Global Optimization for Bilevel Portfolio Design: Economic Insights from the Dow Jones Index

Julio González-Díaz*[1,2,3], Brais González-Rodríguez[1,2], Marina Leal[4,5], and Justo Puerto[5]

[1] *Department of Statistics, Mathematical Analysis and Optimization and IMAT. University of Santiago de Compostela.*
[2] *MODESTYA Research Group.*
[3] *ITMATI (Technological Institute for Industrial Mathematics).*
[4] *ALOP. Trier University, Germany*
[5] *IMUS. University of Seville, Spain*

Accepted for publication on October 21st, 2020

Abstract

This paper deals with a portfolio selection problem with transaction costs and two levels of decision-making. It is assumed that the decision making structure is twofold: there is a broker-dealer that controls the fees to be charged on the different securities in order to maximize his benefit and there is an investor who chooses his portfolio trying to minimize risk while ensuring a minimum level of return. This structure gives rise to an implicit hierarchical competition that consists in anticipating the rational decision of the other agent in order to optimize the decision-makers' own criteria. We analyze different situations depending on who is first in the hierarchy: the broker-dealer or the investor. We present different nonlinear and nonconvex mathematical programming models for the different situations and develop an extensive computational study in which we discuss the ensuing economic insights for the models based on Dow Jones index data.

1 Introduction

Mathematical optimization problems are pervasive in the fields of economics and management science, and the importance of developing realistic models to delve into the understanding of complex economic settings has long been recognized. In particular, bilevel optimization models have attracted a lot of attention since the pioneering work by von Stackelberg (1934). However, realistic models often result in difficult optimization problems and, thus, a compromise is required between realism and solvability of the model. Quite often the analyst is confronted with a nonlinear and nonconvex optimization problem, in which finding globally optimal solutions may be an extremely challenging task.

One of the contributions of this paper is to illustrate how proper modelling skills may allow to efficiently solve complex optimization problems, enabling the development of qualitative and quantitative economic analysis in problems that, otherwise, would be hard to tackle. In order to do so, we formally study two novel bilevel portfolio design problems and show how one can rely on well-established approaches to reformulate them as single-level problems that can be fully solved with state-of-the-art optimization software.

Historically, the main criterion to design an optimal portfolio was to find the configuration of assets that generated the highest expected return. However, this perspective changed in

*Corresponding author. Email address: julio.gonzalez@usc.es.

1952, when Harry Markowitz introduced a new variable along with the expected return: the risk of each portfolio (Markowitz, 1952). Thereafter, analysts began to incorporate a risk-return trade-off in their models.

The model proposed by Markowitz only focuses on finding an optimal portfolio from the investor's point of view. The literature is plenty of papers with similar approaches, as for instance Benati (2003, 2015), Castro et al. (2011), Kolm et al. (2014), Mansini et al. (2014), Cesarone (2020), Cesarone et al. (2020), Perrin and Roncalli (2019) and Puerto et al. (2020), to mention a few. However, real markets are generally more complex, since there is another decision-maker who can set fees on the transactions of the securities to profit from anticipating the rational behavior of investors. Transaction costs, those incurred by the investors when buying and selling assets on financial markets, have been widely studied in papers such as Kellerer et al. (2000), Baule (2010), Baumann and Trautmann (2013), Woodside-Oriakhi et al. (2013), Mansini et al. (2014), and Valle et al. (2014), among others. In this work we study these situations in which the investor, when deciding his optimal portfolio, has to consider the transaction fee to pay to an agent: the broker-dealer, hereafter, the broker. The broker makes decisions regarding fees associated to the assets, trying to maximize the resulting profit.[1]

One of the main contributions of this paper is methodological: introducing a single-period hierarchical portfolio problem, with continuous choices, that accounts for decisions on transaction fees. The broker fixes these fees while the investor chooses his portfolio. The timing of these two decisions is crucial and bilevel optimization is necessary to understand the impact of different hierarchical structures. We discuss different models representing this situation, which arise depending on the order in which choices are made. The main model, and arguably the most realistic one, is the B-L model (Broker as Leader) in which, first, transaction fees are determined by the broker and then, after observing them, the investor chooses his portfolio. Two additional models are considered and can be seen as instrumental benchmarks to gain understanding on the B-L model: i) the I-L model (Investor as Leader) in which the broker chooses the transaction fees after observing the portfolio chosen by the investor and ii) the SW model, a *simultaneous*-choice model in which the goal is to maximize social welfare.

Importantly, regarding broker's fees, we allow for quite general market structures. The usual functioning of these markets, where the broker selects a fee, common to all assets, to be charged on top of the fees fixed by the stock market, is just a particular case. The extra generality enables the study of natural departures from this baseline setting and the analysis of the impact of the novel strategic aspects they may introduce. For instance, in our analysis we let the broker associate different fees to different assets, giving him extra freedom to tailor the fees to the attractiveness of the different assets. These models and the associated solution techniques may be seen as a first building block in the evaluation of this new competitive situation. An important challenge for future research is to enhance the models so that they can help to understand the effects on long-term dynamics of the ensuing competition between different brokers.

In Markowitz's seminal paper, the risk measure under consideration was the variance, which, despite being a sound measure of dispersion, is nowadays known to have important drawbacks as a risk measure. Since then, many different risk measures have been introduced and analyzed, such as Gini's Mean Difference (Bey and Howe, 1984), Mean Absolute Deviation (Konno and Yamazaki, 1991), Value at Risk (Stambaugh, 1996), and Conditional Value at Risk (Rockafellar and Uryasev, 2000), to name a few. At the same time, a theoretical body around risk measures was developed and the notion of *coherency*, introduced in Artzner et al.

[1]This pricing aspect has been modeled as a bilevel problem in the literature in many different applications. See, for instance, Shioda et al. (2011), Labbé and Violin (2013), Grimm et al. (2019), Maravillo et al. (2019), Qiu et al. (2020) and the references therein.

(1997, 1999), was identified as a natural requirement. A coherent risk measure must satisfy the properties of monotonicity, sub-additivity, homogeneity, and translational invariance.[2]

Coherency is one of the main reasons to have developed our analysis for the Conditional Value at Risk (CVaR), also known as Expected Shortfall, which is the weighted average of the extreme losses in the tail of the distribution of the returns. The other main reason is that CVaR falls into the set of risk measures whose optimization can be formulated as a linear optimization problem, as shown in Rockafellar and Uryasev (2000). Reviews of other LP solvable risk measures can be found in Mansini et al. (2003, 2014).

A first approach to the type of bilevel models dealt with in this paper can be seen in Leal et al. (2020). Nevertheless, there are a number of differences with respect to our approach, the main one being that transaction fees are assumed to be chosen from a discrete set in Leal et al. (2020), and solution approaches revolve around Mixed Integer Linear Programming reformulations and Bender's decomposition techniques (Benders, 1962). Our analysis allows for continuous transaction fees and, hence, the resulting optimization problems are of a completely different nature: nonlinear and nonconvex polynomial optimization problems.[3] From the computational point of view, finding the global optimum of these problems is known to be an NP-hard problem.[4] However, moderate size problems can be handled by state-of-the-art optimization techniques and solvers. Our computational analysis builds on two such solvers: BARON (Sahinidis, 2019) and RAPOSa (González-Rodríguez et al., 2020). In addition, we have also addressed the issue of scalability of our solution techniques as compared with other solution methods for larger size instances. To this end, we have studied solution techniques built in different local solvers: Ipopt (Wächter and Biegler, 2006), Knitro (Byrd et al., 2006) and MINOS (Murtagh and Saunders, 1978). The comparative analysis of the quality of solutions found and of the running times is reported in Appendix A. Another important difference with respect to Leal et al. (2020) is that we also model, and numerically study, situations in which there are multiple followers in the bilevel problems, which leads to richer economic interactions and potential for additional economic insights. One of the main findings in the numerical analysis is that the outcomes of the B-L model Pareto dominate those of the I-L model and, further, B-L outcomes are Pareto efficient.[5].

The organization of this paper is as follows. First, in Section 2 we describe the baseline individual optimization problems for the investor and the broker. Second, in Section 3 we present the joint optimization problems: the two hierarchical Stackelberg models and the *simultaneous*-choice model; further, the solution techniques for these models are also developed. Then, in Section 4 we present a case study based on data from the Dow Jones Index and provide some economic insights.

2 Baseline Models for Broker-Investor Interactions

The portfolio optimization problem considered in this paper is based on a single-period model of investment and incorporates a pricing aspect on the transaction costs. Borrowing from Leal et al. (2020), we assume the existence of two types of decision-makers: investors and brokers, with a hierarchical decision structure.

The investor faces the classic problem of allocating his capital among various financial securities, each of which will generate some uncertain returns whose random distribution is assumed to be known by the investor. Building upon the mean-variance approach initiated

[2]Refer to Rockafellar (2014) for a recent review on the topic.

[3]In this context, by a polynomial optimization problem we refer to optimization problems in which both the objective function and the constraints are given by polynomials. Refer, for instance, to Lasserre (2015).

[4]A class of problems for which no polynomial-time algorithm to find the global solution is known.

[5]An outcome is Pareto efficient if no outcome makes one agent better off without hurting the other one. An outcome that is not Pareto efficient is Pareto dominated. Refer to Pareto (1971) for an English translation of Pareto's pioneering work in the late eighties

in Markowitz (1952), we assume that the goal of the investor is to minimize his risk subject to achieving a given expected return. On the other hand, the broker must determine the transaction costs associated with the different securities, with the goal of maximizing his profit. Thus, the decision variables of the investor are the proportions of his capital to invest in each security and those of the broker are the fees on the different securities. These decisions determine the amount paid by the investor to the broker and the risk and expected return of the investor. Note that the net return for the investor is the result of subtracting, for each security, the broker's fee from the security's return.

Formally, the main element of all our models is the set of securities S. The rate of return of each security $j \in S$ is uncertain and we model it through a random variable R_j. Following the standard approach in the field of decision making under uncertainty, we assume that these random variables take values on a finite set T of scenarios.[6] Given a security $j \in S$ and a scenario $t \in T$, r_{jt} denotes the realization of the rate of return R_j in scenario t. Each scenario $t \in T$ has associated probability π_t and $\sum_{t \in T} \pi_t = 1$.

The broker has to choose the transaction costs associated with the securities in S, which we call prices and denote by p. Thus, for each $j \in S$, $p_j \in [0,1]$ represents the proportion of the amount invested in security j that must be paid to the broker. It is natural to consider that there are some limits on the prices that can be chosen by the broker, so we assume that there is a set of feasible prices P, which, moreover, is taken to be a bounded polyhedron. In actual cases, the associated constraints may be imposed by market regulations, by the board of the broker's company given some market analysis, or by a combination of both.

The investor has to choose a portfolio x so that, for each $j \in S$, $x_j \in [0,1]$ represents the weight of security j in the portfolio. We assume that all capital is invested, so $\sum_{j \in S} x_j = 1$; if needed, a safe security with zero rate of return and zero price can be added to set S to represent the proportion of money that is not invested. Each portfolio x defines a random variable $R_x = \sum_{j \in S} R_j x_j$ that represents the rate of return of the portfolio.

In some of the models we discuss we allow for multiple investors, differing only in their degrees of risk aversion. Importantly, throughout our analysis, although each of these investors may be thought of as a single agent, he can also represent a discrete or continuum set of agents, all of them with the same preferences. At optimality, since all the agents with the same degree of risk aversion have the same preferences, we can assume that they choose the same portfolio and, thus, in our mathematical formulation we "aggregate" them in a single investor.

Broker's Problem

The goal of the broker is to maximize his profit so, given a portfolio x, this can be achieved by solving the linear optimization problem

$$\max_{p} \quad \sum_{j \in S} p_j x_j \tag{$\mathrm{B^P}$}$$

$$\text{s.t.} \quad p \in P. \tag{$\mathrm{B^P}$.a}$$

The main difference between ($\mathrm{B^P}$) and (PricP) in Leal et al. (2020) relies on the continuous character of prices. This leads to a totally different family of optimization problems in terms of properties and solution techniques: the model in Leal et al. (2020) is combinatorial whereas ($\mathrm{B^P}$) is continuous.

Investor's Problem

We model the investor's optimization problem as one in which he wants to reduce the risk associated with his portfolio while attaining a certain expected return. As we have already

[6]See, for instance, Chapter 1 in Birge and Louveaux (2011) and Chapter 4 in King and Wallace (2012).

discussed in the Introduction, the risk measure under consideration is the Conditional Value at Risk (Rockafellar and Uryasev, 2000, 2002), which is the opposite of the expected return on the portfolio when considering only the worst $\alpha\%$ of cases. The smaller α is the more concerned is the investor with the lower tail of the distribution, *i.e.*, the more risk averse. Given a level α and a discrete random variable Y defined on the set of scenarios T, we denote the corresponding Conditional Value at Risk by $\mathrm{CVaR}_\alpha(Y)$; the larger is its value, the riskier is the random variable Y.

Given a portfolio x and a price profile p, let Y denote the random variable that, for each $t \in T$, gives the net rate of return for the investor: $y_t = \sum_{j\in S}(r_{jt}x_j - p_j x_j)$. Suppose that we have an investor with risk level α and minimum required expected profit given by $E_{\min}$. Then, given p, he wants to solve the following optimization problem:

$$
\begin{aligned}
\min_{x,y} \quad & \mathrm{CVaR}_\alpha(Y) && (\mathrm{I}_0^{\mathrm{P}})\\
\text{s.t.} \quad & y_t = \sum_{j\in S}(r_{jt}x_j - p_j x_j), \quad t \in T && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{a})\\
& \sum_{t\in T} \pi_t y_t \geq E_{\min} && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{b})\\
& \sum_{j\in S} x_j = 1 && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{c})\\
& x_j \geq 0, \quad j \in S. && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{d})
\end{aligned}
$$

An important property of CVaR is that its computation is equivalent to the solution of a linear programming problem. Following Rockafellar and Uryasev (2000, 2002), we have that, for $\alpha \in (0,1)$, $\mathrm{CVaR}_\alpha(Y)$ can be computed as

$$
\begin{aligned}
\min_{\eta,d} \quad & -\eta + \frac{1}{\alpha}\sum_{t\in T} d_t \pi_t && (\mathrm{CVaR}^{\mathrm{P}})\\
\text{s.t.} \quad & d_t \geq \eta - y_t, \quad t \in T && (\mathrm{CVaR}^{\mathrm{P}}.\mathrm{a})\\
& d_t \geq 0, \quad t \in T. && (\mathrm{CVaR}^{\mathrm{P}}.\mathrm{b})
\end{aligned}
$$

The combination of the elements in problems ($\mathrm{I}_0^{\mathrm{P}}$) and ($\mathrm{CVaR}^{\mathrm{P}}$) leads to the full formulation of the investor's problem:

$$
\begin{aligned}
\min_{x,y,\eta,d} \quad & -\eta + \frac{1}{\alpha}\sum_{t\in T} d_t \pi_t && (\mathrm{I}^{\mathrm{P}})\\
\text{s.t.} \quad & y_t = \sum_{j\in S}(r_{jt}x_j - p_j x_j), \quad t \in T && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{a})\\
& \sum_{t\in T} \pi_t y_t \geq E_{\min} && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{b})\\
& \sum_{j\in S} x_j = 1 && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{c})\\
& x_j \geq 0, \quad j \in S && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{d})\\
& d_t \geq \eta - y_t, \quad t \in T && (\mathrm{CVaR}^{\mathrm{P}}.\mathrm{a})\\
& d_t \geq 0, \quad t \in T. && (\mathrm{CVaR}^{\mathrm{P}}.\mathrm{b})
\end{aligned}
$$

Observe that ($\mathrm{I}_0^{\mathrm{P}}$.a) gives the expected return in each scenario, accounting for the transaction costs and ($\mathrm{I}_0^{\mathrm{P}}$.b) ensures the minimum expected return. Constraints ($\mathrm{I}_0^{\mathrm{P}}$.c) and ($\mathrm{I}_0^{\mathrm{P}}$.d) define the portfolio. Finally, the objective function and constraints ($\mathrm{CVaR}^{\mathrm{P}}$.a) and ($\mathrm{CVaR}^{\mathrm{P}}$.b) come from the optimization problem to compute $\mathrm{CVaR}_\alpha(Y)$. Different choices of parameters α and $E_{\min}$ allow one to model different investor risk profiles. Note that we again have a linear optimization problem.

Dual Problems

Now that we have formally defined both the broker and the investor problems, the next step is to put them together in a joint optimization problem. We do so in the next section by defining two bilevel problems, depending on who is the leader, the broker or the investor. These problems are then reformulated as single level problems by relying on strong duality in linear programming.[7] In order to do so, we need the formulations of the duals of problems (B^{P}) and (I^{P}), which we present below.

To formulate the broker's dual problem we can assume, without loss of generality, that the polyhedron P can be defined as $Ap \leq b$, with these constraints being indexed over the set P^{cons}. Then, the dual variables are $v_s \geq 0$ with $s \in P^{\mathrm{cons}}$ and the dual constraints are $v^\intercal A - x^\intercal = 0$, leading to the dual problem

$$\begin{aligned} \min_{v} \quad & \sum_{s \in P^{\mathrm{cons}}} b_s v_s & (\mathrm{B}^{\mathrm{D}}) \\ \text{s.t.} \quad & v^\intercal A - x^\intercal = 0 & (\mathrm{B}^{\mathrm{D}}.\mathrm{a}) \\ & v_s \geq 0, \quad s \in P^{\mathrm{cons}}. & (\mathrm{B}^{\mathrm{D}}.\mathrm{b}) \end{aligned}$$

The formulation of the investor's dual problem is more involved. The dual variables are

- δ, such that $\delta_t \in \mathbb{R}$ for each $t \in T$, associated with constraints ($\mathrm{I}_0^{\mathrm{P}}.\mathrm{a}$).
- $\mu \leq 0$, associated with constraint ($\mathrm{I}_0^{\mathrm{P}}.\mathrm{b}$).
- $\beta \in \mathbb{R}$, associated with constraint ($\mathrm{I}_0^{\mathrm{P}}.\mathrm{c}$).
- γ, such that $\gamma_t \leq 0$ for each $t \in T$, associated with constraints ($\mathrm{CVaR}^{\mathrm{P}}.\mathrm{a}$).

Then, we get the dual problem

$$\begin{aligned} \max_{\delta,\mu,\beta,\gamma} \quad & -\beta - E_{\min}\mu & (\mathrm{I}^{\mathrm{D}}) \\ \text{s.t.} \quad & \beta + \sum_{t \in T} (r_{jt} - p_j)\delta_t \geq 0, \quad j \in S & (\mathrm{I}^{\mathrm{D}}.\mathrm{a}) \\ & \sum_{t \in T} \gamma_t = -1 & (\mathrm{I}^{\mathrm{D}}.\mathrm{b}) \\ & \gamma_t \geq -\frac{\pi_t}{\alpha}, \quad t \in T & (\mathrm{I}^{\mathrm{D}}.\mathrm{c}) \\ & \gamma_t - \delta_t + \pi_t \mu = 0, \quad t \in T & (\mathrm{I}^{\mathrm{D}}.\mathrm{d}) \\ & \gamma_t \leq 0, \quad t \in T & (\mathrm{I}^{\mathrm{D}}.\mathrm{e}) \\ & \mu \leq 0. & (\mathrm{I}^{\mathrm{D}}.\mathrm{f}) \end{aligned}$$

3 Joint Optimization Models for Broker-Investor Interactions

In this section we present different models for the interaction between brokers and investors. First, we consider bilevel optimization models in which either i) we have a leading broker with multiple investors acting after prices have been set or ii) we have a leading investor with multiple brokers setting prices once the investment made on each of them has been chosen. Although enriching the above models to include multiple leaders might lead to additional insights, the resulting optimization problems are substantially more complex to solve and go beyond the scope of this paper. On top of the aforementioned bilevel models, we also discuss a social welfare maximization problem, in which we look at the set of Pareto effcient

[7] Refer, for instance, to Chapter 6 in Bazaraa et al. (2004).

combinations of prices and portfolios. As we already argued in the Introduction, the model with a leading broker is the main object of interest in our analysis and the other two models are instrumental for comparison. In particular, the Pareto frontier represents a good benchmark to assess the quality of the outcomes obtained in the bilevel models.

3.1 B-L Model: One Broker-Leader, Several Investors-Followers

This model considers the situation where the broker makes his decision first and then a set M of investors, with possibly different degrees of risk aversion, choose their portfolios.[8] More precisely, once the leader's decision p is revealed, each investor $i \in M$ chooses his portfolio x^i. Investors may have different risk levels, α^i, and minimum expected returns, $E^i_{\min}$. The associated bilevel optimization problem is

$$\max_{p,(x^i,y^i,\eta^i,d^i)_{i\in M}} \quad \sum_{i\in M}\sum_{j\in S} p_j x^i_j \tag{B-L-bilevel}$$
$$\text{s.t.} \quad p \in P \tag{B$^{\mathrm{P}}$.a}$$
$$\forall i \in M,\ x^i \in \arg\min_{x^i,y^i,\eta^i,d^i} -\eta^i + \frac{1}{\alpha^i}\sum_{t\in T}\pi_t d^i_t \tag{I$^{\mathrm{P}}$}$$
$$\text{s.t.} \quad y^i_t = \sum_{j\in S}(r_{jt}x^i_j - p_j x^i_j), \quad t \in T \tag{I$^{\mathrm{P}}_0$.a}$$
$$\sum_{t\in T}\pi_t y^i_t \ge E^i_{\min} \tag{I$^{\mathrm{P}}_0$.b}$$
$$\sum_{j\in S} x^i_j = 1 \tag{I$^{\mathrm{P}}_0$.c}$$
$$x^i_j \ge 0, \quad j \in S \tag{I$^{\mathrm{P}}_0$.d}$$
$$d^i_t \ge \eta^i - y^i_t, \quad t \in T \tag{CVaR$^{\mathrm{P}}$.a}$$
$$d^i_t \ge 0, \quad t \in T. \tag{CVaR$^{\mathrm{P}}$.b}$$

The broker's leading problem has nested followers' subproblems in which investors choose their portfolios. Once the prices are set by the broker, the followers' subproblems are continuous linear programs. Hence, applying strong duality, we can obtain a single-level reformulation in which each subproblem is replaced by the feasibility constraints of the primal, (I$^{\mathrm{P}}$), the feasibility constraints of the dual, (I$^{\mathrm{D}}$), and the strong duality constraints, (B-L-SD),

[8] Recall that each investor type may itself be thought of as a set of investors sharing the same preferences.

namely, the equality between the objective functions of (I^{P}) and (I^{D}):

$$
\begin{aligned}
\max_{p,(x^i,y^i,\eta^i,d^i,\delta^i,\mu^i,\beta^i,\gamma^i)_{i\in M}} \quad & \sum_{i\in M}\sum_{j\in S} p_j x_j^i && \text{(B-L)}\\
\text{s.t.} \quad & p \in P, && (\mathrm{B}^{\mathrm{P}}.\mathrm{a})\\
& y_t^i = \sum_{j\in S}(r_{jt}x_j^i - p_j x_j^i), \quad t\in T,\ i\in M && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{a})\\
& \sum_{t\in T}\pi_t y_t^i \geq E_{\min}^i, \quad i\in M && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{b})\\
& \sum_{j\in S} x_j^i = 1, \quad i\in M, && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{c})\\
& x_j^i \geq 0, \quad j\in S,\ i\in M, && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{d})\\
& d_t^i \geq \eta^i - y_t^i, \quad t\in T,\ i\in M, && (\mathrm{CVaR}^{\mathrm{P}}.\mathrm{a})\\
& d_t^i \geq 0, \quad t\in T,\ i\in M && (\mathrm{CVaR}^{\mathrm{P}}.\mathrm{b})\\
& \beta^i + \sum_{t\in T}(r_{jt}-p_j)\delta_t^i \geq 0, \quad j\in S,\ i\in M && (\mathrm{I}^{\mathrm{D}}.\mathrm{a})\\
& \sum_{t\in T}\gamma_t^i = -1, \quad i\in M && (\mathrm{I}^{\mathrm{D}}.\mathrm{b})\\
& \gamma_t^i \geq -\frac{\pi_t}{\alpha^i}, \quad t\in T,\ i\in M && (\mathrm{I}^{\mathrm{D}}.\mathrm{c})\\
& \gamma_t^i - \delta_t^i + \pi_t\mu^i = 0, \quad t\in T,\ i\in M && (\mathrm{I}^{\mathrm{D}}.\mathrm{d})\\
& \gamma_t^i \leq 0, \quad t\in T,\ i\in M && (\mathrm{I}^{\mathrm{D}}.\mathrm{e})\\
& \mu^i \leq 0, \quad i\in M && (\mathrm{I}^{\mathrm{D}}.\mathrm{f})\\
& \eta^i - \frac{1}{\alpha^i}\sum_{t\in T}\pi_t d_t^i = \beta^i + E_{\min}^i\mu^i, \quad i\in M. && \text{(B-L-SD)}
\end{aligned}
$$

Problem (B-L) is a polynomial optimization problem, since there are quadratic terms in the formulation. Moreover, some of these terms appear in equality constraints such as ($\mathrm{I}_0^{\mathrm{P}}.\mathrm{a}$), which implies that (B-L) is a nonconvex optimization problem. Thus, local optimality does not imply global optimality, and the solution of this kind of problems requires the use of specialized global optimization solvers, as we discuss in Section 4.1.

3.2 I-L Model: One Investor-Leader, Several Brokers-Followers

This model deals with the less realistic situation in which the investor makes his decision first and a set K of brokers react optimally to the leader's decision. More precisely, the investor first decides how much to invest with each broker $k \in K$, x^k, so that $\sum_{k\in K}\sum_{j\in S} x_j^k = 1$ and then each broker sets prices. In this case, the bilevel model that represents this hierarchical situation is formulated as follows:

$$
\begin{aligned}
\min_{\eta,d,(x^k,y^k,p^k)_{k\in K}} \quad & -\eta + \frac{1}{\alpha}\sum_{t\in T}\pi_t d_t && \text{(I-L-BILEVEL)}\\
\text{s.t.} \quad & y_t^k = \sum_{j\in S}(r_{jt}x_j^k - p_j x_j^k), \quad t\in T,\ k\in K && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{a})\\
& \sum_{k\in K}\sum_{t\in T}\pi_t y_t^k \geq E_{\min} && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{b})\\
& \sum_{j\in S}\sum_{k\in K} x_j^k = 1 && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{c})\\
& x_j^k \geq 0, \quad j\in S,\ k\in K && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{d})\\
& d_t \geq \eta - \sum_{k\in K} y_t^k, \quad t\in T && (\mathrm{CVaR}^{\mathrm{P}}.\mathrm{a})\\
& d_t \geq 0, \quad t\in T && (\mathrm{CVaR}^{\mathrm{P}}.\mathrm{b})\\
& \qquad \forall k\in K,\ p^k \in \arg\max_{p^k}\sum_{j\in S} p_j^k x_j^k && (\mathrm{B}^{\mathrm{P}})\\
& \qquad\qquad \text{s.t.} \quad p^k \in P^k. && (\mathrm{B}^{\mathrm{P}}.\mathrm{a})
\end{aligned}
$$

This situation is similar to the one discussed for the B-L model. We have a bilevel problem such that, once the decisions of the investor are fixed, the resulting subproblems for the brokers are linear. Thus, we can again rely on strong duality to combine the feasibility constraints of problems (B^{P}) and (B^{D}) with the strong duality constraints, (I-L-SD), to transform the bilevel problem into an equivalent single level problem:

$$
\begin{aligned}
\min_{\eta,d,y,x} \quad & -\eta + \frac{1}{\alpha}\sum_{t\in T}\pi_t d_t && \text{(I-L)}\\
\text{s.t.} \quad & y_t^k = \sum_{j\in S}(r_{jt}x_j^k - p_j x_j^k), \quad t\in T,\ k\in K && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{a})\\
& \sum_{k\in K}\sum_{t\in T}\pi_t y_t^k \geq E_{\min} && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{b})\\
& \sum_{j\in S}\sum_{k\in K} x_j^k = 1 && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{c})\\
& x_j^k \geq 0, \quad j\in S,\ k\in K && (\mathrm{I}_0^{\mathrm{P}}.\mathrm{d})\\
& d_t \geq \eta - \sum_{k\in K} y_t^k, \quad t\in T && (\mathrm{CVaR}^{\mathrm{P}}.\mathrm{a})\\
& d_t \geq 0, \quad t\in T && (\mathrm{CVaR}^{\mathrm{P}}.\mathrm{b})\\
& p^k \in P^k, \quad k\in K && (\mathrm{B}^{\mathrm{P}}.\mathrm{a})\\
& (v^k)^\intercal A^k - (x^k)^\intercal = 0, \quad k\in K && (\mathrm{B}^{\mathrm{D}}.\mathrm{a})\\
& v_s^k \geq 0, \quad s\in P^{\mathrm{k,cons}},\ k\in K && (\mathrm{B}^{\mathrm{D}}.\mathrm{b})\\
& \sum_{j\in S} p_j^k x_j^k = \sum_{s\in P^{\mathrm{k,cons}}} b_s^k v_s^k, \quad k\in K. && \text{(I-L-SD)}
\end{aligned}
$$

This optimization problem is again a nonconvex polynomial optimization problem, whose solution requires the use of global optimization solvers.

3.3 SW Model: Social Welfare Maximization

In this section we study the situation in which the broker and the investor "cooperate" and try to jointly optimize their respective objective functions. We refer to this situation as the Social Welfare Model, SW. Despite being a less realistic model, it represents a good benchmark to get a better understanding of the efficiency of the outcomes obtained by the B-L model and facilitate its comparison with the outcomes of the I-L model. As a by-product, the model also serves to quantify the potential benefits of cooperation in this setting. Since one of the main goals of this model is to facilitate a comparison of the outcomes of the bilevel models, we assume that there is only one broker and one investor.

The classic approach to study this kind of multi-objective problems, introduced more than a century ago by Italian economist Vilfredo Pareto (refer to Pareto (1971) for an English translation), consists in characterizing the two dimensional Pareto frontier. Each coordinate in this set represents one of the objective functions to optimize and the Pareto efficient points are those feasible pairs (z_1, z_2) such that, for any other feasible pair in which one of the agents is better off, the other agent is worst off. A standard approach to compute the Pareto frontier consists in optimizing with respect to one of the objective functions while requiring that the other attains a minimum level. For each value of the minimum level we obtain a Pareto efficient allocation, whereas the full Pareto frontier is generated by varying the minimum level within the range of possible values for the corresponding objective function. Therefore, if we let B_0 be a minimum level of the broker's profit, we are interested in the following optimization problem:

$$\max_{\eta,d,y,x,p} \ \eta - \frac{1}{\alpha}\sum_{t\in T}\pi_t d_t \tag{SW$_{B_0}$}$$

$$\text{s.t.} \ \sum_{j\in B} p_j x_j \geq B_0 \tag{SW$_{\mathrm{PE}}$}$$

$$y_t = \sum_{j\in S} r_{jt}x_j - \sum_{j\in S} p_j x_j, \quad t \in T, \tag{I$_0^{\mathrm{P}}$.a}$$

$$\sum_{t\in T}\pi_t y_t \geq E_{\min}, \tag{I$_0^{\mathrm{P}}$.b}$$

$$d_t \geq \eta - y_t, \quad t \in T, \tag{CVaR$^{\mathrm{P}}$.a}$$

$$d_t \geq 0, \quad t \in T, \tag{CVaR$^{\mathrm{P}}$.b}$$

$$\sum_{j\in S} x_j = 1, \tag{I$_0^{\mathrm{P}}$.c}$$

$$x_j \geq 0, \quad j \in S, \tag{I$_0^{\mathrm{P}}$.d}$$

$$p \in P. \tag{B$^{\mathrm{P}}$.a}$$

In the above model, the objective function together with constraints (CVaR$^{\mathrm{P}}$.a) and (CVaR$^{\mathrm{P}}$.b), correctly define the CVaR of the returns y_t. Constraint (B$^{\mathrm{P}}$.a) ensures the feasibility of the prices and constraints (I$_0^{\mathrm{P}}$.b), (I$_0^{\mathrm{P}}$.c), and (I$_0^{\mathrm{P}}$.d) ensure that the chosen portfolio is feasible and achieves an expected return greater than $E_{\min}$. Then, constraints (I$_0^{\mathrm{P}}$.a) contain the interaction between prices and returns. Finally, constraint (SW$_{\mathrm{PE}}$) ensures the minimum level of the broker's profit and by varying B_0 we can construct the Pareto frontier. Note that there is a Pareto frontier for each value of $E_{\min}$.

Yet again, we are confronted with a nonconvex polynomial optimization problem, whose solution requires the use of global optimization solvers.

4 Numerical Results: Case Study for the Dow Jones Index

This section is devoted to present the numerical results associated to a case study that builds upon real data taken from the Dow Jones Index and try to hint at potential economic insights that might be worth studying further. In Section 4.1 below we start by discussing how the parameters required by the optimization models presented in Section 2 and Section 3 are obtained from the data on the Dow Jones Index. Further, we also discuss how the resulting instances of these models have been solved by using state-of-the-art global optimization solvers.

Once the data and models for the case study are in place, we proceed to discuss the results. First, we separately analyze the B-L model in Section 4.2. Next, we present the results for the I-L model and a comparison of the results for the two models in Section 4.3. Finally, in Section 4.4 we discuss the results for the SW model, along with some concluding comments on the overall analysis.

It is important to note that we have performed similar analyses to the one we present below, but taking alternative financial indices (such as the Spanish IBEX) and different estimates on future realizations of the returns of their securities. We have found that the qualitative results and economic insights that we present below for the Dow Jones' companies also appeared quite consistently in all other analyses.

4.1 Instances and Solution Technique

There are two main ingredients common to all the models we have discussed: the set of securities in which to invest and the set of constraints P, that limits the fees the broker can impose on those securities. Regarding the latter, for the case study in this section we assume that the set P is given by the following constraints:

$$\begin{array}{rcll} \sum_{j\in S} p_i & \leq & 0.3 & \\ p_j & \leq & 0.1, & j \in S \\ p_j & \geq & 0, & j \in S. \end{array}$$

The above constraints are just a simple example of the polyhedral set P, which facilitates the ensuing analysis. There is a maximum fee that can be imposed on each security and there is also a limit on the sum of the fees on the different securities. One natural consequence of these constraints is that the broker prefers to face investors which do not diversify much. If the investment is made on three or less securities, then the broker can get a profit of 0.1 by putting a fee of 0.1 in all those securities. On the contrary, if the investor puts his money in more than three securities, then the broker cannot go for a maximum fee of 0.1 in all of them and has to choose on which ones to concentrate. We do not claim that this choice for the set P is particularly realistic, but our approach could be readily applied to any other polyhedral set P.[9] As we have already argued, in actual cases these constraints may be imposed by a regulator, by some board of the company based on considerations regarding their competitors, or by both of them.

Regarding the set of securities, we consider those associated with the 30 companies of the Dow Jones Index: AAPL, AXP, BA, CAT, CSCO, CVX, DIS, DWDP, GS, HD, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, NKE, PFE, PG, TRV, UNH, UTX, V, VZ, WBA, WMT, and XOM.[10]

For the above securities we have tracked the weekly market return during a six-month period, from August 15th, 2018, to March 17th, 2019. This results in a total of 30 vectors of joint realizations for the securities under study. Then, we have taken these past realizations and assumed them to be equally likely future scenarios. We do not claim that these estimates for future realizations represent optimal or accurate predictions in any sense. We

[9] In particular, the usual market functioning in which the fee has to be common to all assets is achieved, for instance, by adding the constraints $p_i = p_1$ for all $i \neq 1$. Adding these constraints to the set P in our case study would result in a straightforward decision problem for the broker, in which he would evenly split his "total budget", 0.3, among the different assets. The resulting outcomes would be worse for the broker than those obtained in our analysis, where he can associate different fees to the different assets.

[10] Information on the associated companies can be checked, for instance, at https://finance.yahoo.com/.

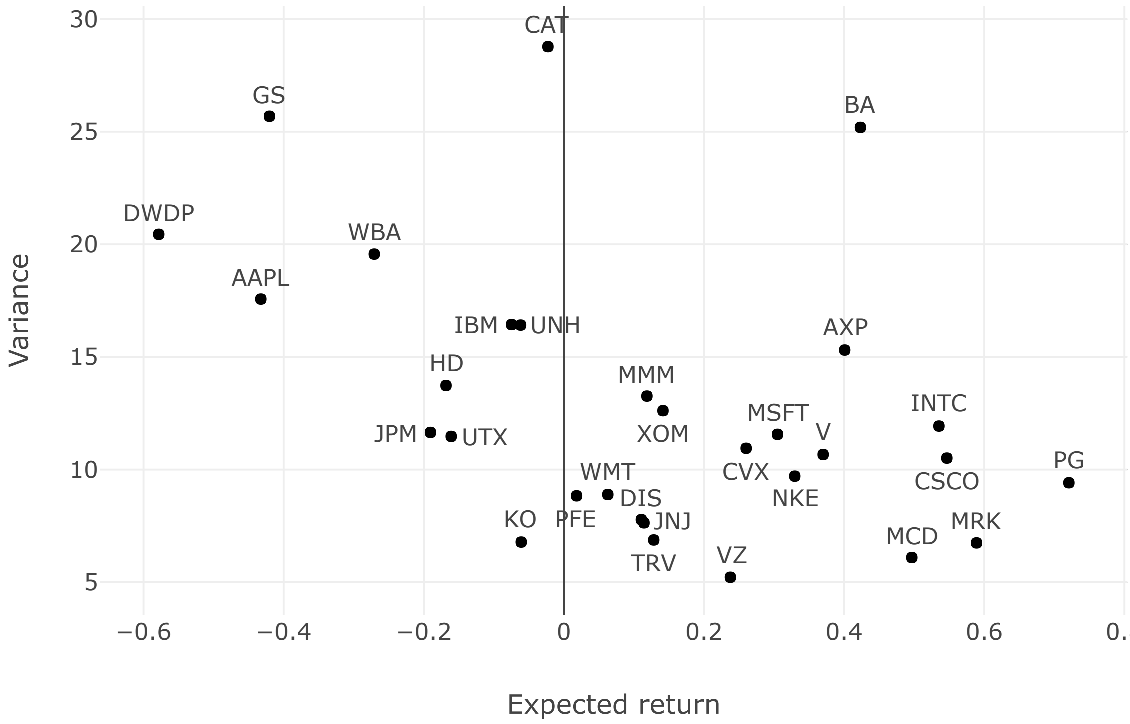


Figure 1: Expected return and variance for the Dow Jones securities.

just consider they represent realistic scenarios (they have appeared in the past) which, for the sake of our analysis, ease the exposition and deliver similar qualitative results to those that would be obtained after more rigorous scenario estimations based, for instance, on time series techniques. In Figure 1 we represent the expected value and variance of the R_j random variables associated with each of the 30 Dow Jones securities in the 30 scenarios we have considered.

The mathematical programming problems associated to the "one level" formulations of the B-L, I-L, and SW models have been formulated in AMPL modeling language (Fourer et al., 1990). Then, each of the resulting instances has been solved on a server of ITMATI[11] using two global optimization solvers: BARON (Tawarmalani and Sahinidis, 2005; Sahinidis, 2019), a general solver for mixed integer nonlinear programming problems, and RAPOSa (González-Rodríguez et al., 2020), a solver that builds upon the Reformulation Linearization Technique (Sherali and Tuncbilek, 1992) and that has been specifically designed for polynomial programming problems, a class to which all the models discussed in this paper belong to.[12] Further, we have analyzed the issue of the scalability of our methodology. In this regard, we have compared the above solution approach with different methodologies built in local solvers: Ipopt (Wächter and Biegler, 2006), Knitro (Byrd et al., 2006) and MINOS (Murtagh and Saunders, 1978). The results are reported in Appendix A. As expected, global optimization solvers such as BARON and RAPOSa deliver the best solutions for small instances but, as problem sizes increase, they find it harder to close the optimality gap and global optimality is no longer guaranteed. In spite of that, BARON finds in most of the cases the best solution. On the other hand, local optimizers are much faster and always deliver solutions without even approaching the time limit (one hour). With the exception of MINOS, which performs rather poorly, the quality of the solutions obtained by these local solvers is quite acceptable, with Knitro being competitive with BARON for the largest instances.

[11] Technological Institute for Industrial Mathematics (http://www.itmati.com/en).

[12] The AMPL files associated to the models and data discussed in this case study can be downloaded from `http://shorturl.at/chv17`.

There are a couple of important parameters associated with all the models we have presented, so it is important to clarify their values in the analysis:

Risk profile of the investor. We consider four different risk profiles for the investors, 0.05, 0.25, 0.50, and 0.99, with the first one being the most risk averse and the last one being essentially an expected utility maximizer.

Minimum expected return. In all the optimization models there is a constraint that sets the minimum expected return, $E_{\min}$, that the investor is willing to accept in order to make his investment. What we do in our analysis is to solve a set of 32 problems associated with different values of $E_{\min}$ ranging from -0.1 to 0.72 (the highest expected return of any security).

4.2 B-L Model

We start discussing the results assuming that there is only an investor profile which, as we have argued before, can be thought of as a continuum of agents with the same degree of risk aversion. Next, we present results for the model in which the population of followers is composed of investors with different degrees of risk aversion.

4.2.1 B-L Model. One Follower

For each risk profile $\alpha \in \{0.05, 0.25, 0.50, 0.99\}$, we solve the associated B-L model. In Figure 2 we present, for each value of α, a line representing the objective function CVaR_α for each value of $E_{\min}$. This figure seems to suggest a counterintuitive result. Namely, for all levels of $E_{\min}$, the more risk averse the investor is, the more risk he must assume in the optimal solution. However, note that each line represents the CVaR_α associated with the optimal solution for that level α, so the degrees of riskiness implied by the different lines are not really comparable. For instance, in the line associated with type $\alpha = 0.05$ we can see that, in order to get an expected return of 0.2, the investor has to be willing to risk an average loss of almost 3 in the tail with the worst 5% realizations (given his security choices).

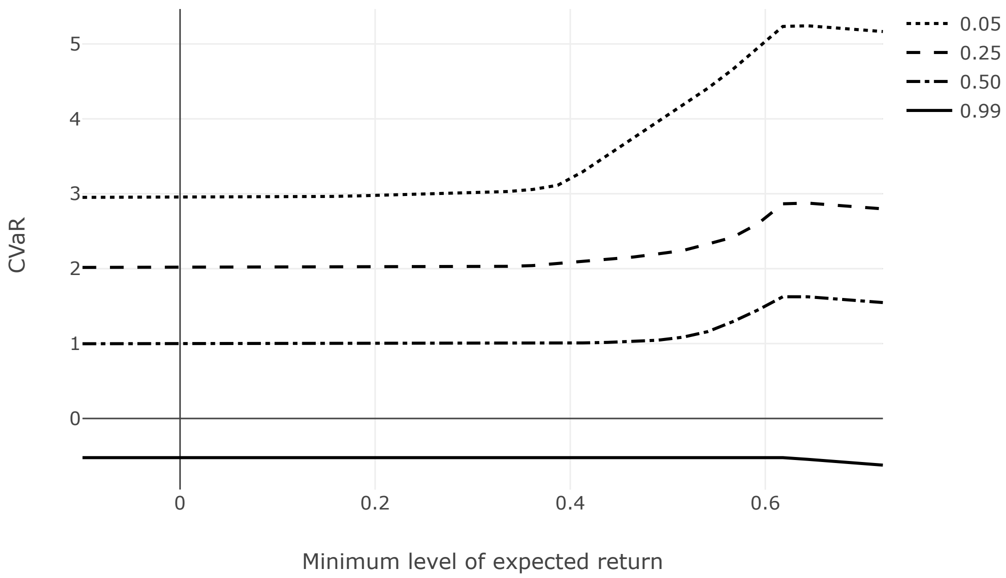
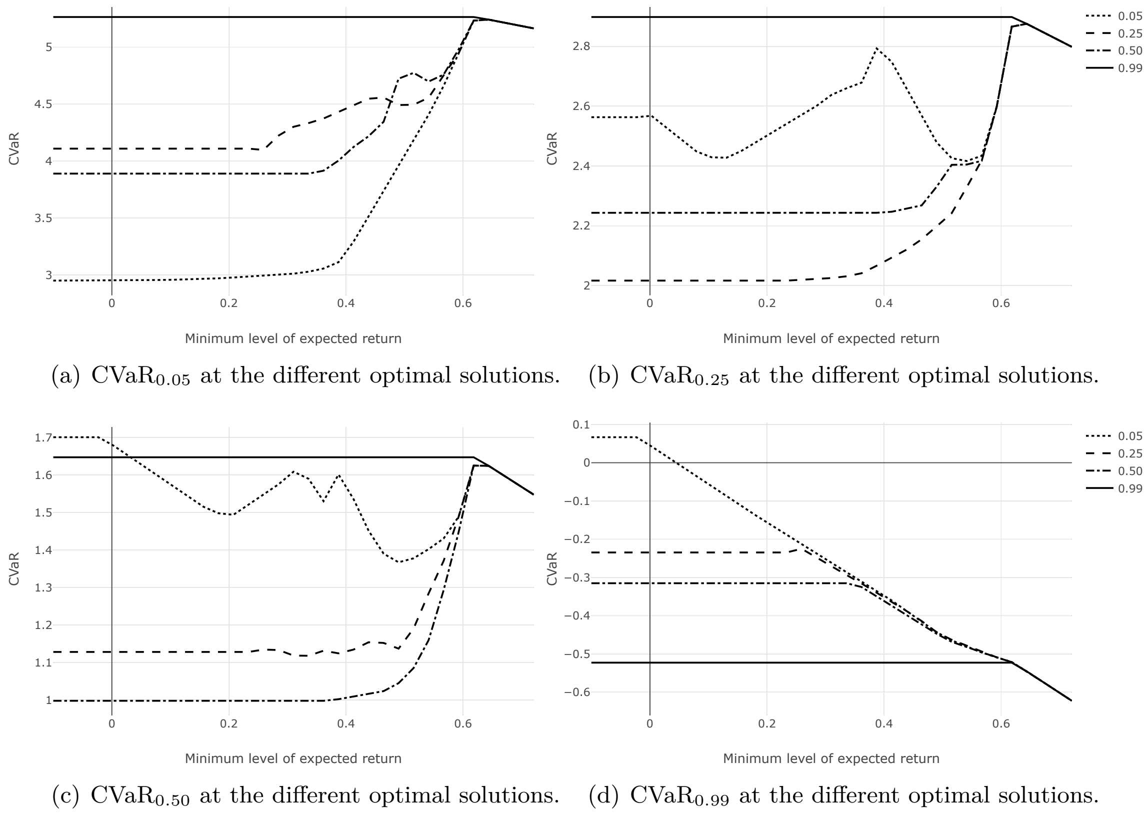


Figure 2: CVaR_α for each risk profile and each value of $E_{\min}$.

In order to obtain comparable results across risk profiles, in Figure 3 we present four subfigures, one for each value of α. For instance, Figure 3(a) contains the $\mathrm{CVaR}_{0.05}$ associated to the optimal solutions of the different risk profiles (again, for each level of $E_{\min}$). We now find, as expected, that the line corresponding to each risk profile α is the one minimizing the corresponding CVaR_α. Interestingly, setting aside the 0.99 type, which is usually the one assuming the highest risk according to all other risk profiles (he just goes for the security with the highest expected return regardless of the value of $E_{\min}$), there is no clear pattern characterizing how the other risk profiles are ranked. For instance, in Figure 3(a), the solutions for type 0.25 have a higher $\mathrm{CVaR}_{0.05}$ than the solutions for type 0.50 for values of $E_{\min}$ below 0.45, but the situation reverses after that point.

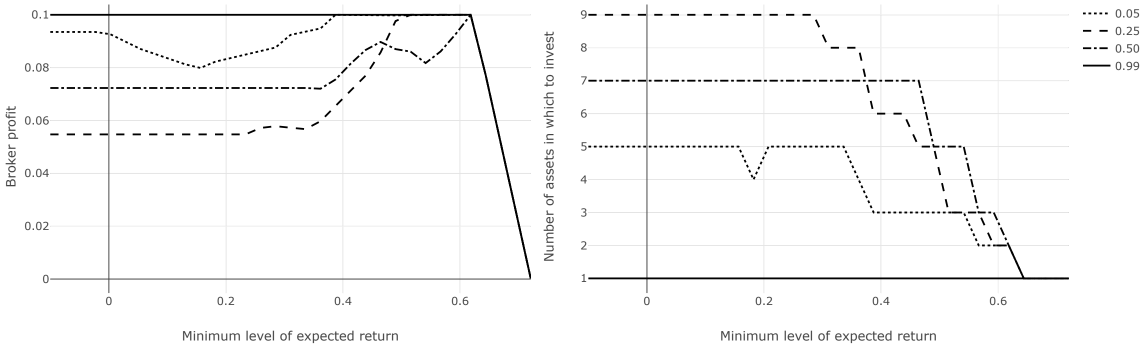


(a) $\mathrm{CVaR}_{0.05}$ at the different optimal solutions. (b) $\mathrm{CVaR}_{0.25}$ at the different optimal solutions.

(c) $\mathrm{CVaR}_{0.50}$ at the different optimal solutions. (d) $\mathrm{CVaR}_{0.99}$ at the different optimal solutions.

Figure 3: Comparison of CVaR levels at the optimal solutions of the different risk profiles.

Going back to Figure 2, we can also see that, for each given risk profile, the CVaR at optimality remains constant for small values of $E_{\min}$, the reason being that the associated constraint is not demanding (indeed, not even binding at optimality). Once the threshold on the minimum level of expected return starts to matter, we see that the risk associated with the optimal solutions starts to increase; the risk-return trade-off starts to kick in. Then, once the value of $E_{\min}$ gets larger that 0.62 we see that the risk starts decreasing for all risk types, which may be counterintuitive at first. This is because, for sufficiently large values of $E_{\min}$, to get a large enough expected return, the investor has to put all his money in the security PG, the one with the highest return, 0.72 (see Figure 1). The broker anticipates this investment and puts a fee of 0.1 on security PG, resulting in an expected return of 0.62 for the investor. Beyond that level, in order to let the agent have feasible solutions in his "subproblem", the broker must reduce the fees and, indeed, we see in Figure 4(a) how the broker's profit decreases from that point onwards. We can also see in Figure 4(a) that for moderate values of $E_{\min}$, those at which the associated constraint is already demanding, the profit of the broker tends to be increasing. The reason for this is that the investor cannot diversify so much in order to get the target expected return and becomes more predictable for the broker. This can be seen in Figure 4(b), where we see that the number of securities in which the agent invests decreases as $E_{\min}$ increases (until eventually the investment is concentrated on security PG).



(a) Broker profit at optimality.

(b) Number of securities in the optimal portfolio.

Figure 4: Broker profit and number of securities at optimality for different risk profiles.

To conclude the discussion regarding the B-L model with one follower, note that the broker seems to prefer to face investors with "extreme" degrees of risk aversion (0.05 and 0.99 in our analysis), since these types of investors tend to have less "appealing" securities in which to invest, making them more predictable (see Figure 4).

4.2.2 B-L Model. Multiple Followers

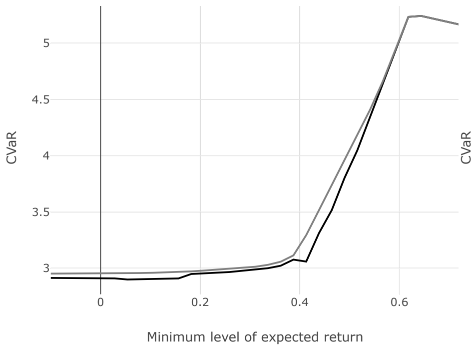
In this section we solve again the B-L model, but now with different investor profiles choosing simultaneously their portfolios at the second level, once the broker has fixed the fees on the different securities. We consider three different profiles, with $\alpha \in \{0.05, 0.50, 0.99\}$. The main reason for not including $\alpha = 0.25$ in the analysis is that the optimization problem with three investors on the second level is already quite involved, with the optimizers experiencing already some minor numerical difficulties. Most instances required more than one hour to be solved to optimality and adding a fourth profile just made time requirements even more demanding and numerical issues more accused.[13]

Since the qualitative results are analogous to those for the case with a single follower, we have moved the associated figures to Appendix B (figures 14, 15, and 16). The interesting part of the analysis comes when we compare the results obtained for the two cases: single follower and multiple followers.

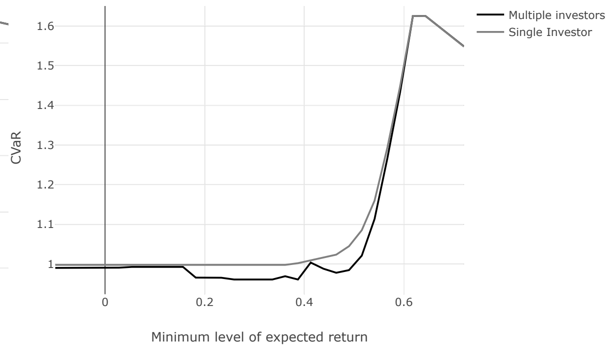
Intuitively, the situation with investors with different risk profiles acting simultaneously in the second level should be beneficial for them and reduce the profit of the broker, since investors with different profiles may go for different portfolios and the broker cannot put a maximal fee of 0.1 in all the involved securities. In the model with a single investor, the broker could tailor his fees specifically to each risk profile, and now has to divide his "total budget" of 0.3 among them. Accordingly, in Figure 5 we can see that, when the investors act independently, they must assume higher risks (higher CVaR) to ensure a given expected return. Further, this difference disappears when $E_{\min}$ is sufficiently large so that all the investors must essentially go for security PG and then the CVaR values for both models coincide.

Figure 6 shows the profit for the broker which, as expected, is smaller in the multiple follower model. The effect is very clean for the investor with risk profile $\alpha = 0.99$. Since this investor just wants to go for security PG regardless of the value of $E_{\min}$, in the case of a single follower the broker can extract for him his maximum possible benefit, 0.1. However, in

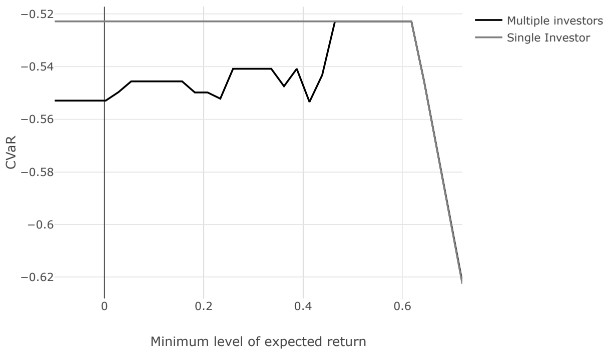
[13]Indeed, the reader will notice that, in the figures in this section, the lines for the model with multiple followers are not as smooth as the ones for the model with a single follower. This is because the precision obtained in the final solutions is smaller in the former.



(a) $\text{CVaR}_{0.05}$ risk profiles.



(b) $\text{CVaR}_{0.50}$ risk profiles.



(c) $\text{CVaR}_{0.99}$ risk profiles.

Figure 5: CVaR_α for the models with one follower and multiple followers.

the case with multiple followers, the broker cannot extract so much profit from this investor, because of the trade-off with the fees on the securities of the other investors. Additionally, we can see that, although the profit of the broker tends to increase with the value of $E_{\min}$, the profit is not a monotone function of it. The reason for this is that, for relatively small values of $E_{\min}$, it is not so clear how the diversification on the different securities will affect the benefit of the broker (it is not just a matter of the total number of securities in which the investor invests, but also about how evenly the money is split among them).[14]

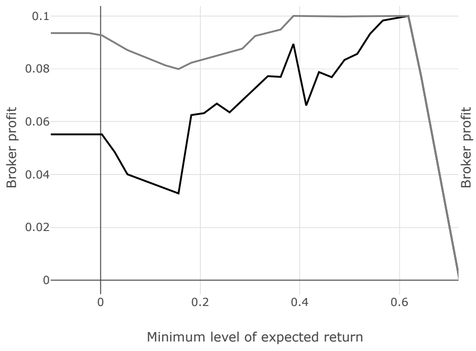
The above discussion of the B-L model hints at the following insight: given a market with multiple brokers and multiple investors, even if each of them can choose asset specific fees, as long as these fees cannot be different for different investors, then investors are better-off by concentrating all on the same broker, limiting his power to tailor the fees to specific risk profiles. Validating such an insight would require the analysis of a model with multiple leading brokers, which goes beyond the scope of this paper.

4.3 I-L Model and Comparison of B-L and I-L Models

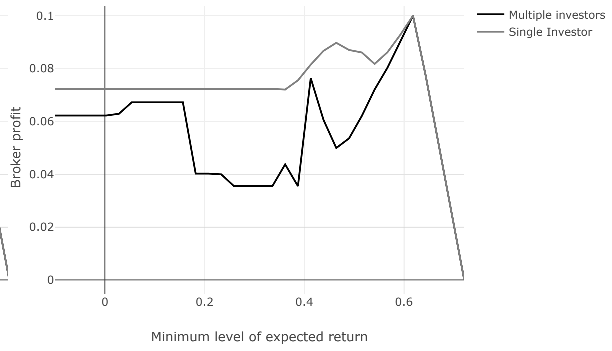
We now move to the setting in which the investor is the leader, with the broker choosing the fees with full knowledge of the investment made on each security. Arguably, this is a less realistic model which, intuitively, should favor the broker. For the sake of exposition, we present the figures that allow to compare the B-L and I-L models. Before proceeding, it is worth discussing a couple of "technical" differences between the results obtained for the different models:

- For the I-L model we do not consider the case of multiple followers. The reason is that,

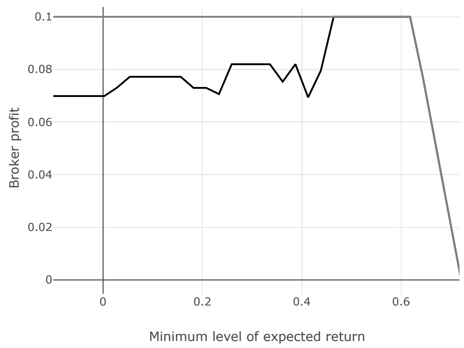
[14] For the sake of completeness, in Figure 15(b) in Appendix B we represent the number of securities in which each risk profile invests as a function of $E_{\min}$.



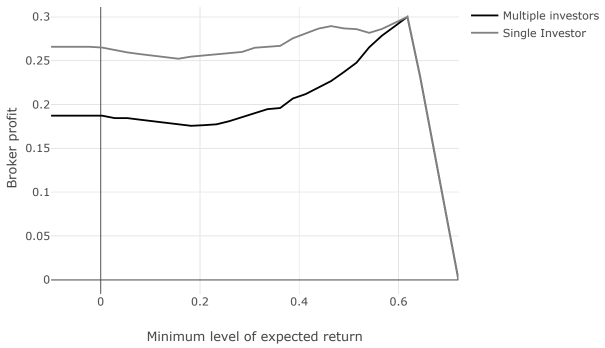
(a) Profit from $\mathrm{CVaR}_{0.05}$ risk profiles.



(b) Profit from $\mathrm{CVaR}_{0.50}$ risk profiles.



(c) Profit from $\mathrm{CVaR}_{0.99}$ risk profiles.



(d) Total profit in each model.

Figure 6: Profit of the broker for the models with one follower and multiple followers.

as long as all the brokers face the same constraints, there is no additional richness and the investor cannot do better than what he would do in the model with a single follower. On the other hand, although allowing for different feasible sets for the different brokers might lead to different results and additional insights, such an analysis is beyond the scope of this paper.

- The value of $E_{\min}$ is taken in the interval $[-0.1, 0.62]$ (recall that in the B-L model the interval was $[-0.1, 0.72]$). This difference is driven by the order in which the investor and the broker take their actions. In the I-L model, once the investor has chosen his portfolio, the broker will choose fees in order to maximize his profit. In particular, in order to get an expected value of at least 0.62, the investor has to put all the money in security PG (which has an expected return of 0.72), after which the broker would associate a 0.1 fee to security PG, leading to an expected return of precisely 0.62. The situation in the B-L model was different since the broker chooses first and can "credibly" let the investor gain more than 0.62 by associating a fee smaller than 0.1 to security PG.

We start the analysis with Figure 7, which contains a comparison of the CVaR values obtained for the I-L (gray) and B-L (black) models and shows that, for the investor, both models result in essentially the same objective function. Yet, if we were to zoom in into the different lines, we would observe that the CVaR for the B-L is often slightly higher than the one for the I-L model (and never lower). Qualitatively this is quite natural, since the broker chooses his fees being fully informed of the investor's portfolio, the investor's objective function gets worse with respect to the B-L model.

Interestingly, although the difference in the CVaR values is very small, the difference is more noticeable in the associated portfolios. In particular, it seems that in the I-L model the investor tends to diversify more in order to reduce the fees paid to the broker (even if

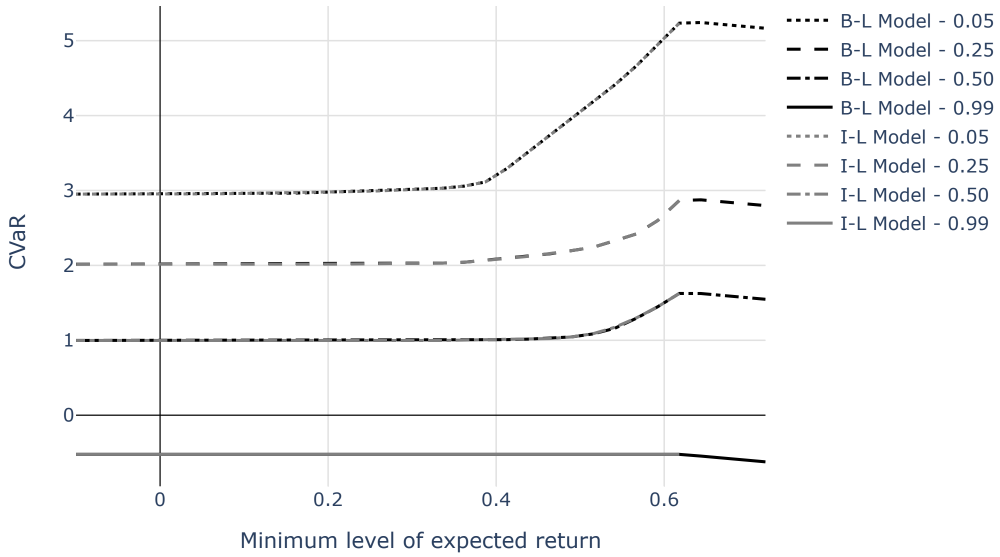
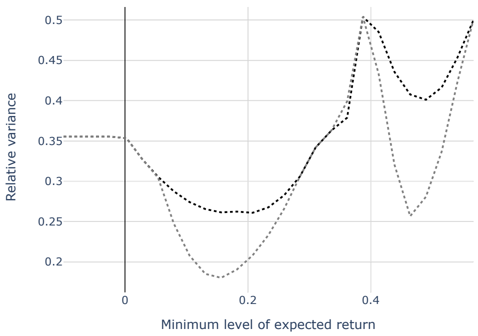
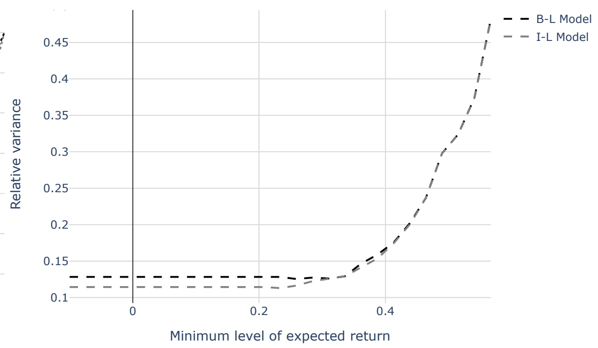


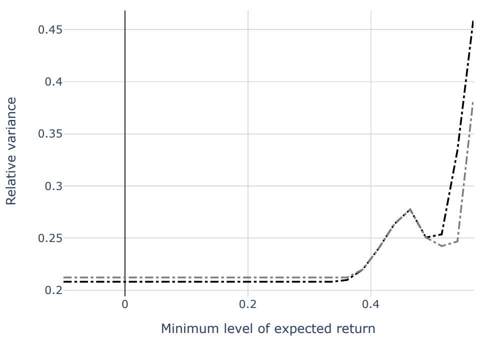
Figure 7: CVaR_α for each risk profile and each value of $E_{\min}$ in the B-L and I-L models.



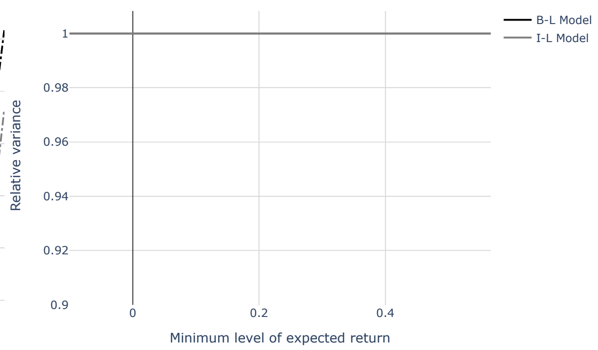
(a) $\mathrm{CVaR}_{0.05}$ risk profiles.



(b) $\mathrm{CVaR}_{0.25}$ risk profiles.



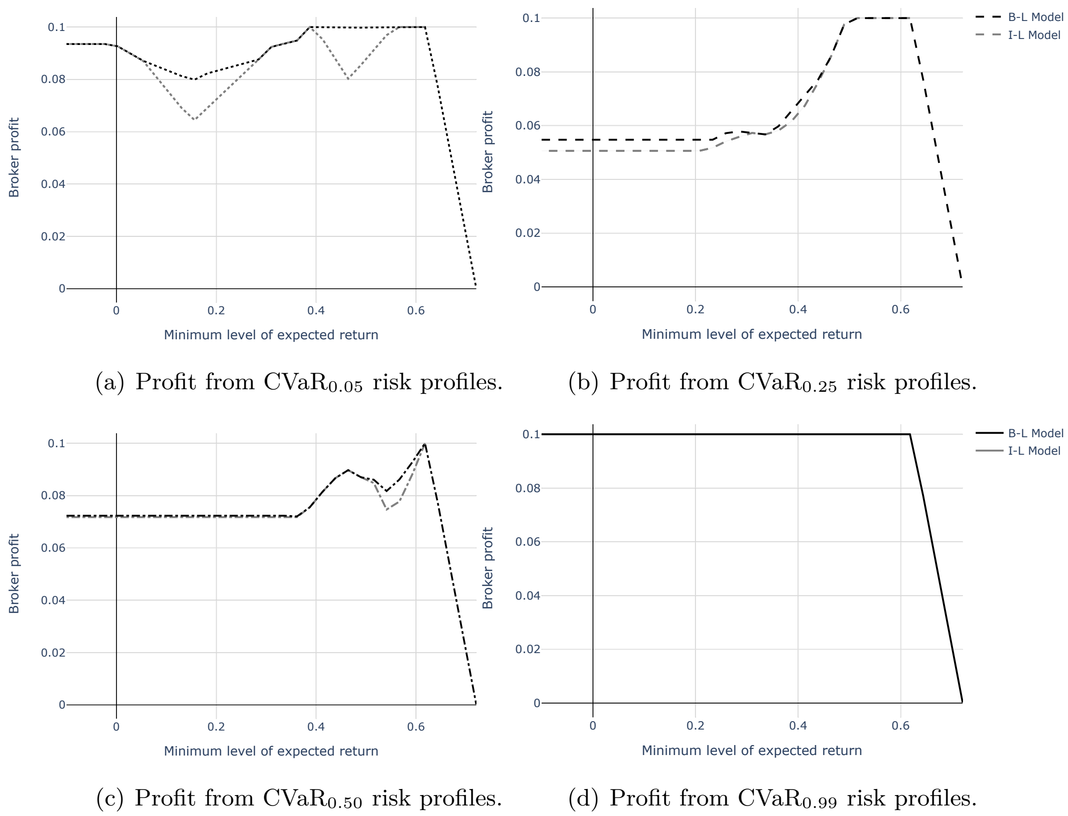
(c) $\mathrm{CVaR}_{0.50}$ risk profiles.



(d) $\mathrm{CVaR}_{0.99}$ risk profiles.

Figure 8: Relative variances of the optimal portfolios in B-L and I-L models.

it requires to choose a portfolio with slightly worse returns). This difference is illustrated in Figure 8, where we represent the relative variance of the optimal portfolios. More precisely, the variance of portfolio x is computed as $\sum_{j\in S}(x_j - 1/30)^2$. In this case, the largest possible variance is achieved when all the investment is made on the same asset, resulting on a value of 0.0322.[15] On the other hand, the variance would be 0 if all assets get investment 1/30, which corresponds with full diversification. In Figure 8 we represent the relative variances of the portfolios, computed as the variances divided by 0.0322, the maximum variance. Since the smaller relative variance, the larger the diversification, the figure shows that, indeed, the I-L tends to exhibit more diversification. Finally, this additional diversification in the I-L model should also result in a decrease of the profit of the broker, which we show in Figure 9.



(a) Profit from $\mathrm{CVaR}_{0.05}$ risk profiles. (b) Profit from $\mathrm{CVaR}_{0.25}$ risk profiles.

(c) Profit from $\mathrm{CVaR}_{0.50}$ risk profiles. (d) Profit from $\mathrm{CVaR}_{0.99}$ risk profiles.

Figure 9: Profit of the broker in B-L and I-L models.

The above discussion and the associated figures show that the outcomes of the B-L model Pareto dominate those of the I-L model. Thus, not only the B-L model seems more realistic, but also seems to lead to outcomes that are more efficient from the social point of view.

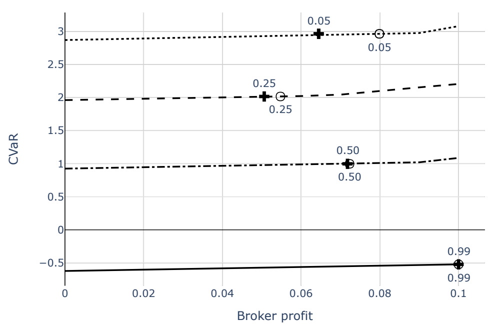
4.4 SW Model

The final discussion in the previous section makes it natural to study the SW model in order to understand not only how the outcomes of the B-L and I-L models relate to each other in terms of Pareto domination, but also how they perform in terms of Pareto efficiency.

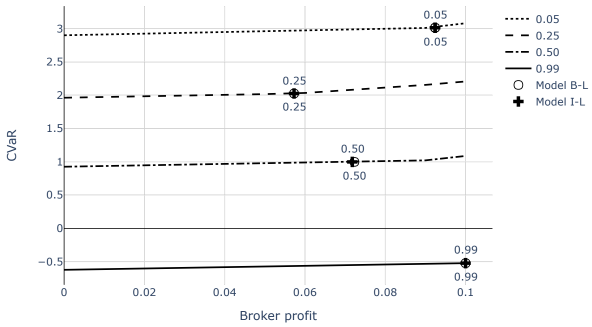
Recall that if we fix a value for $E_{\min}$ then, varying the value of B_0 in model (SW_{B_0}), we obtain the Pareto frontier associated with minimum return $E_{\min}$. In Figure 10 we present the Pareto frontiers for four different values of $E_{\min}$, evenly distributed among those values

[15] Given a feasible portfolio $x \in [0,1]^{30}$, the condition $\sum_{j\in S} x_j = 1$ implies that the average investment is 1/30. Thus, when the portfolio concentrates on a single asset we get a term of the form $(1-1/30)^2$ which leads to the highest possible variance.

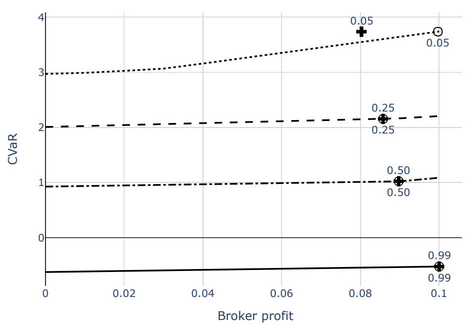
in the grid used in the previous sections (0.156, 0.310, 0.464, and 0.617). In this figure we also represent the outcomes obtained in the B-L and I-L models for the same values of $E_{\min}$, so we can get a better understanding of the efficiency of these models.



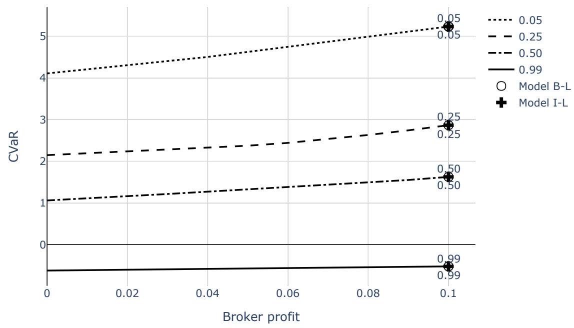
(a) Pareto frontier for $E_{\min} \approx 0.156$.



(b) Pareto frontier for $E_{\min} \approx 0.310$.



(c) Pareto frontier for $E_{\min} \approx 0.464$.



(d) Pareto frontier for $E_{\min} \approx 0.617$.

Figure 10: Pareto frontiers and solutions for B-L and I-L models.

As expected, the results in Figure 10 confirm the findings in the previous section. The outcomes of B-L and I-L models are generally close to each other but, for some instances, the B-L outcome Pareto dominates the I-L outcome. The clearest cases in Figure 10 are obtained for $E_{\min} = 0.156$ and $E_{\min} = 0.464$ with $\alpha = 0.05$. Last, but not least, note also that the B-L outcomes always lie in the Pareto frontier, whereas the I-L outcomes are sometimes inefficient. Thus, our numerical results show some form of domination of the B-L model with respect to the I-L model. An interesting line of research, that goes beyond the scope of this paper, would be to explore to what extent these numerical results can be backed up with theoretical results, not only for the models discussed in this paper but, possibly, with respect to more general versions of them.

As a conclusion of the discussion in the last two sections on the comparison of the B-L model with the I-L and SW models, we want to highlight once again the fact that it is precisely the most realistic model, B-L, the one leading to efficient outcomes. This is in a sense reassuring for the way in which these markets currently operate. In particular, the results suggest that the I-L model might lead to inefficient markets in which investors might choose to “give up” some rent in order to be less predictable.

5 Conclusions

This paper deals with a single-period portfolio selection problem with transaction cost and two levels of decision-making. On the one hand, there is a broker that controls the fee to be charged to the different securities in order to maximize his benefit and, on the other hand, there is an investor that chooses his portfolio trying to minimize the risk while ensuring a certain expected return. This structure gives rise to an implicit competition in order to anticipate the rational decision of the other party so as to optimize the decision-makers' own criteria. We have presented different models depending on the order in which choices are made. This produces three situations: i) the main one, in which the broker is leader and the investor follows with his decision (B-L model), ii) the investor acts first and the broker is the follower (I-L model), and iii) the social welfare model (SW) in which both parties collaborate to maximize the aggregated objective functions. We have developed mathematical programming formulations to model the three of them. As opposed to the models in Leal et al. (2020), we have assumed continuous sets of possible transaction costs, which leads to nonlinear and nonconvex optimization problems. The first two are bilevel programs that can be reformulated into single level polynomial optimization problems and the third model also belongs to this class. We solved them using two global optimization solvers: BARON and RAPOSa. To illustrate the economic insights that can be gained with these models, we have developed a case study based on data from the Dow Jones index with weekly market returns during a six months period from August 15th, 2018, to March 17th, 2019. These results report a comparison among the models from different angles. One of the most interesting findings of our analysis is that there seems to be some form of domination of the B-L model over the I-L model with respect to the broker-dealer profit and the CVaR risk obtained by the two decision-makers, with the B-L model delivering Pareto efficient outcomes. This paper opens up some lines for future research such as the extensions to multiple leader-follower models, the consideration of multiple period situations, and the study of whether or not some of the economic insights from the case study can lead to theoretical results, not only for the models discussed in this paper, but also to some of their extensions mentioned above.

Acknowledgements

We are grateful to two anonymous referees for their constructive suggestions. The research of the first and second authors has been funded by FEDER and the Spanish Ministry of Science and Technology through projects MTM2014-60191-JIN and MTM2017-87197-C3-3-P. The research of the third and fourth authors has been funded by the Spanish Ministry of Science and Technology through project MTM2016-74983-C2-1-R, and projects FEDER-US-1256951, CEI-3-FQM331 and *NetmeetData*: Ayudas Fundación BBVA a equipos de investigación científica 2019. Brais González-Rodríguez acknowledges support from the Spanish Ministry of Education through FPU grant 17/02643.

References

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1997): "Thinking coherently," *Risk*, 10, 68–91.

——— (1999): "Coherent measures of risk," *Mathematical Finance*, 9, 203–227.

Baule, R. (2010): "Optimal portfolio selection for the small investor considering risk and transaction costs," *OR Spectrum*, 32, 61–76.

Baumann, T. and N. Trautmann (2013): "Portfolio-optimization models for small investors," *Mathematical Methods of Operations Research*, 77, 345–356.

Bazaraa, M. S., J. J. Jarvis, and H. D. Sherali (2004): *Linear Programming and Network Flows*, Wiley-Interscience.

Benati, S. (2003): "The optimal portfolio problem with coherent risk measure constraints," *European Journal of Operational Research*, 150, 572–584.

——— (2015): "Using medians in portfolio optimization," *Journal of the Operational Research Society*, 66, 720–731.

Benders, J. F. (1962): "Partitioning procedures for solving mixed-variables programming problems," *Numerische Mathematik*, 4, 238–252.

Bey, R. P. and K. M. Howe (1984): "Gini's Mean Difference and Portfolio Selection: An Empirical Evaluation," *Journal of Financial and Quantitative Analysis*, 19, 329–338.

Birge, J. R. and F. Louveaux (2011): *Introduction to Stochastic Programming*, Springer-Verlag, Berlin-Vienna.

Byrd, R. H., J. Nocedal, and R. A. Waltz (2006): "Knitro: An Integrated Package for Nonlinear Optimization," .

Castro, F., J. Gago, I. Hartillo, J. Puerto, and J. M. Ucha (2011): "An algebraic approach to Integer Portfolio problems," *European Journal of Operational Research*, 210, 647–659.

Cesarone, F. (2020): *Computational Finance. MatLab oriented modeling*, Routledge, Rome.

Cesarone, F., A. Scozzari, and F. Tardella (2020): "An optimization–diversification approach to portfolio selection," *Journal of Global Optimization*, 76, 245–265.

Fourer, R., D. M. Gay, and B. W. Kernighan (1990): "AMPL: A Mathematical Programing Language," *Management Science*, 36, 519–554.

González-Rodríguez, B., J. Ossorio-Castillo, J. González-Díaz, A. M. González-Rueda, D. Rodríguez-Penas, and D. Rodríguez-Martínez (2020): "Computational advances in polynomial optimization: RAPOSa, a freely available global solver," Tech. rep., Optimization Online, 7942.

Grimm, V., T. Kleinert, F. Liers, M. Schmidt, and G. Zöttl (2019): "Optimal price zones of electricity markets: a mixed-integer multilevel model and global solution approaches," *Optimization methods and software*, 34, 406–436.

Kellerer, H., R. Mansini, and M. G. Speranza (2000): "Selecting portfolios with fixed cost and minimum transaction lots," *Annals of Operations Research*, 99, 287–304.

King, A. J. and S. W. Wallace (2012): *Modeling with Stochastic Programming*, Springer-Verlag, Berlin-Vienna.

Kolm, P. N., R. Tutuncu, and F. J. Fabozzi (2014): "60 Years of portfolio optimization: Practical challenges and current trends," *European Journal of Operational Research*, 234, 356–371.

Konno, H. and H. Yamazaki (1991): "Mean-Absolute Deviation Portfolio Optimization Model and Its Applications to Tokyo Stock Market," *Management Science*, 37, 501–623.

Labbé, M. and A. Violin (2013): "Bilevel programming and price setting problems," *4OR*, 11, 1–30.

Lasserre, J. B. (2015): *An Introduction to Polynomial and Semi-Algebraic Optimization*, Cambridge University Press.

Leal, M., D. Ponce, and J. Puerto (2020): "Portfolio problems with two levels decision-makers: Optimal portfolio selection with pricing decisions on transaction costs," *European Journal of Operational Research*, 284, 712–727.

Mansini, R., W. Ogryczak, and M. G. Speranza (2003): "On LP Solvable Models for Portfolio Selection," *Informatica*, 14, 37–62.

——— (2014): "Twenty years of linear programming based on portfolio optimization," *European Journal of Operational Research*, 234, 518–535.

Maravillo, H., J.-F. Camacho-Vallejo, J. Puerto, and M. Labbé (2019): "A market regulation bilevel problem: a case study of the Mexican petrochemical industry," *Omega*, In press.

Markowitz, H. (1952): "Portfolio selection," *The Journal of Finance*, 7, 77–91.

Murtagh, B. A. and M. A. Saunders (1978): "Large-scale linearly constrained optimization," *Mathematical Programming*, 14, 41–72.

Pareto, V. (1971): *Manual of political economy*, A.M. Kelley, New York.

Perrin, S. and T. Roncalli (2019): *Machine Learning Optimization Algorithms & Portfolio Allocation*, ISTE Press. Elsevier, 66 pages, Emmanuel Jurczenko ed.

Puerto, J., M. Rodríguez-Madrena, and A. Scozzari (2020): "Clustering and portfolio selection problems: A unified framework," *Computers and Operational Research*, 117, 104891.

Qiu, R., J. Xu, R. Ke, Z. Zeng, and Y. Wang (2020): "Carbon pricing initiatives-based bi-level pollution routing problem," *European Journal of Operational Research.*

Rockafellar, R. T. (2014): "Coherent Approaches to Risk in Optimization Under Uncertainty," *INFORMS Tutorials in Operations Research*, published online: 14 Oct 2014; 38-61.

Rockafellar, R. T. and S. Uryasev (2000): "Optimization of conditional value-at-risk," *Journal of Risk*, 2, 21–42.

——— (2002): "Conditional value-at-risk for general loss distributions," *Journal of Banking & Finance*, 26, 1443–1471.

Sahinidis, N. V. (2019): *BARON 19.3.24: Global Optimization of Mixed-Integer Nonlinear Programs,* User's Manual.

Sherali, H. D. and C. H. Tuncbilek (1992): "A global optimization algorithm for polynomial programming problems using a Reformulation-Linearization Technique," *Journal of Global Optimization*, 2, 101–112.

Shioda, R., L. Tunçel, and T. G. Myklebust (2011): "Maximum utility product pricing models and algorithms based on reservation price," *Computational Optimization and Applications*, 48, 157–198.

Stambaugh, F. (1996): "Risk and value at risk," *European Management Journal*, 14, 612–621.

Tawarmalani, M. and N. V. Sahinidis (2005): "A polyhedral branch-and-cut approach to global optimization," *Mathematical Programming*, 103, 225–249.

Valle, C. A., N. Meade, and J. E. Beasley (2014): "Absolute return portfolios," *Omega*, 45, 20–41.

von Stackelberg, H. (1934): *Marktform und Gleichgewicht (Market Structure and Equilibrium)*, Springer-Verlag, Berlin-Vienna.

Wächter, A. and L. T. Biegler (2006): "On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming," *Mathematical Programming*, 106, 25–57.

Woodside-Oriakhi, M., C. Lucas, and J. E. Beasley (2013): "Portfolio rebalancing with an investment horizon and transaction costs," *Omega*, 41, 406–420.

A Comparison of different solution methods

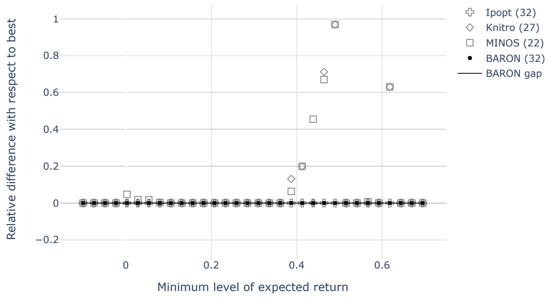
This appendix is devoted to compare different solution methods applied to the problems in this paper. Additionally, we have also addressed the issue of scalability of our solution technique as compared with other solution methods for larger size instances. To this end, we have studied solution techniques built in different local solvers: Ipopt (Wächter and Biegler, 2006), Knitro (Byrd et al., 2006) and MINOS (Murtagh and Saunders, 1978). The comparative analysis of the quality of solutions found and of the running times is shown in figures 11, 12 and 13.

Figure 11 compares four solution methods built in different solvers (three of them local Ipopt, Knitro and MINOS and one global BARON) over our benchmark problem B-L with one investor and different risk profiles ($\alpha \in \{0.05, 0.25, 0.50, 0.99\}$). After the name of each solver we show the number of problems, out of 32, in which the corresponding solver found the best solution. On the x-axis we represent the different instances (as the minimum level of expected return varies). On the y-axis we represent the relative difference in the quality of the solution of each solver compared to the best solver.[16] To provide additional information on the quality of the solutions, we also represent the final optimality gap obtained by BARON. The performance of the different methods is quite similar and most of them find optimal or near optimal solutions within the time limit. Despite the good performance of all solvers, BARON is the only one finding the best solution for all problems. Indeed, given the reported gaps, we know that BARON found the global optimum for these problems. Note that MINOS is the solver with the worst performance.

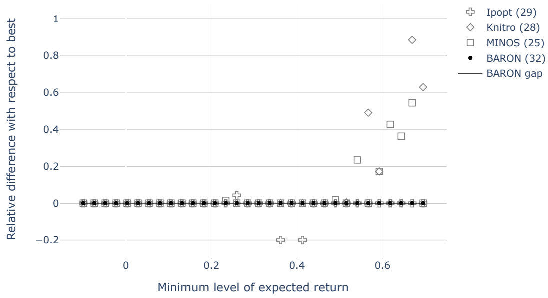
Figure 12 analyzes the issue of scalability of solutions methods for larger problems. We report results for problems with 3 to 20 investor profiles. Again, after the name of each solver we show the number of problems, out of 32, in which the corresponding solver found the best solution. Increasing the number of investors augments the complexity of these problems. As it can be seen in the graphs, our original solution method (based on the global solver BARON) is quite robust and in almost all cases it provides the best results in terms of gap with respect to the best solution found. Yet, we can see how, as problem sizes increase, BARON finds it more difficult to close the optimality gap and, indeed, for the larger instances Knitro's results are competitive with those obtained by BARON.

Finally, for the more difficult problem with 20 investor profiles, we report the running time required for the different methods until they stop with a possibly local optimal solution (local solvers report solution without guaranteed global optimality). The three local solvers always report a solution quickly (in all cases less than 200 seconds) whereas BARON always took one hour trying to certify optimality. Comparing the local solvers the most efficient for these problems seems to be again Knitro, which is the fastest with only one exception.

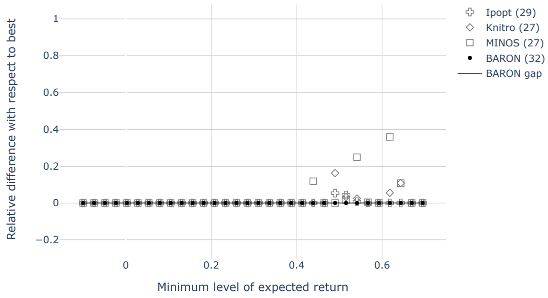
[16] For instance, a value 0.5 for a solver indicates that it obtained an objective function worse, by 50%, than the best solution found.



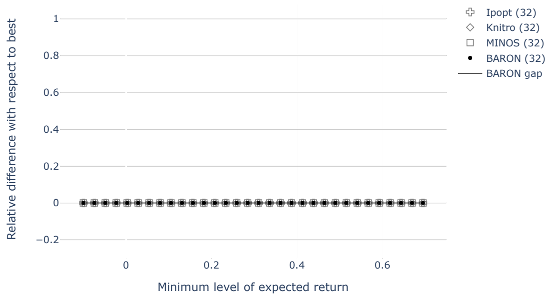
(a) One investor with $\alpha = 0.05$.



(b) One investor with $\alpha = 0.25$.

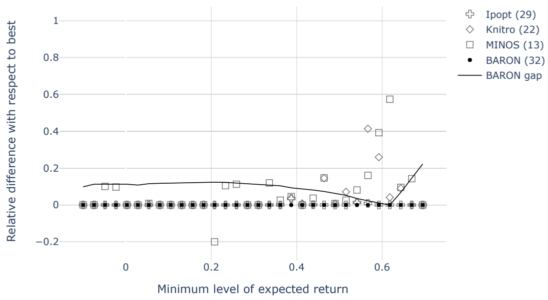


(c) One investor with $\alpha = 0.50$.

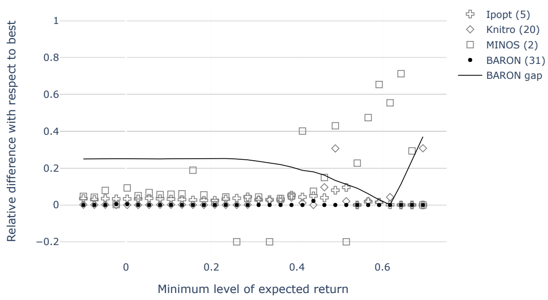


(d) One investor with $\alpha = 0.99$.

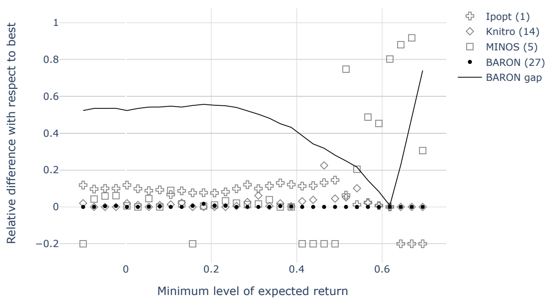
Figure 11: Comparison between solvers in BoT model with one investor and different risk profiles.



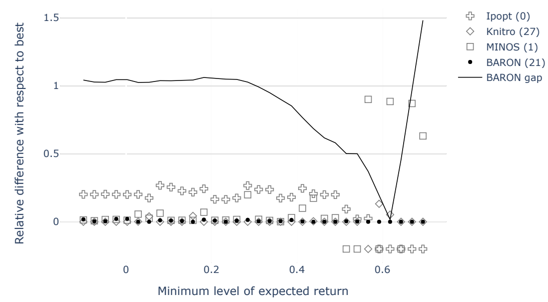
(a) Three investors.



(b) Five investors.



(c) Ten investors.



(d) Twenty investors.

Figure 12: Comparison between solvers in BoT model when the number of investors increases.

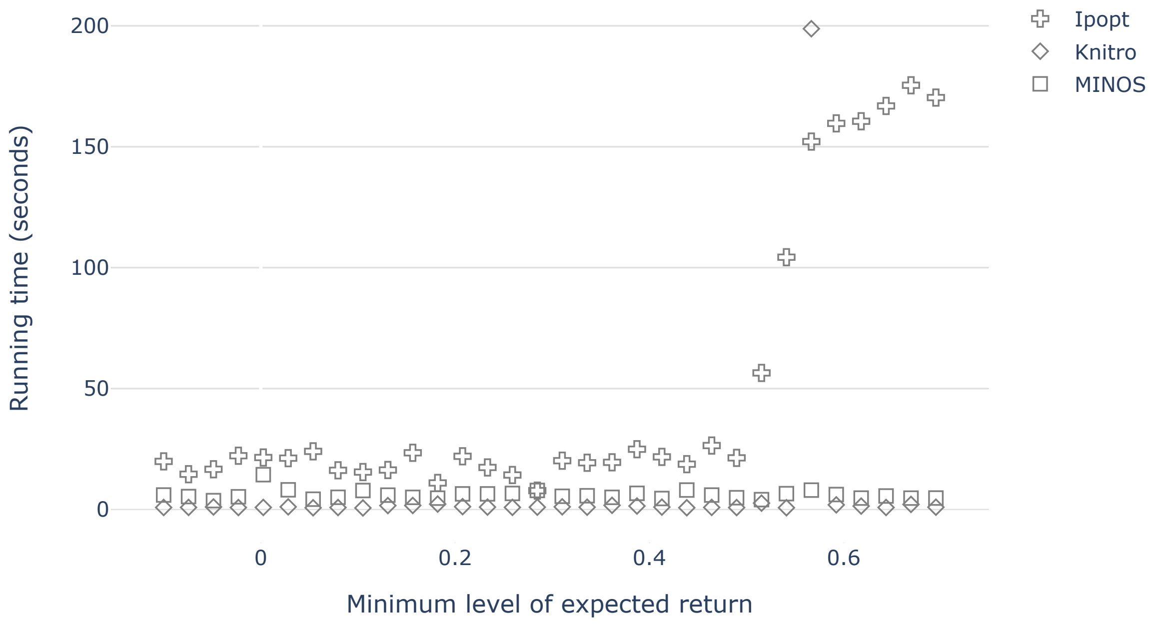


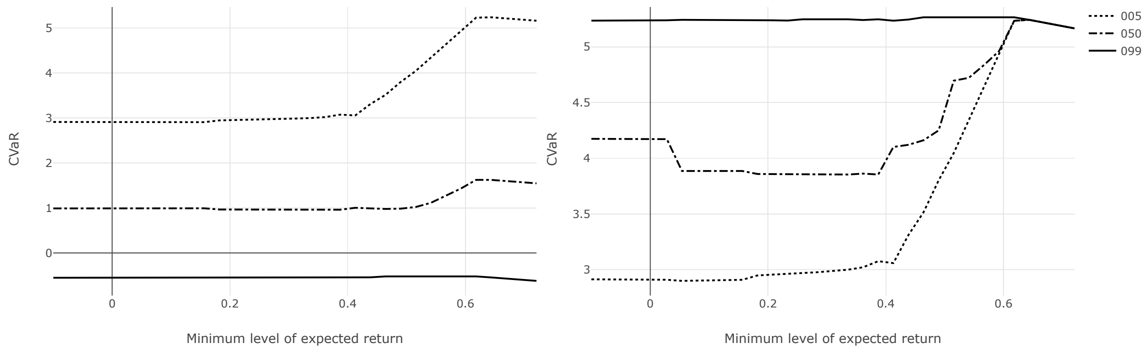
Figure 13: Running time in BoT model with 20 investors.

B Additional Graphs

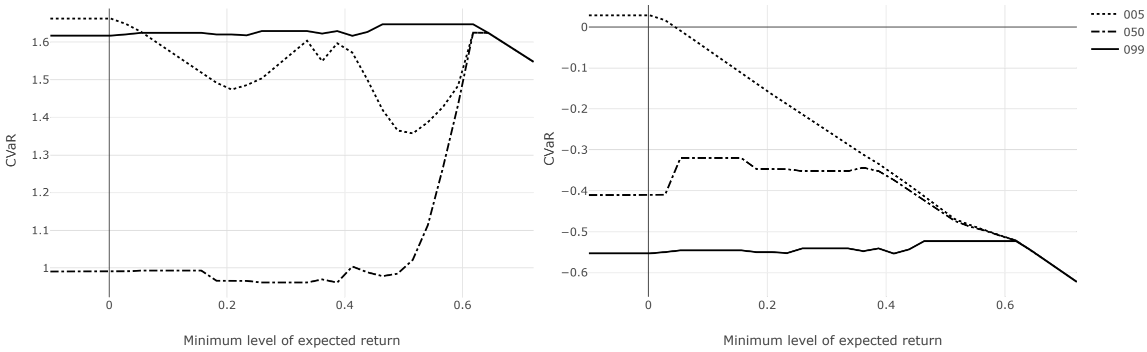
In this section we present the Figures associated with the B-L model with multiple followers. These parallel those discussed in Section 4.2.1 for the case with one follower.

In Figure 14 we show the objective function of each risk profile as a function of the value of $E_{\min}$ and also the CVaR_α at the different optimal solutions for each $\alpha \in \{0.05, 0.50, 0.99\}$. In Figure 15 we show the broker-dealer profit and also the number of number of securities chosen at optimality for the different risk profiles.

In Figure 16 we represent the number of securities in which each risk profile invests as a function of $E_{\min}$.

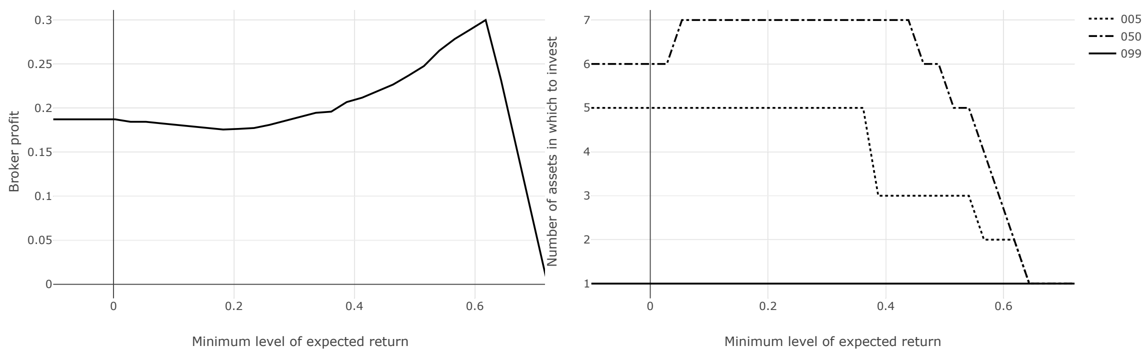


(a) CVaR$_\alpha$ for each profile and value of $E_{\min}$. (b) CVaR$_{0.05}$ at the different optimal solutions.



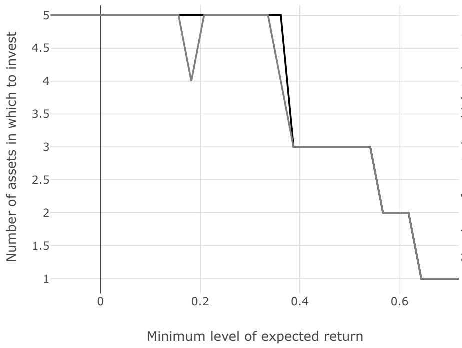
(c) CVaR$_{0.50}$ at the different optimal solutions. (d) CVaR$_{0.99}$ at the different optimal solutions.

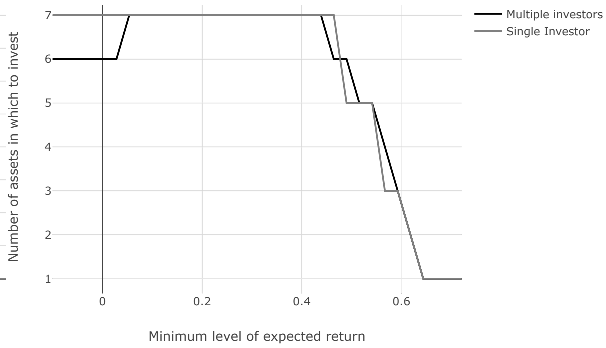
Figure 14: Comparison of CVaR$_\alpha$ levels at the optimal solutions of the different risk profiles.



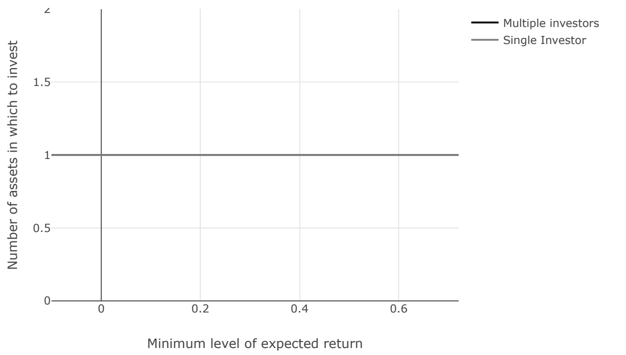
(a) Broker profit at optimality. (b) Number of securities in the optimal portfolio.

Figure 15: Broker profit and number of securities at optimality for different risk profiles.





(a) Securities for $\mathrm{CVaR}_{0.05}$ risk profiles. (b) Securities for $\mathrm{CVaR}_{0.50}$ risk profiles.



(c) Securities for $\mathrm{CVaR}_{0.99}$ risk profiles.

Figure 16: Comparison of the number of securities as a function of $E_{\min}$.