping* $\tilde{\mathbf{F}}$ *transforms bounded sets into bounded sets.*

Corollary 4.6. *If* $\|\mathbf{q}^{F*}\| \leq \frac{\|\tilde{\mathbf{F}}_\mathbf{0}\|}{C}$ *then*

$$\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| \leq 3\|\tilde{\mathbf{F}}_\mathbf{0}\|.$$

Proof: From estimate (48) we get

$$\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| \leq \max\{\|\tilde{\mathbf{F}}_\mathbf{0}\|, \frac{1}{4}\Big(\sqrt{\|\tilde{\mathbf{F}}_\mathbf{0}\|} + \sqrt{\|\tilde{\mathbf{F}}_\mathbf{0}\| + 8\|\tilde{\mathbf{F}}_\mathbf{0}\|}\Big)^2 - \|\tilde{\mathbf{F}}_\mathbf{0}\|\}$$
$$= \max\{\|\tilde{\mathbf{F}}_\mathbf{0}\|, 4\|\tilde{\mathbf{F}}_\mathbf{0}\| - \|\tilde{\mathbf{F}}_\mathbf{0}\|\} = 3\|\tilde{\mathbf{F}}_\mathbf{0}\|.$$

Now we are in a position to prove the following result:

Proposition 4.1. *If* $C \leq \frac{1}{3}$ *then* $\tilde{\mathbf{F}}$ *maps the ball with center* $\mathbf{0}$ *and radius* $\frac{\|\tilde{\mathbf{F}}_\mathbf{0}\|}{C}$ *into itself.*

Proposition 4.2. *The mapping* $\tilde{\mathbf{F}} : \mathbb{R}^{e_f} \to \mathbb{R}^{e_f}$ *is Hölder continuous with exponent* $1/2$ *on bounded sets. In particular it is continuous at any point in* $\mathbb{R}^{e_f}$*.*

Proof: From estimate (48) we deduce that given any $R > 0$ there exists $M > 0$ such that if $\|\mathbf{q}^{F*}\| < R$ then $\|\tilde{\mathbf{F}}(\mathbf{q}^{F*})\| < M$. Therefore, (45) yields

$$\|\tilde{\mathbf{F}}(\mathbf{q}_1^{F*}) - \tilde{\mathbf{F}}(\mathbf{q}_2^{F*})\|^2 \leq 2MC\|\mathbf{q}_1^{F*} - \mathbf{q}_2^{F*}\| \quad \forall \mathbf{q}_i^{F*} \text{ with } \|\mathbf{q}_i^{F*}\| < R, \quad i = 1, 2$$

and then

$$\|\tilde{\mathbf{F}}(\mathbf{q}_1^{F*}) - \tilde{\mathbf{F}}(\mathbf{q}_2^{F*})\| \leq \sqrt{2MC}\|\mathbf{q}_1^{F*} - \mathbf{q}_2^{F*}\|^{1/2} \quad \forall \mathbf{q}_i^{F*} \text{ with } \|\mathbf{q}_i^{F*}\| < R, \quad i = 1, 2.$$

From the above results one can prove the main result of this article.

Theorem 4.2. *Under the assumptions:*

$$\tilde{\mathcal{B}}^t = (\mathcal{A}_F^t - \tilde{\mathcal{W}}\mathcal{F}\Lambda^t)\mathcal{D}^t \text{ is injective,} \tag{50}$$

$$C \leq \frac{1}{3} \tag{51}$$

the nonlinear system of equations (25) has a solution for given vectors $\boldsymbol{\alpha}^R$, $\mathbf{c}^D$, $\mathbf{q}^V$ *and* $\mathbf{u}^U$.

Proof: It is a straightforward consequence of the above results and the Brouwer fixed-point theorem (see, for instance, [16]) applied to mapping $\tilde{\mathbf{F}}$.

Remark 4.4. *It is worth mentioning that assumptions (50) and (51) only depend on intrinsic structural characteristics of the network but not on any flow data, so it can be checked a priori for each specific network. Moreover, they are plausible because for real networks, the numbers* w_i *defined in (19) are largely smaller than one as far as the height difference between the two nodes of each edge is not too big. Indeed, we notice that in the definition of* w_i *(see (19)),* R *is about* 440 *in the SI,* θ *is larger than* $273K$ *and* Z *is close to one. In fact we have checked them for several networks.*

Remark 4.5. *The numerical resolution of the model introduced in Section 3 can be done by using Newton-like methods. Moreover, this model can be used to optimize the network design and operation. The latter has been done, for instance, in [9] where the self-consumption of gas in the compression stations is minimized under some constraints regarding the security of supply and the maximum and minimum pressure at the nodes.*

5 Example

In this section we illustrate the above results on a small gas network. The topology of this network is shown in Figure 1, and physical data for nodes and edges are given in Table 1. Briefly, it has eleven edges including one compressor, one pressure control valve and one flow control valve, and eleven nodes, the pressure being imposed at two of them. In addition, this network has a cycle.

Figure 1: The network topology.

	NODES		EDGES				
Num.	Node type	Height [m]	Edge type	Initial node	Final node	Length [m]	Diameter [m]
1	imp. press.	16	structural	1	2	47900	0.6350
2	consumer	17	PCV	2	3	50	0.6350
3	structural	17	structural	3	4	91000	0.4064
4	consumer	469	structural	2	5	50400	0.5080
5	consumer	256	compressor	5	6	50	0.6350
6	structural	256	structural	6	7	98200	0.5080
7	consumer	145	structural	7	9	87300	0.5080
8	consumer	19	structural	9	8	24900	0.5080
9	consumer	104	structural	8	5	52500	0.5080
10	structural	104	FCV	10	9	50	0.6350
11	imp. press.	44	structural	11	10	26600	0.5080

Table 1: Physical information about the nodes and the edges.

Besides, we consider the following realistic mean values for each edge:

- the temperature of the gas θ: 288.15 K.
- the constant of the gas R: 476.7388 J/(kg K).
- the compressibility factor Z: 0.8248.
- the friction factor λ: 0.0123.

Based on the above data, one can compute matrix $\tilde{\mathcal{B}}$ and constant C and check that assumptions of Theorem 4.2 are satisfied. Indeed, $\tilde{\mathcal{B}}^t$ is one-to-one and $C = 0.1861 < 1/3$.

6 Conclusions

In this paper we have proved the existence of solution to a steady-state mathematical model of a gas transportation network on non-flat topography. The model is introduced in a detailed way and includes all elements existing in such networks: emission and consumption points, pipes, compressor stations, flow control valves and pressure control valves. The existence is proved in two steps. The first one uses convex analysis and the second one a fixed-point theorem.

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AppendixA The mathematical model

In this appendix we recall the model analyzed in this paper. Since the length of a pipeline is much greater than the diameter we can obtain a 1D model by integrating on the cross-sections of the pipeline the 3D Navier-Stokes equations for the isothermal flow of a real gas. For this purpose, the curve joining the centers of the cross-sections is parametrized with respect to the arc length to be called x; we denote by L the length of this curve, i.e., the length of the pipeline. Moreover, some approximations will be done. In particular, the viscous force and thermal conduction terms involving second-order derivatives in the pipeline direction will be neglected, but the other terms will not and they will be modelled. In particular, for the tangential viscous force the Darcy-Weisbach formula will be used (see below).

One-dimensional model

- Mass conservation equation:

$$A\frac{\partial \rho}{\partial t}(x,t)+\frac{\partial q}{\partial x}(x,t)=0, \tag{A.1}$$

where

  - A is the area of the cross-sections (m^2).
  - $\rho(x,t)$ is the average density on section x at time t (kg/m^3).
  - $q(x,t)$ is the mass flow rate across the x section at time t (kg/s).

The mass-weighted average velocity on section x is defined by

$$v\,(x,t)=\frac{q\,(x,t)}{A\rho\,(x,t)}.$$

Let us recall that equation (A.1) is exact because no approximations are needed to get it.

- Linear momentum equation:

$$\begin{aligned}&\frac{\partial(\rho v_1)}{\partial t}(x,t)+\frac{\partial(\rho v^2)}{\partial x}(x,t)+\frac{\partial p}{\partial x}(x,t)\\ +&\frac{\lambda\rho(x,t)}{2D}|v(x,t)|v(x,t)-g\rho(x,t)h'(x)=0.\end{aligned} \tag{A.2}$$

where

  - $p(x,t)$ is the average pressure on the x cross-section, at time t (N/m^2).
  - g is the gravity acceleration (m/s^2)
  - $h(x)$ is the height of the x cross-section (m)
  - D is the diameter of the pipe (m)
  - λ is the friction factor between the gas and the pipe walls; it is a non-dimensional number depending on the diameter of the pipe, the rugosity of its wall and the Reynolds number of the flow.

The computation of λ can be made by using the Colebrook's equation (see [17]):

$$\frac{1}{\sqrt{\lambda}}=-2\log_{10}\left(\frac{2.51}{\mathrm{Re}\sqrt{\lambda}}+\frac{k}{3.7D}\right)=-2\log_{10}\left(\frac{2.51\pi D\eta}{4\,|q|\,\sqrt{\lambda}}+\frac{k}{3.7D}\right), \tag{A.3}$$

where k is the roughness coefficient of the pipe (m).

The last term in (A.2) arises from the gravity force. In fact, the correct expression of that term is $g\rho(x,t)\sin(\pi-\alpha(x))$ (see Figure A.2) but, if the slope of the pipeline is small, i.e., if

$$|h'(x)| \ll 1$$

then

$$\sin(\pi-\alpha(x)) \approx \tan(\pi-\alpha(x)) = -\tan\alpha(x) = -h'(x).$$

Figure A.2: The gravity force term.

Isothermal steady state model

From this point forward, we will suppose that the flow is in steady-state so partial derivatives with respect to time are null. Hence, the system of equations becomes

$$\frac{dq}{dx}(x) = 0, \tag{A.4}$$

$$A\frac{dp}{dx}(x) + \frac{\lambda(q(x))}{2DA}\frac{1}{\rho(x)}|q(x)|\,q(x) + Ag\rho(x)\frac{dh(x)}{dx} = 0, \tag{A.5}$$

$$Z(p(x),\theta(x))\,\rho(x)\,R\theta(x) = p(x), \tag{A.6}$$

where θ is the absolute temperature, which is supposed to be known lengthwise the pipe. The last equation is the equation of state for real gases. Notations are as follows:

$$R = \frac{\mathcal{R}}{M},$$

where $\mathcal{R}$ is the universal gas constant (J/(k-mol K), and M is the molar mass (kg/k-mol). The compressibility factor, Z, depends on pressure and temperature. It can be determined using different equations, like the van der Waals equation, but in the gas transportation industry the AGA8 model is widely used. This model is an empirical equation proposed by the American Gas Association [18], namely,

$$Z = \hat{Z}(p, \theta) = 1 + p_r 0.257 - 0.533 \frac{p_r}{\theta_r}, \tag{A.7}$$

being $p_r := p/p_c$, $\theta_r := \theta/\theta_c$, and p_c and θ_c the *critical pressure* and the *critical temperature*, respectively. Let us recall that above the critical temperature it is impossible to liquefy a gas, while the critical pressure is the minimum pressure required to liquefy a gas at its critical temperature. For natural gas the critical temperature is around 170 K and the critical pressure around 5 MPa. Notice that the first equation implies that the mass flow is constant along the pipe, $q(x) = q \ \forall x \in (0, L)$.

Approximate solution of the model

The approximate solution of the previous problem requires, on the one hand, to know the boundary conditions (for example, the pressures at the ends of the pipe) and, on the other hand, the use of numerical methods. Nevertheless, for numerical simulation of gas transmission networks a simplified model is used which is deduced by integrating the equation (A.5) between the ends of the pipe, $x = 0$ and $x = L$ and making certain approximations.

Firstly, the mean density in the section is replaced with the following expression, deduced from the equation of state for real gases:

$$\rho(x) = \frac{p(x)}{Z(p(x), \theta(x)) \ R\theta(x)}.$$

Thus, we obtain

$$\begin{aligned} Ap(x) \frac{dp}{dx}(x) + \frac{\lambda(q)}{2DA} Z(p(x), \theta(x)) \ R\theta(x) |q| q \\ + Ag \frac{p^2(x)}{Z(p(x), \theta(x)) \ R\theta(x)} h'(x) = 0. \end{aligned} \tag{A.8}$$

Integrating this equation from $x = 0$ to $x = L$ and dividing by $A/2$ yields,

$$\begin{aligned} p^2(L) - p^2(0) = -\frac{\lambda(q)}{DA^2} R |q| q \int_0^L Z(p(x), \theta(x)) \theta(x) \, dx \\ - \frac{2g}{R} \int_0^L \frac{p^2(x)}{Z(p(x), \theta(x)) \theta(x)} h'(x) \, dx. \end{aligned} \tag{A.9}$$

At this point, let us rewrite equation (A.9) by using the new variable $u(x) := p^2(x)$,

$$u(L) - u(0) = -\frac{\lambda(q)}{DA^2} R|q|q \int_0^L Z(p(x), \theta(x)) \theta(x)\, dx - \frac{2g}{R} \int_0^L \frac{u(x)}{Z(p(x), \theta(x)) \theta(x)} \frac{dh(x)}{dx}\, dx. \tag{A.10}$$

Now the integrals of the second member are approximated using average values of pressure and temperature in the pipe, denoted by p_m and θ_m, respectively, whose expression will be specified below:

$$\int_0^L Z(p(x), \theta(x)) \theta(x)\, dx \approx Z(p_m, \theta_m) \theta_m L,$$

$$\int_0^L \frac{u(x)}{Z(p(x), \theta(x)) \theta(x)} \frac{dh(x)}{dx}\, dx \approx \frac{u_m}{Z(p_m, \theta_m) \theta_m}(h(L) - h(0)).$$

Replacing in (A.10), we finally obtain,

$$u(0) - u(L) = \frac{\lambda(q) L}{DA^2} R\theta_m |q| q Z(p_m, \theta_m) + \frac{2g}{R\theta_m} \frac{u_m}{Z(p_m, \theta_m)} (h(L) - h(0)), \tag{A.11}$$

Assuming that the section of the pipe is circular ($A = \pi D^2/4$), we have,

$$u(0) - u(L) = G(p_m, \theta_m, q) + \frac{2g}{R\theta_m} \frac{u_m}{Z(p_m, \theta_m)} (h(L) - h(0)) \tag{A.12}$$

where

$$G(p_m, \theta_m, q) := \frac{16\lambda(q) L}{\pi^2 D^5} R\theta_m |q| q Z(p_m, \theta_m), \tag{A.13}$$

or, introducing $\mu(q) := \lambda(q)|q|q$,

$$G(p_m, \theta_m, q) = \frac{16\mu(q) L}{\pi^2 D^5} R\theta_m Z(p_m, \theta_m), \tag{A.14}$$

and the average u_m can be computed by the following alternatives:

$$u_m := \frac{u(0) + u(L)}{2}, \tag{A.15}$$

$$u_m := \frac{2}{3}\left(u(0) + u(L) - \frac{u(0)\, u(L)}{u(0) + u(L)}\right), \tag{A.16}$$

and similar expressions for θ_m. Let us recall that the absolute temperatures at points $x = 0$ and $x = L$ are assumed to be known.

AppendixB Short background of graph theory

In this appendix we recall some elementary results from graph theory. Further details can be found, for instance, in reference [19].

The topology of a gas network can be modeled by a *direct graph*. A direct graph $\mathcal{G}$ is a pair of finite sets $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, so that $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$. Thus, each element from $\mathcal{E}$ is an ordered pair of elements from $\mathcal{N}$. The elements of $\mathcal{N}$ are called *nodes* and the ones of $\mathcal{E}$ *edges*. A graph can be represented on a plane: the nodes are points and the edges are arcs.

Let us denote by n the number of nodes and by e the number of edges. For $j = 1, \cdots, e$, $\mathcal{M}_{1,j}$ and $\mathcal{M}_{2,j}$ denote the numbers of the first and second nodes, respectively, of edge number j.

Matrix $\mathcal{M}$, of order $2 \times e$, is called *connectivity matrix* of the direct graph.

The *incidence matrix* of the direct graph is the matrix $\mathcal{A}$, of order $n \times e$, defined as follows:

$$\mathcal{A}_{ij} = \begin{cases} 0 & \text{if node } N_i \text{ does not belong to edge } E_j, \\ 1 & \text{if node } N_i \text{ belongs to edge } E_j \text{ and also } i = \mathcal{M}_{1,j}, \\ -1 & \text{if node } N_i \text{ belongs to edge } E_j \text{ and also } i = \mathcal{M}_{2,j}. \end{cases}$$

A set of edges $r_1, \ldots, r_m$ of a graph is a *path* between nodes a and b if the following conditions hold: 1) Two consecutive edges of the set r_i, r_{i+1} always have a common node. 2) There is not any node of the graph belonging to more than two edges of the set. 3) Node a is a node of only one edge of the set and the same is true for b. A graph is said *connected* if there is a path between any two nodes. The following result can be proved: the graph $\mathcal{G}$ has exactly m connected components if and only if $\text{rank}(\mathcal{A}) = n - m$. From this result we deduce that if $\mathcal{G}$ is a connected graph with n nodes and e edges then $n - 1 \leq e$.

From this point forward we will assume that $\mathcal{G}$ is a connected graph. A subgraph $\mathcal{S}$ of a graph $\mathcal{G}$ is said a *cycle* if the following conditions hold: 1) $\mathcal{S}$ is connected. 2) Each node of $\mathcal{S}$ belongs to exactly two edges of $\mathcal{S}$.

Let $\mathcal{G}$ be a graph with n nodes and e edges. Let l be the number of its cycles which have been provided with an orientation. The *cycle matrix* $\mathcal{C} \in \mathrm{M}_{e \times l}$ is defined as follows: for each $k \in \{1, \ldots, l\}$,

$$\mathcal{C}_{ij} = \begin{cases} 0, & \text{if edge } j \text{ does not belong to cycle } k, \\ -1, & \text{if edge } j \text{ belongs to cycle } k \text{ but they have opposed orientations,} \\ 1, & \text{if edge } j \text{ belongs to cycle } k \text{ and they have the same orientation.} \end{cases}$$

One can prove that $\mathcal{AC} = 0$, and therefore $\text{im}(\mathcal{C}) \subset \ker(\mathcal{A})$ and $\text{rank}(\mathcal{C}) \leq e - n + 1$. In fact, one can prove a stronger result, namely, $\text{rank}(\mathcal{C}) = e - n + 1$. The dimension of $\text{rank}(\mathcal{C}) = \ker(\mathcal{A})$ is equal to the number of so-called *fundamental cycles* of the graph.

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