Airport games: The core and its center

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Abstract

An approach to define a rule for an airport problem is to associate to each problem a cooperative game, an airport game, and using game theory to come out with a solution. In this paper we study the rule that is the average of all the core allocations: the core-center (González-Díaz and Sánchez-Rodríguez, 2007). The structure of the core is exploited to derive insights on the core-center. First, we provide a decomposition of the core in terms of the cores of the downstream-substraction reduced games. Then, we analyze the structure of the faces of the core of an airport game that correspond to the no-subsidy constraints to find that the faces of the core can be seen as new airport games, the face games, and that the core can be decomposed through the no-subsidy cones (those whose bases are the cores of the no-subsidy face games). As a consequence, we provide two methods for computing the core-center of an airport problem, both with interesting economic interpretations: one expresses the core-center as a ratio of the volume of the core of an airport game for which a player is cloned over the volume of the original core, the other defines a recursive algorithm to compute the core-center through the no-subsidy cones. Finally, we prove that the core-center is not only an intuitive appealing game-theoretic solution for the airport problem but it has also a good behavior with respect to the basic properties one expects an airport rule to satisfy. We examine some differences between the core-center and, arguably, the two more popular game theoretic solutions for airport problems: the Shapley value and the nucleolus.

Keywords: cooperative TU games, core, core-center, airport games, face games.

1 Introduction

The airport problem, introduced by Littlechild and Owen (1973), is a classic cost allocation problem that has been widely studied. To get a better idea of the attention it has generated one can refer to the survey by Thomson (2013). One standard approach to study this problem consists of associating a cooperative game with it and take advantage of all the machinery developed for cooperative games to gain insights in the original problem. The core, introduced by Gillies (1953), stands as one of the most studied solution concepts in the

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theory of cooperative games. Its properties have been thoroughly analyzed and, when a new class of games is studied, one of the first questions to ask is whether or not the games in that class have a nonempty core. This is because of the desirable stability requirements underlying core allocations.

Importantly, the cooperative game associated with an airport problem with n agents has a special structure that can be exploited to facilitate the analysis of different solutions. In particular, $2^n - 1$ parameters are needed to define a general n-player cooperative game, whereas for an airport game one just needs n. This special structure simplifies the geometry of the core of such games, since they turn to be defined by $2n - 1$ inequality constraints instead of the usual $2^n - 2$.

When the core of a game is nonempty, there is a set of alternatives at which agents' payoffs differ that are coalitionally stable. The core-center (González-Díaz and Sánchez-Rodríguez, 2007), selects the expected value of the uniform distribution over the core of the game: the center of gravity of the core. Therefore, the core-center is an intuitive appealing game-theoretic solution for the airport problem since it represents the "average behavior" of all the stable allocations. There are two important issues concerning the core-center of the airport game that we want to address. First, the computation of the core-center of a general balanced game is very complex. Second, existing rules for the airport problem are evaluated and compared with the core-center in terms of the properties they satisfy or violate. In both cases, the corresponding analysis must be carried out by a detailed examination of the core structure.

The core of an n-player airport game is a $(n - 1)$-dimensional convex polytope, so its $(n - 1)$-Lebesgue measure (its volume) can be seen as the "amount" of stable allocations. Naturally, the mathematical expression of the core-center of an airport game is given in terms of integrals over the core of the game. We provide a decomposition of the core in terms of the cores of the downstream-substraction reduced games that allows us to find explicit integral formulae for the core-center of an airport game. Building upon this expression, we establish our main result. For each player j, consider the airport problem obtained when agent j makes a clone of himself, that is, replicates his cost. Then, what the core-center assigns to agent j (in the original problem) is the ratio of the number of stable allocations in the game with the clone of player j over the original stable allocations. An important implication of this result is the possibility to implement general volume computation algorithms for convex polytopes to develop methods that effectively compute the core-center of an airport problem. Furthermore, we can easily check that the core-center satisfies many desirable properties: homogeneity, equal treatment of equals, order preservation for contributions and benefits, and last-agent cost additivity among others.

To each agent j, we can associate a face of the core polytope that corresponds to the j-th no-subsidy constraint. Each no-subsidy face is the cartesian product of the cores of two reduced airport games. This particular facial structure of the core of an airport game allows us to derive several results. The rate of change of the number of stable allocations with respect to a parameter cost c_i is proportional to the amount of stable allocations of the j-face game. The variation of what the core-center assigns to player j with respect to the cost parameter c_i depends on the relative position of the core-center of the game and the core-center of the j-face games. As a consequence, we derive a necessary and sufficient condition for the monotonicity of the core-center with respect to the cost parameters in terms of its relative position with respect to the centroids of the no-subsidy faces of the core. Applying this characterization, González-Díaz et al. (2015) show that the core-center satisfies some important monotonicity properties: individual cost monotonicity, downstream-cost monotonicity, weak cost monotonicity, and population monotonicity.

The cones rooted at the origin and whose bases are the cores of the no-subsidy face games are called the no-subsidy cones. The core of the airport game can be decomposed as the union of the no-subsidy cones. Using this decomposition we present the sketch of a recursive algorithm to compute the core-center through the no-subsidy cones. At the end of this recursive process, the core-center is a weighted sum of the core-centers of reduced two-player airport games (geometrically, the midpoints of all the core edges corresponding to the no-subsidy constraints).

In summary, besides the intuition provided by its own definition, the core-center is a well behaved rule and it may be an interesting addition to the list of solutions for the class of airport problems. In that respect, we point out some differences between the core-center and, arguably, the two more popular game theoretic solutions for airport problems: the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969). For instance, we define a natural property, unequal treatment of unequals, and show that, whereas the Shapley value and the core-center satisfy it, the nucleolus does not.

The paper is structured as follows. In Section 2 we present the basic concepts and notations. Then, in Section 3 we obtain the fundamental integral representation of the core-center as the ratio of volumes. The basic properties of the core-center are examined in Section 4. In Section 5 the structure of the faces of the core of an airport game is exploited to obtain a necessary and sufficient condition for the monotonicity of the core-center and an expression that relates the core-center of the game with the centroids of the no-subsidy faces of the core. We conclude in Section 6 with some summarizing remarks and further comments.

2 Preliminaries

We assume that there is an infinite set of potential players, indexed by the natural numbers. Then, in each given problem only a finite number of them are present. Let $\mathcal{N}$ be the set of all finite subsets of $\mathbb{N} = \{1, 2, \dots\}$.

A *cost game* with transferable utility is a pair (N, c), where $N \in \mathcal{N}$ and $c\colon 2^N \to \mathbb{R}$ is a function assigning, to each coalition S, its cost $c(S)$. By convention $c(\emptyset) = 0$. Let $\mathcal{V}^N$ be the domain of all cooperative cost games with player set N. Given a coalition of players S, $|S|$ denotes its cardinality. Given $N \in \mathcal{N}$ and $S \subseteq N$, a vector $x \in \mathbb{R}^N$ is referred to as an *allocation* and $x(S) = \sum_{i \in S} x_i$; also, $e_S \in \{0,1\}^N$ is defined as $e^i_S = 1$ if $i \in S$ and $e^i_S = 0$ otherwise. An allocation is *efficient* if $x(N) = c(N)$. A cost game $c \in \mathcal{V}^N$ is *concave* if, for each $i \in N$ and each S and T such that $S \subseteq T \subseteq N \backslash \{i\}$, $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$.

For most of the discussion and results, we have a fixed n-player set $N = \{1, 2, \dots, n\}$. A *solution* is a correspondence ψ defined on some subdomain of cost games that associates to each game $c \in \mathcal{V}^N$ in the subdomain a subset $\psi(c)$ of efficient allocations. If a solution is single-valued then it is referred to as an *allocation rule*.

Given a cost game $c \in \mathcal{V}^N$, the *imputation set*, $I(c)$, consists of the *individually rational* and efficient allocations, *i.e.*, $I(c) = \{x \in \mathbb{R}^n : x(N) = c(N) \text{ and } x_i \leq c(\{i\}) \text{ for all } i \in N\}$. The *core* (Gillies, 1953), is defined as $C(c) = \{x \in I(c) : x(S) \leq c(S) \text{ for each } S \subset N\}$.

An *airport problem* (Littlechild and Owen, 1973) with set of agents $N \in \mathcal{N}$ is a non-negative vector $c \in \mathbb{R}^N$, with $c_i \geq 0$ for each $i \in N$. Let $\mathcal{C}^N$ denote the domain of all airport problems with agent set N. Throughout the paper, given an airport problem $c \in \mathbb{R}^N$, we make the standard assumption that for each pair of agents i and j, if $i < j$ then $c_i \leq c_j$. An allocation for an airport problem is given by a non-negative vector $x \in \mathbb{R}^N$ such that $x(N) = c_n$. An allocation rule selects an allocation for each airport problem in a given subdomain. A complete survey on airport problems is Thomson (2013).

Given an allocation x, the difference $c_i - x_i$ between agent i's cost parameter and his contribution can be seen as his profit at x. A basic requirement is that at an allocation x no group $N' \subset N$ of agents should contribute more that what it would have to pay on its own, $\max\{c_i : i \in N'\}$. Otherwise, the group would unfairly "subsidize" the other agents. The constraints $\sum_{j \leq i} x_j \leq c_i$ are called the *no-subsidy constraints*.

To each airport problem $c \in \mathcal{C}^N$ one can associate a cost game $c \in \mathcal{V}^N$ defined, for each $S \subseteq N$, by setting $c(S) = \max\{c_i : i \in S\}$; such a game is called an *airport game*. We have denoted by the same letter c both the airport problem and the associated cost game. It should be clear by the context to which one we are referring to. Airport games are concave and their core coincides with the set of allocations satisfying the no-subsidy constraints:

$$C(c) = \Big\{x \in \mathbb{R}^n : x \geq 0,\ x(N) = c_n \text{ and } \sum_{j \leq i} x_j \leq c_i \text{ for each } i < n\Big\}.$$

The core of the airport game is contained in the efficiency hyperplane $x_1 + \dots + x_n = c_n$ and it is defined by, at most, $2n - 2$ inequality constraints, instead of the maximum number of $2^n - 2$ inequality constraints of an arbitrary cost game. This makes the structure of the core of an airport game more tractable. In particular, whenever $c_1 > 0$, the core of an airport game is a $(n-1)$-dimensional convex polytope. Then, in what follows, we will always assume that $c_1 > 0$. Further, because of the no-subsidy constraints, any core payoff for the highest cost agent (agent n) can be obtained by adding the incremental cost $c_n - c_{n-1}$ to any core allocation of the airport game where agents $n-1$ and n have the same cost c_{n-1}. Therefore,

$$C(c) = (c_n - c_{n-1})e_{\{n\}} + C(c - (c_n - c_{n-1})e_{\{n\}}).$$

Now, suppose that the agent with the lowest cost leaves the game paying x_1, with $0 \leq x_1 \leq c_1$. Since $0 \leq c_2 - x_1 \leq c_3 - x_1 \leq \cdots \leq c_n - x_1$, we have a new airport problem $c^{1,x_1} = (c_2 - x_1, \ldots, c_n - x_1) = c_{N\backslash\{1\}} - x_1 e_{N\backslash\{1\}}$ with an associated reduced cost game $c^{1,x_1} \in \mathcal{V}^{N\backslash\{1\}}$. The problem $c^{1,x_1} \in \mathcal{C}^{N\backslash\{1\}}$ is known as the *downstream-substraction reduced problem* of $c \in \mathcal{C}^N$ with respect to $N\backslash\{1\}$ and x_1 (Thomson, 2013). In general, given $i \in N$, and $0 \leq x_i \leq c_i$, the downstream-substraction reduced problem of $c \in \mathcal{C}^N$ with respect to $N\backslash\{i\}$ and x_i, $c^{i,x_i} \in \mathcal{C}^{N\backslash\{i\}}$, is defined by

$$c_k^{i,x_i} = \begin{cases} c_k - x_i & k > i \\ \min\{c_i - x_i, c_k\} & k < i \end{cases}.$$

The next proposition, whose proof is straightforward, relates the core of the airport game $c \in \mathcal{V}^N$ and the core of the $c^{1,x_1} \in \mathcal{V}^{N\backslash\{1\}}$ reduced games.

Proposition 1. *Let $c \in \mathcal{C}^N$ be an airport problem. Then*

$$C(c) = \left\{(x_1, x_{N\backslash\{1\}}) \in \mathbb{R}^n : 0 \leq x_1 \leq c_1, x_{N\backslash\{1\}} \in C(c^{1,x_1})\right\} = \bigcup_{0 \leq x_1 \leq c_1} \{x_1\} \times C(c^{1,x_1}).$$

Geometrically, when $0 < x_1 < c_1$, the core of the downstream-substraction reduced problem of c with respect to $N\backslash\{1\}$ and x_1, $C(c^{1,x_1})$, is a cross-section of the core $C(c)$. Now, if we repeatedly apply the above decomposition, the core of the airport game can be covered with the cores of reduced games of s agents, $1 \leq s \leq n-1$. In particular, we have the following result for $s = n-1$.

Corollary 1. *Given an airport problem $c \in \mathcal{C}^N$, the allocation $(x_1, \ldots, x_n)$ belongs to $C(c)$ if, and only if, for each $j \in N\backslash\{n\}$, $0 \leq x_j \leq c_j - \sum_{i<j} x_i$ and $x_n = c_n - \sum_{i<n} x_i$.*

González-Díaz and Sánchez-Rodríguez (2008) associate a *face game* with each face of the core and show that these games have interesting properties for the class of convex (concave) games. In terms of cost games, given a coalition $T \subset N$, the face F_T of the core contains the allocations that are worst for T and best for $N\backslash T$. The downstream-substraction reduced games for the cases $x_1 = c_1$ and $x_1 = 0$ are the face games for agent 1 and coalition $N\backslash\{1\}$, respectively.

In general, the core of an airport game $c \in \mathcal{V}^N$ can be described using reduced airport games with respect to other players. Indeed, for each $i \in N$ we have that

$$C(c) = \bigcup_{0 \leq x_i \leq c_i} \{x_i\} \times C(c^{i,x_i}). \tag{1}$$

3 The core-center and its integral representation

One of the main goals of this paper is to gain insights on the structure of the core of an airport game by studying its center of gravity. Given a cooperative game $c \in \mathcal{V}^N$, its *core-center*, $\mu(c)$, is defined as the expected value of the uniform distribution over the core of the game (González-Díaz and Sánchez-Rodríguez, 2007). The core-center treats all the stable allocations equally and picks up their mean value. In this section we develop some analytic tools that exploit the structure of the core of an airport game. Given a convex set A we denote by $\mu(A)$ its center of gravity.

In the case of an airport game $c \in \mathcal{V}^N$ the core is an $(n-1)$-manifold contained in the efficiency hyperplane, $x_1 + \cdots + x_n = c_n$. The latter is, therefore, the tangent space at each point of the manifold. The vector

$(1,1,\ldots,1) \in \mathbb{R}^n$ is normal to the manifold at each point and it has length $\sqrt{n}$. The transformation $g\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$, $g(x_1,\ldots,x_{n-1}) = (x_1,\ldots,x_{n-1}, c_n - x_1 - \cdots - x_{n-1})$ defines a coordinate system for $C(c)$, so that $g^{-1}(C(c))$ is the projection of the core onto $\mathbb{R}^{n-1}$ that simply "drops" the n-th coordinate. Let $\hat{C}(c) = g^{-1}(C(c)) \subset \mathbb{R}^{n-1}$. This transformation is illustrated in Figure 1.

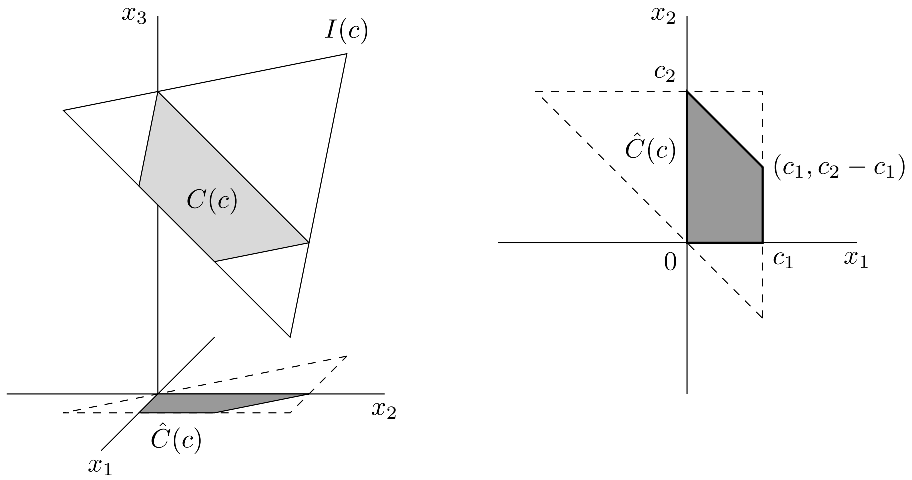

Figure 1: The core $C(c)$ and its projection $\hat{C}(c)$ in the three-player case.

Given $1 \le r \le n$, let m_r be the r-dimensional Lebesgue measure. Then, the $(n-1)$-dimensional measure of the core is given by

$$m_{n-1}(C(c)) = \int_{C(c)} dm_{n-1} = \int_{g^{-1}(C(c))} \sqrt{n}\; dm_{n-1} = \sqrt{n}\; m_{n-1}(\hat{C}(c)).$$

Hence, the volume of the core as a subset of $\mathbb{R}^n$ is $\sqrt{n}$ times the volume of its projection onto $\mathbb{R}^{n-1}$. Analogously, for each $i \in N\backslash\{n\}$, the corresponding component of the core-center is given by

$$\begin{aligned}\mu_i(c) &= \frac{1}{m_{n-1}(C(c))}\int_{C(c)} x_i\; dm_{n-1} = \frac{1}{m_{n-1}(C(c))}\int_{\hat{C}(c)} \sqrt{n}x_i\; dm_{n-1}\\ &= \frac{1}{m_{n-1}(\hat{C}(c))}\int_{\hat{C}(c)} x_i\; dm_{n-1}.\end{aligned}$$

Example 1. *Let $N = \{1,2\}$ and $c = (c_1,c_2) \in \mathcal{C}^N$. Clearly, the core of the airport game is the segment $C(c) = \{\lambda(0,c_2) + (1-\lambda)(c_1, c_2 - c_1), 0 \le \lambda \le 1\} \subset \mathbb{R}^2$. Then, $\mu_1(c) = \frac{\int_0^{c_1} x_1 dx_1}{\int_0^{c_1} dx_1} = \frac{c_1}{2}$ and $\mu_2(c) = c_2 - \frac{c_1}{2}$.*

The above integral expression for the core-center can be written in terms of iterated integrals. First, we introduce some notation. Given $0 < c_1 \le c_2 \le \cdots \le c_{n-1} \le c_n$ and $j \in \{1,\ldots,n-1\}$, we define

$$\begin{aligned}U^j_{n-1}(c_1,\ldots,c_{n-1}) &= \int_0^{c_1}\int_0^{c_2-x_1}\cdots\int_0^{c_{n-1}-\sum_{k=1}^{n-2} x_k} x_j\; dx_{n-1}\ldots dx_2 dx_1,\\ V_{n-1}(c_1,\ldots,c_{n-1}) &= \int_0^{c_1}\int_0^{c_2-x_1}\cdots\int_0^{c_{n-1}-\sum_{k=1}^{n-2} x_k} dx_{n-1}\ldots dx_2 dx_1, \text{ and}\\ \hat{\mu}_j(c_1,\ldots,c_{n-1}) &= \frac{U^j_{n-1}(c_1,\ldots,c_{n-1})}{V_{n-1}(c_1,\ldots,c_{n-1})},\end{aligned}$$

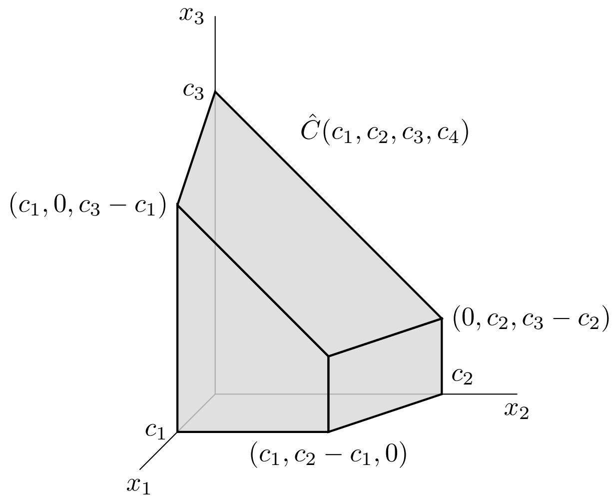

Figure 2: The domain of integration of $V_3(c_1, c_2, c_3)$.

with the convention that $V_0 = 1$. Clearly, U_{n-1}^j is a homogeneous function of degree n, V_{n-1} is a homogeneous function of degree $n-1$, and $\hat{\mu}_j$ is a homogeneous function of degree 1. Figure 2 shows $\hat{C}(c)$ in the four-player case. Now, applying Corollary 1, it is easy to derive the following result.

Theorem 1. *Let $c \in \mathcal{C}^N$ be an airport problem. Then, $m_{n-1}(\hat{C}(c)) = V_{n-1}(c_1, \ldots, c_{n-1})$. Moreover, for each $j \in N \backslash \{n\}$, $\mu_j(c) = \hat{\mu}_j(c_1, \ldots, c_{n-1})$ and $\mu_n(c) = \hat{\mu}_{n-1}(c_1, \ldots, c_{n-1}) + (c_n - c_{n-1})$.*

Note that all the coordinates of the core-center, except the last one, are independent of c_n. In addition, all the coordinates μ_j are homogeneous functions of degree 1.

Certainly, the decomposition in expression (1) gives rise to alternative integral expressions for the core-center, one for each $i \in N \backslash \{1\}$, although we will not use them in this paper.

Let $c \in \mathcal{C}^N$ be an airport problem. From the integral expressions derived previously it is clear that the functions V_{n-1} and U_{n-1}^j, $j \in N \backslash \{n\}$, can be differentiated with respect to the c_i costs, with $i \in N \backslash \{n\}$. As a first consequence, we obtain a result that is fundamental for the analysis in this paper, namely, a representation of the core-center as a ratio of volumes of cores of airport games. We start by proving two auxiliary results.

Lemma 1. *Given $0 < c_1 \leq \cdots \leq c_k$, $k \in \mathbb{N}$, we have that $V_1(c_1) = c_1$, $V_2(c_1, c_2) = \frac{c_2^2}{2} - \frac{(c_2 - c_1)^2}{2}$, and, if $k \geq 3$,*

$$V_k(c_1, \ldots, c_k) = \frac{c_k^k}{k!} - \frac{(c_k - c_1)^k}{k!} - \sum_{i=2}^{k-1} \frac{(c_k - c_i)^{k-i+1}}{(k-i+1)!} V_{i-1}(c_1, \ldots, c_{i-1}).$$

Proof. The expressions for $V_1(c_1)$ and $V_2(c_1, c_2)$ are straightforward. The result also holds for $k = 3$, since $V_3(c_1, c_2, c_3) = \int_0^{c_1} V_2(c_2 - x_1, c_3 - x_1) dx_1 = \int_0^{c_1} \Big(\frac{(c_3 - x_1)^2}{2} - \frac{(c_3 - c_2)^2}{2} \Big) dx_1 = \frac{c_3^3}{3!} - \frac{(c_3 - c_1)^3}{3!} - \frac{(c_3 - c_2)^2}{2} c_1$. We proceed by induction. Let $k \in \mathbb{N}$, $k > 3$, and assume that the result holds for $k - 1$. Then,

$$\begin{aligned} V_k(c_1, \ldots, c_k) &= \int_0^{c_1} V_{k-1}(c_2 - x_1, \ldots, c_k - x_1) dx_1 \\ &= \int_0^{c_1} \Big(\frac{(c_k - x_1)^{k-1}}{(k-1)!} - \frac{(c_k - c_2)^{k-1}}{(k-1)!} - \sum_{i=2}^{k-2} \frac{(c_k - c_{i+1})^{k-i}}{(k-i)!} V_{i-1}(c_2 - x_1, \ldots, c_i - x_1) \Big) dx_1. \end{aligned}$$

We compute separately the integral of each addend:

$$\int_0^{c_1} \frac{(c_k - x_1)^{k-1}}{(k-1)!} dx_1 = \frac{c_k^k}{k!} - \frac{(c_k - c_1)^k}{k!}, \quad \int_0^{c_1} \frac{(c_k - c_2)^{k-1}}{(k-1)!} dx_1 = \frac{(c_k - c_2)^{k-1}}{(k-1)!} V_1(c_1), \quad \text{and}$$
$$\int_0^{c_1} V_{i-1}(c_2 - x_1, \ldots, c_i - x_1) dx_1 = V_i(c_1, \ldots, c_i).$$

By the linearity of the integral,

$$\begin{aligned} V_k(c_1, \ldots, c_k) &= \frac{c_k^k}{k!} - \frac{(c_k - c_1)^k}{k!} - \frac{(c_k - c_2)^{k-1}}{(k-1)!} V_1(c_1) - \sum_{i=2}^{k-2} \frac{(c_k - c_{i+1})^{k-i}}{(k-i)!} V_i(c_1, \ldots, c_i) \\ &= \frac{c_k^k}{k!} - \frac{(c_k - c_1)^k}{k!} - \sum_{i=1}^{k-2} \frac{(c_k - c_{i+1})^{k-i}}{(k-i)!} V_i(c_1, \ldots, c_i), \end{aligned}$$

which, after a simple rearrangement of the indices, coincides with the desired expression. □

Lemma 2. *Let $0 < c_1 \leq \cdots \leq c_k$, $k \in \mathbb{N}$, $x_1 \leq c_1$, and denote $u_k(x_1) = V_k(c_1 - x_1, \ldots, c_k - x_1)$. Then,*

$$\frac{du_k}{dx_1}(x_1) = -V_{k-1}(c_2 - x_1, \ldots, c_k - x_1).$$

Proof. The proof is by induction. The property holds for $k = 1$, since $u_1(x_1) = V_1(c_1 - x_1) = c_1 - x_1$ and $\frac{du_1}{dx_1}(x_1) = -1$. It also holds for $k = 2$, because $u_2(x_1) = \frac{(c_2 - x_1)^2}{2} - \frac{(c_2 - c_1)^2}{2}$ and $\frac{du_2}{dx_1}(x_1) = -(c_2 - x_1)$. Now, let $k \geq 3$ and suppose that the property is true for u_i, $1 \leq i \leq k - 1$. Then, according to Lemma 1,

$$u_k(x_1) = V_k(c_1 - x_1, \ldots, c_k - x_1) = \frac{(c_k - x_1)^k}{k!} - \frac{(c_k - c_1)^k}{k!} - \sum_{i=2}^{k-1} \frac{(c_k - c_i)^{k-i+1}}{(k-i+1)!} u_{i-1}(x_1).$$

Differentiating with respect to x_1,

$$\frac{du_k}{dx_1}(x_1) = -\frac{(c_k - x_1)^{k-1}}{(k-1)!} - \sum_{i=2}^{k-1} \frac{(c_k - c_i)^{k-i+1}}{(k-i+1)!} \frac{du_{i-1}}{dx_1}(x_1).$$

By the induction hypothesis, if $i \geq 2$, $\frac{du_{i-1}}{dx_1}(x_1) = -V_{i-2}(c_2 - x_1, \ldots, c_{i-1} - x_1)$. Therefore,

$$\frac{du_k}{dx_1}(x_1) = -\frac{(c_k - x_1)^{k-1}}{(k-1)!} + \frac{(c_k - c_2)^{k-1}}{(k-1)!} + \sum_{i=3}^{k-1} \frac{(c_k - c_i)^{k-i+1}}{(k-i+1)!} V_{i-2}(c_2 - x_1, \ldots, c_{i-1} - x_1).$$

Renumbering the terms,

$$\begin{aligned} \frac{du_k}{dx_1}(x_1) &= -\left(\frac{(c_k - x_1)^{k-1}}{(k-1)!} - \frac{(c_k - c_2)^{k-1}}{(k-1)!} - \sum_{i=2}^{k-2} \frac{(c_k - c_{i+1})^{k-i}}{(k-i)!} V_{i-1}(c_2 - x_1, \ldots, c_i - x_1) \right) \\ &= -V_{k-1}(c_2 - x_1, \ldots, c_k - x_1), \end{aligned}$$

where the last equality follows directly from Lemma 1. □

Now, we can show that, in fact, each coordinate of the core-center is the ratio of two volumes. To be precise, consider for each player $j \in N \backslash \{n\}$ the airport problem $(c_1, \ldots, c_j, c_j, \ldots, c_{n-1})$, obtained when agent j makes a clone of himself, that is, replicates his cost.[1] Then, what the core-center assigns to agent j (in the original problem) is the percentage of stable allocations in the game with the clone of player j over the original stable allocations. Mathematically, the j-th coordinate of the core-center is the ratio of the volumes of the core of the airport game obtained by replicating agent j and the core of the original game.

[1] The idea of replication has been already used in the literature, see for instance Debreu and Scarf (1963), Thomson (1988).

Theorem 2. *Let $c \in \mathcal{C}^N$ be an airport problem and $j \in N \setminus \{n\}$. Then:*

1. $U^j_{n-1}(c_1, \ldots, c_{n-1}) = V_n(c_1, \ldots, c_j, c_j, \ldots, c_{n-1})$
2. $\hat{\mu}_j(c_1, \ldots, c_{n-1}) = \dfrac{V_n(c_1, \ldots, c_j, c_j, \ldots, c_{n-1})}{V_{n-1}(c_1, \ldots, c_{n-1})}$

Proof. First observe that

$$V_n(c_1, \ldots, c_j, c_j, \ldots, c_{n-1}) = \int_0^{c_1} \cdots \int_0^{c_j - \sum_{k=1}^{j-1} x_k} V_{n-j}\big(c_j - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\big)\ dx_j \ldots dx_1.$$

If $u_{n-j}(x_j) = V_{n-j}\Big(c_j - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\Big)$ then, by Lemma 2,

$$\frac{du_{n-j}}{dx_j}(x_j) = -V_{n-j-1}\Big(c_{j+1} - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\Big).$$

Integrating by parts,

$$\int_0^{c_j - \sum_{k=1}^{j-1} x_k} V_{n-j}\big(c_j - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\big)\ dx_j =$$

$$\Big[x_j V_{n-j}\big(c_j - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\big)\Big]_0^{c_j - \sum_{k=1}^{j-1} x_k} + \int_0^{c_j - \sum_{k=1}^{j-1} x_k} x_j V_{n-j-1}\big(c_{j+1} - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\big)\ dx_j.$$

The bracketed expression vanishes, since $V_{n-j}(0, c_{j+1} - c_j, \ldots, c_{n-1} - c_j) = 0$. Consequently,

$$\int_0^{c_j - \sum_{k=1}^{j-1} x_k} V_{n-j}\Big(c_j - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\Big)\ dx_j =$$

$$\int_0^{c_j - \sum_{k=1}^{j-1} x_k} x_j V_{n-j-1}\Big(c_{j+1} - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\Big)\ dx_j. \quad (2)$$

Finally,

$$\begin{aligned} V_n(c_1, \ldots, c_j, c_j, \ldots c_{n-1}) &= \int_0^{c_1} \cdots \int_0^{c_j - \sum_{k=1}^{j-1} x_k} x_j V_{n-j-1}\Big(c_{j+1} - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\Big)\ dx_j \ldots dx_1 \\ &= U^j_{n-1}(c_1, \ldots, c_{n-1}). \end{aligned}$$

This last result follows from the previous property and the definition of $\hat{\mu}$. □

Corollary 2. *Let $c \in \mathcal{C}^N$ be an airport problem. Then*

$$\mu_n(c) = \frac{V_n(c_1, \ldots, c_n)}{V_{n-1}(c_1, \ldots, c_{n-1})}.$$

Proof. It follows from the combination of Theorems 1 and 2 and simple calculus. □

Apart from being a key tool for the ensuing analysis, Theorem 2 is interesting on its own. There are no known efficient deterministic algorithms for computing the centroid of a convex body. Therefore, the issue of computing the core-center of a general balanced game is very complex. The second statement in Theorem 2 says that the j-th coordinate of the core-center of an airport game can be computed just by using the volume of its core and the volume of the core of the airport game obtained by replicating agent j. Then, Theorem 2 opens the door to implementing volume computation algorithms for convex polytopes to compute, in a relatively easy way, the core-center of an airport game.[2]

Geometrically, we must note that $\mu_j(c) = \frac{V_n(c_1,\ldots,c_j,c_j,\ldots,c_{n-1})}{V_{n-1}(c_1,\ldots,c_{n-1})} = \sqrt{\frac{n+1}{n}}\frac{m_n(C(c_1,\ldots,c_j,c_j,\ldots,c_n))}{m_{n-1}(C(c_1,\ldots,c_n))}$ and that $C(c)$ is a face of $C(c_1,\ldots,c_j,c_j,\ldots,c_n)$, in fact, the intersection of $C(c_1,\ldots,c_j,c_j,\ldots,c_n)$ with the hyperplane $x_{j+1} = 0$. Then $\mu_j(c)$ is a measure of the ratio between the volume of the core of the problem with a clone of agent j and a particular face of it, the core of the original problem.

From a game-theoretic perspective, recall that $c(S \cup \{j\}) - c(S)$ measures the marginal contribution of player $j \in N$ to a coalition $S \subset N$. We can interpret the ratio $\frac{V_n(c_1,\ldots,c_j,c_j,\ldots,c_{n-1})}{V_{n-1}(c_1,\ldots,c_{n-1})}$ as another way of computing the contribution of player $j \in N$ to the game: it measures the change on the set of stable allocations when a clone of player j is added to the initial set of players.

4 General properties of the core-center

Next we study some basic properties that are satisfied by the core-center of the airport game. For a complete survey on airport problems the reader is referred to Thomson (2013). We say that a rule ψ satisfies:

- *Non-negativity* if, for each $c \in \mathcal{C}^N$ and each $i \in N$, $\psi_i(c) \geq 0$.
- *Cost boundedness* if, for each $c \in \mathcal{C}^N$ and each $i \in N$, $\psi_i(c) \leq c_i$.
- *Efficiency* if for each $c \in \mathcal{C}^N$, $\sum_{i \in N} \psi_i(c) = c_n$.
- *No-subsidy* if, for each $c \in \mathcal{C}^N$ and each $S \subset N$, $\sum_{i \in S} \psi_i(c) \leq \max_{i \in S} c_i$.
- *Homogeneity* if, for each $c \in \mathcal{C}^N$ and each $\alpha > 0$, $\psi(\alpha c) = \alpha \psi(c)$.
- *Equal treatment of equals* if, for each $c \in \mathcal{C}^N$ and each pair $\{i, j\} \subset N$ with $c_i = c_j$, $\psi_i(c) = \psi_j(c)$.
- *Continuity* if, for each sequence $\{(c^\nu)\}_{\nu \in \mathbb{N}}$ of elements of $\mathcal{C}^N$ and each $c \in \mathcal{C}^N$, if $c^\nu \to c$ then $\psi(c^\nu) \to \psi(c)$.

Proposition 2. *The core-center satisfies non-negativity, cost-boundedness, efficiency, no-subsidy, homogeneity, equal treatment of equals, and continuity.*

Proof. The first five properties follow from the fact that any core allocation satisfies them. A couple of comments on the last two properties are needed. González-Díaz and Sánchez-Rodríguez (2007) prove that the core-center treats symmetric players equally and that it is a continuous function of the values of the characteristic function. In our context, equal treatment of equals holds because agents with the same cost parameter are symmetric players in the associated airport game. Similarly, continuity holds because the values of the characteristic function are continuous with respect to the cost parameters. Then, the core-center satisfies continuity since it is a composition of continuous functions. □

Equal treatment of equals states that if two agents have the same cost they should pay equal amounts. Now, we focus on the reverse property: if two agents have different costs then they should pay different amounts.

[2] The reader interested in the difficulties and algorithms for computing volumes of convex polytopes may refer to Lasserre (1983), Lasserre and Zeron (2001) and references therein. Regarding the computational aspects of cooperative game theory, the survey paper Bilbao et al. (2000) and the book by Chalkiadakis et al. (2011) provide excellent starting points to the field.

- A rule ψ satisfies *unequal treatment of unequals* if, for each $c \in \mathcal{C}^N$ and each pair $\{i, j\} \subset N$ with $c_i \neq c_j$, $\psi_i(c) \neq \psi_j(c)$.

Obviously, the Shapley value satisfies unequal treatment of unequals. Nevertheless, the nucleolus violates this property. Observe that, for the airport problem $(1.1, 1.5, 3) \in \mathcal{C}^{\{1,2,3\}}$ the nucleolus is $(0.5, 0.5, 2)$.

Proposition 3. *The core-center satisfies unequal treatment of unequals.*

Proof. Let $c = (c_1, \ldots, c_n) \in \mathcal{C}^N$. We have to prove that for all $i \in \{1, \ldots, n-1\}$, $\mu_i(c) \neq \mu_{i+1}(c)$ whenever $c_i \neq c_{i+1}$. If $i = n-1$, the result is clear since $\mu_n(c) = \mu_{n-1}(c) + (c_n - c_{n-1})$. Now, if $i \in \{1, \ldots, n-2\}$ then, by Theorem 2,

$$\mu_{i+1}(c) - \mu_i(c) = \frac{V_n(c_1, \ldots, c_i, c_{i+1}, c_{i+1}, c_{i+2}, \ldots, c_{n-1}) - V_n(c_1, \ldots, c_i, c_i, c_{i+1}, c_{i+2}, \ldots, c_{n-1})}{V_{n-1}(c_1, \ldots, c_{n-1})}.$$

Therefore, $\mu_{i+1}(c) = \mu_i(c)$ if, and only if,

$$V_n(c_1, \ldots, c_i, c_{i+1}, c_{i+1}, c_{i+2}, \ldots, c_{n-1}) = V_n(c_1, \ldots, c_i, c_i, c_{i+1}, c_{i+2}, \ldots, c_{n-1}),$$

that is, if and only if $c_i = c_{i+1}$. □

The order preservation for contributions (benefits) property states that if agent i's cost parameter is at least as large as agent j's cost parameter, agent i should contribute (or benefit) at least as much as agent j does.

- A rule ψ satisfies *order preservation for contributions* if, for each $c \in \mathcal{C}^N$ and each pair $\{i, j\} \subset N$ with $c_i \leq c_j$, $\psi_i(c) \leq \psi_j(c)$.
- A rule ψ satisfies *order preservation for benefits* if, for each $c \in \mathcal{C}^N$ and each pair $\{i, j\} \subset N$ with $c_i \leq c_j$, $c_i - \psi_i(c) \leq c_j - \psi_j(c)$.

Proposition 4. *The core-center satisfies order preservation for contributions.*

Proof. Let $c \in \mathcal{C}^N$. Trivially, by Theorem 1, $\mu_{n-1}(c) \leq \mu_n(c)$. Now, take two consecutive agents i and $i+1$ where $i < n-1$. Then, by Theorem 2, $\mu_i(c) \leq \mu_{i+1}(c)$ if, and only if,

$$V_n(c_1, \ldots, c_i, c_i, c_{i+1}, \ldots, c_{n-1}) \leq V_n(c_1, \ldots, c_i, c_{i+1}, c_{i+1}, \ldots, c_{n-1}),$$

which is immediate since $c_i \leq c_{i+1}$. □

Proposition 5. *The core-center satisfies order preservations for benefits.*

Proof. Let $c \in \mathcal{C}^N$. Recall that $\mu_{n-1}(c) - \mu_n(c) = c_{n-1} - c_n$. Let $i < n-1$. We have to prove that $\mu_{i+1}(c) - \mu_i(c) \leq c_{i+1} - c_i$. Applying Theorem 2, the difference $\mu_{i+1}(c) - \mu_i(c)$ can be written as follows:

$$\mu_{i+1}(c) - \mu_i(c) = \frac{\displaystyle\int_0^{c_1} \int_0^{c_2 - x_1} \cdots \int_0^{c_i - \sum_{j=1}^{i-1} x_j} \int_{c_i - \sum_{j=1}^{i} x_j}^{c_{i+1} - \sum_{j=1}^{i} x_j} \cdots \int_0^{c_{n-1} - \sum_{j=1}^{n-1} x_j} dx_n \ldots dx_2 dx_1}{V_{n-1}(c_1, \ldots, c_{n-1})}$$

$$= \frac{\displaystyle\int_0^{c_1} \int_0^{c_2 - x_1} \cdots \int_{c_i - \sum_{j=1}^{i} x_j}^{c_{i+1} - \sum_{j=1}^{i} x_j} V_{n-1-i}\big(c_{i+1} - \sum_{j=1}^{i+1} x_j, \ldots, c_{n-1} - \sum_{j=1}^{i+1} x_j\big) dx_{i+1} dx_i \ldots dx_1}{V_{n-1}(c_1, \ldots, c_{n-1})}.$$

Now, by the mean-value theorem for integrals, there exists a point $\xi \in (c_i - \sum_{j=1}^{i} x_j, c_{i+1} - \sum_{j=1}^{i} x_j)$ such that

$$\int_{c_i - \sum_{j=1}^{i} x_j}^{c_{i+1} - \sum_{j=1}^{i} x_j} V_{n-1-i}\big(c_{i+1} - \sum_{j=1}^{i+1} x_j, \ldots, c_{n-1} - \sum_{j=1}^{i+1} x_j\big) dx_{i+1} =$$

$$(c_{i+1} - c_i) V_{n-1-i}\big(c_{i+1} - \sum_{j=1}^{i} x_j - \xi, \ldots, c_{n-1} - \sum_{j=1}^{i} x_j - \xi\big).$$

But, since $c_s - \sum_{j=1}^{i} x_j - \xi \leq c_s - c_i$ for all $i+1 \leq s \leq n-1$, we have that

$$V_{n-1-i}\big(c_{i+1} - \sum_{j=1}^{i} x_j - \xi, \ldots, c_{n-1} - \sum_{j=1}^{i} x_j - \xi\big) \leq V_{n-1-i}(c_{i+1} - c_i, \ldots, c_{n-1} - c_i).$$

Therefore,

$$\begin{aligned} \mu_{i+1}(c) - \mu_i(c) &\leq (c_{i+1} - c_i) \frac{\displaystyle\int_0^{c_1} \cdots \int_0^{c_i - \sum_{j=1}^{i-1} x_j} V_{n-1-i}(c_{i+1} - c_i, \ldots, c_{n-1} - c_i) dx_i \ldots dx_1}{V_{n-1}(c_1, \ldots, c_{n-1})} \\ &= (c_{i+1} - c_i) \frac{\displaystyle\int_0^{c_1} \cdots \int_0^{c_i - \sum_{j=1}^{i-1} x_j} V_{n-1-i}(c_{i+1} - c_i, \ldots, c_{n-1} - c_i) dx_i \ldots dx_1}{\displaystyle\int_0^{c_1} \cdots \int_0^{c_i - \sum_{j=1}^{i-1} x_j} V_{n-1-i}(c_{i+1} - \sum_{j=1}^{i} x_j, \ldots, c_{n-1} - \sum_{j=1}^{i} x_j) dx_i \ldots dx_1} \\ &\leq c_{i+1} - c_i \end{aligned}$$

where the last inequality holds because, $\sum_{j=1}^{i} x_i \leq c_i$ implies that $c_s - c_i \leq c_s - \sum_{j=1}^{i} x_i$ for $i+1 \leq s \leq n-1$. □

The next property demands that the payment required to the last agent should increase by an amount equal to the increase in his cost parameter.

- A rule ψ satisfies *last-agent cost additivity* if for each pair $\{c, c'\}$ of elements of $\mathcal{C}^N$, each $\gamma \geq 0$, and each $i \in N$ such that $c_i = \max\{c_j : j \in N\}$, whenever $c'_{N\setminus\{i\}} = c_{N\setminus\{i\}}$ and $c'_i = c_i + \gamma$, then $\psi_{N\setminus\{i\}}(c') = \psi_{N\setminus\{i\}}(c)$ and $\psi_i(c') = \psi_i(c) + \gamma$.

Proposition 6. *The core-center satisfies last-agent cost additivity.*

Proof. Let $c, c' \in \mathcal{C}^N$ be airport games satisfying the hypothesis of last-agent cost additivity. Then

$$(x_1, \ldots, x_{n-1}, c_n - \sum_{i=1}^{n-1} x_i) \in C(c) \Leftrightarrow (x_1, \ldots, x_{n-1}, c'_n - \sum_{i=1}^{n-1} x_i) \in C(c').$$

Hence, $\hat{C}(c) = \hat{C}(c')$ and $\mu_{N\setminus\{n\}}(c') = \mu_{N\setminus\{n\}}(c)$. By Theorem 1, $\mu_n(c') = \mu_{n-1}(c') + (c'_n - c'_{n-1}) = \mu_{n-1}(c) + (c_n + \gamma - c_{n-1}) = \mu_n(c) + \gamma$. □

A property, introduced in Mirás-Calvo et al. (2016), demands that the last two agents must have equal benefits.

- A rule ψ satisfies *last two agents equal benefits* if for each airport problem $c \in \mathcal{C}^N$, then $c_n - \psi_n(c) = c_{n-1} - \psi_{n-1}(c)$.

We have seen in Theorem 1 that the core-center satisfies last two agents equal benefits. In fact, see Mirás-Calvo et al. (2016), a rule[3] ψ satisfies last two agents equal benefits if, and only if, for each $c \in \mathcal{C}^N$,

$$\psi_{n-1}(c) = \frac{1}{2}\Big(c_{n-1} - \sum_{i=1}^{n-2} \psi_i(c)\Big).$$

Also, one can easily check that if a rule ψ satisfies efficiency, equal treatment of equals, and last-agent cost additivity, then it satisfies last two agents equal benefits. Therefore, in order to have the core-center of a n-player airport game one needs to compute just $n-2$ coordinates. For example in a 3-player game, the computation of the first coordinate gives the entire allocation as the following example shows.

Example 2. *Let $N = \{1,2,3\}$ and $c = (c_1, c_2, c_3) \in \mathcal{C}^N$. Then, $\mu_1(c) = \dfrac{\int_0^{c_1}\int_0^{c_1-x}\int_0^{c_2-x-y} dzdydx}{\int_0^{c_1}\int_0^{c_2-x} dydx} = \dfrac{c_1}{3}\dfrac{3c_2-2c_1}{2c_2-c_1}$. Then $\mu_2(c) = \frac{1}{2}(c_2 - \mu_1(c))$. Finally, by efficiency, $\mu_3(c) = (c_3 - c_2) + \mu_2(c)$.*

5 No-subsidy face games

We devote this section to analyze the structure of the faces of the core of an airport game that correspond to the no-subsidy constraints. We exploit this facial structure to obtain two interesting results: a necessary and sufficient condition for the monotonicity of the core-center with respect to the cost parameters and an expression that relates the core-center of the game with the centroids of the no-subsidy faces of the core. As a consequence, we will provide a recursive algorithm to compute the core-center of an airport game through the no-subsidy cones (those whose bases are the cores of the no-subsidy face games).

Let $c \in \mathcal{C}^N$ be an airport game with agent set N. Denote by F_i, $i \in N\backslash\{n\}$, the i-th no-subsidy face of $\hat{C}(c)$, *i.e.*,

$$F_i = \hat{C}(c) \cap \{x \in \mathbb{R}^{n-1} : x_1 + \cdots + x_i = c_i\} \subset \mathbb{R}^{n-1}.$$

Let $m_{n-2}(F_i)$ be its $(n-2)$-measure and $\mu(F_i)$ be its centroid.

The next decomposition of F_i, that it is easily derived, states that for each player $i \in N\backslash\{n\}$, F_i is the cartesian product of two cores of reduced airport games. The first game is played by the first i players and it is given by the airport problem $(c_1, \ldots, c_i)$. The second game is played by the rest of the players where all the costs parameters are reduced by c_i, which corresponds to the airport problem $(c_{i+1} - c_i, \ldots, c_n - c_i)$. So, players $\{1, \ldots, i\}$ pay c_i and players $\{i+1, \ldots, n\}$ pay $c_n - c_i$ (see Figure 3).

Proposition 7. *For all $i \in N\backslash\{n\}$, we have that*

$$F_i = C(c_1, \ldots, c_i) \times \hat{C}(c_{i+1} - c_i, \ldots, c_n - c_i).$$

The coordinates of the centroid $\mu(F_i)$ are

$$\mu_j(F_i) = \begin{cases} \mu_j(c_1, \ldots, c_i) & \text{if } j \leq i \\ \hat{\mu}_{j-i}(c_{i+1} - c_i, \ldots, c_{n-1} - c_i) & \text{if } i < j \leq n-1 \end{cases}.$$

[3] In this paper, by definition, a rule must choose a stable allocation.

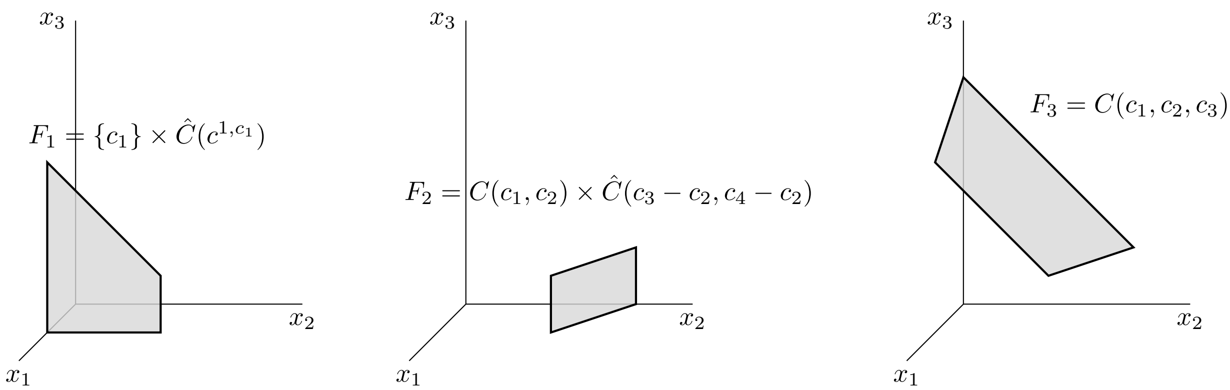

Figure 3: The F_i-faces of $\hat{C}(c)$ in the four-agent case.

Since the faces of the core are again cores of new airport games, Theorem 1 can be used to compute their centroids.

We present a series of results that relate the partial derivatives of the functions V_{n-1} and U^j_{n-1} to the faces of $\hat{C}(c)$ and their centroids.[4]

Proposition 8. *For all $i \in N \backslash \{n\}$,*

$$\frac{\partial V_{n-1}}{\partial c_i}(c_1, \dots, c_{n-1}) = V_{i-1}(c_1, \dots, c_{i-1}) V_{n-1-i}(c_{i+1} - c_i, \dots, c_{n-1} - c_i).$$

Proof. Since $V_{n-1}(c_1, \dots, c_{n-1}) = \int_0^{c_1} \cdots \int_0^{c_i - \sum_{k=1}^{i-1} x_k} V_{n-1-i}\big(c_{i+1} - \sum_{k=1}^{i} x_k, \dots, c_{n-1} - \sum_{k=1}^{i} x_k\big) dx_i \dots dx_1$, a direct computation using Leibniz's rule shows that

$$\begin{aligned}
\tfrac{\partial V_{n-1}}{\partial c_i}(c_1, \dots, c_{n-1}) &= \int_0^{c_1} \cdots \int_0^{c_{i-1} - \sum_{k=1}^{i-2} x_k} \tfrac{\partial}{\partial c_i} \Big(\int_0^{c_i - \sum_{k=1}^{i-1} x_k} V_{n-1-i}\big(c_{i+1} - \sum_{k=1}^{i} x_k, \dots, c_{n-1} - \sum_{k=1}^{i} x_k\big) dx_i \Big) dx_{i-1} \dots dx_1 \\
&= \int_0^{c_1} \cdots \int_0^{c_{i-1} - \sum_{k=1}^{i-2} x_k} V_{n-1-i}(c_{i+1} - c_i, \dots, c_{n-1} - c_i) dx_{i-1} \dots dx_1 \\
&= V_{n-1-i}(c_{i+1} - c_i, \dots, c_{n-1} - c_i) \int_0^{c_1} \cdots \int_0^{c_{i-1} - \sum_{k=1}^{i-2} x_k} dx_{i-1} \dots dx_1 \\
&= V_{n-1-i}(c_{i+1} - c_i, \dots, c_{n-1} - c_i) V_{i-1}(c_1, \dots, c_{i-1}).
\end{aligned}$$

This concludes the proof. □

Proposition 9. *Let $i, j \in N \backslash \{n\}$. Then,*

$$\frac{\partial U^j_{n-1}}{\partial c_i}(c_1, \dots, c_{n-1}) = \begin{cases} V_{i-1}(c_1, \dots, c_{i-1}) U^{j-i}_{n-1-i}(c_{i+1} - c_i, \dots, c_{n-1} - c_i) & i < j \\ U^j_{i-1}(c_1, \dots, c_{i-1}) V_{n-1-i}(c_{i+1} - c_i, \dots, c_{n-1} - c_i) & i > j \\ V_i(c_1, \dots, c_i) V_{n-1-i}(c_{i+1}, \dots, c_{n-1})) & i = j \end{cases}.$$

[4]Since by Theorem 1 the core-center is the ratio of volumes, this part could be rewritten only with V_{n-1} and V_n.

Moreover, $V_i(c_1,\dots,c_i)V_{n-1-i}(c_{i+1},\dots,c_{n-1})$ *can be equivalently written as*

$$\Big(c_iV_{i-1}(c_1,\dots,c_{i-1})-\sum_{k=1}^{i-1}U^k_{i-1}(c_1,\dots,c_{i-1})\Big)V_{n-1-i}(c_{i+1}-c_i,\dots,c_{n-1}-c_i).$$

Proof. The computations when $i\neq j$ are straightforward from the chain rule, Theorem 2 and Proposition 8. The same applies to the first equality in the case $j=i$. The alternative expression is obtained by directly applying Leibniz's rule to the integral formulation of U^j_{n-1}. □

As a consequence, we show that, whenever the $(n-2)$-measure of the face F_i is positive, the rate of change of the number of stable allocations with respect to a cost c_i is proportional to the volume of F_i. In addition, the center of gravity of F_i can be obtained using the partial derivatives computed above.

Proposition 10. *Let* $c\in\mathcal{C}^N$ *be an airport game. For all* $i,j\in N\backslash\{n\}$ *such that* $m_{n-2}(F_i)>0$*,*

1. $\dfrac{\partial V_{n-1}}{\partial c_i}(c_1,\dots,c_{n-1})=\frac{1}{\sqrt{i}}m_{n-2}(F_i)$.

2. $\mu_j(F_i)=\dfrac{\frac{\partial U^j_{n-1}}{\partial c_i}(c_1,\dots,c_{n-1})}{\frac{\partial V_{n-1}}{\partial c_i}(c_1,\dots,c_{n-1})}$.

Proof. Assume that $m_{n-2}(F_i)>0$. Recall that $C(c_1,\dots,c_i)$ is an $(i-1)$-dimensional polytope contained in the hyperplane $x_1+\dots+x_i=c_i$, so

$$\begin{aligned} m_{i-1}\big(C(c_1,\dots,c_i)\big) &= \sqrt{i}\; m_{i-1}\big(\hat{C}(c_1,\dots,c_i)\big)\\ &= \sqrt{i}\; V_{i-1}(c_1,\dots,c_{i-1}). \end{aligned}$$

On the other hand, the measure of $\hat{C}(c_{i+1}-c_i,\dots,c_n-c_i)$ as a subset of $\mathbb{R}^{n-1-i}$ is $V_{n-1-i}(c_{i+1}-c_i,\dots,c_{n-1}-c_i)$. Therefore, the first assertion follows immediately from Proposition 8 and the decomposition of Proposition 7.

The proof of the second property is divided in three cases. First, assume that $i<j$. Then, by Proposition 7,

$$\mu_j(F_i)=\hat{\mu}_{j-i}(c_{i+1}-c_i,\dots,c_{n-1}-c_i)=\frac{U^{j-i}_{n-1-i}(c_{i+1}-c_i,\dots,c_{n-1}-c_i)}{V_{n-1-i}(c_{i+1}-c_i,\dots,c_{n-1}-c_i)}=\frac{\frac{\partial U^j_{n-1}}{\partial c_i}(c_1,\dots,c_{n-1})}{\frac{\partial V_{n-1}}{\partial c_i}(c_1,\dots,c_{n-1})},$$

where the last equality follows from Propositions 8 and 9.

If $i>j$, then, as above, by Propositions 8, 9 and 7, we have

$$\begin{aligned} \mu_j(F_i) &= \mu_j(c_1,\dots,c_i)\\ &= \hat{\mu}_j(c_1,\dots,c_{i-1})=\frac{U^j_{i-1}(c_1,\dots,c_{i-1})}{V_{i-1}(c_1,\dots,c_{i-1})}=\frac{\frac{\partial U^j_{n-1}}{\partial c_i}(c_1,\dots,c_{n-1})}{\frac{\partial V_{n-1}}{\partial c_i}(c_1,\dots,c_{n-1})}. \end{aligned}$$

Finally, when $j=i$, by Proposition 7,

$$\begin{aligned} \mu_i(F_i) &= \mu_i(c_1,\dots,c_i)=c_i-\sum_{k=1}^{i-1}\mu_k(\{1,\dots,i\},(c_1,\dots,c_i))\\ &= c_i-\sum_{k=1}^{i-1}\hat{\mu}_k(c_1,\dots,c_{i-1})=c_i-\sum_{k=1}^{i-1}\frac{U^k_{i-1}(c_1,\dots,c_{i-1})}{V_{i-1}(c_1,\dots,c_{i-1})}. \end{aligned}$$

Therefore,

$$\mu_i(F_i)V_{i-1}(c_1,\ldots,c_{i-1}) = c_i V_{i-1}(c_1,\ldots,c_{i-1}) - \sum_{k=1}^{i-1} U^k_{i-1}(c_1,\ldots,c_{i-1}).$$

Substituting this expression in Proposition 9 and using Proposition 8, the result follows. □

We would like to remark that the first equality of Proposition 10 admits a generalization for any convex polyhedron (Lasserre, 1983).

Relational requirements on rules are of great interest. The monotonicity properties study the behavior of the rules if one single cost c_i increases while the others are held constant. In Thomson (2013), there is a summary of the main monotonicity properties that have been studied in the literature. The next result provides a necessary and sufficient condition for the monotonicity of the core-center in terms of its relative position with respect to the centroids of the no-subsidy faces of the core.

Proposition 11. *Let $c \in \mathcal{C}^N$ be an airport problem and let $i,j \in N\backslash\{n\}$. Then, $\mu_j(c)$ is increasing with respect to c_i if, and only if, $\mu_j(c) \leq \mu_j(F_i)$. Conversely, $\mu_j(c)$ is decreasing with respect to c_i if, and only if, $\mu_j(c) \geq \mu_j(F_i)$.*

Proof. Recall that $\mu_j(c) = \hat{\mu}_j(c_1,\ldots,c_{n-1})$ and, by Theorem 2,

$$\frac{\partial \hat{\mu}_j}{\partial c_i}(c_1,\ldots,c_{n-1}) = \frac{\frac{\partial U^j_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})V_{n-1}(c_1,\ldots,c_{n-1}) - U^j_{n-1}(c_1,\ldots,c_{n-1})\frac{\partial V_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})}{(V_{n-1}(c_1,\ldots,c_{n-1}))^2}.$$

Thus, $\mu_j(c)$ is increasing with respect to c_i if, and only if, the numerator is positive. Now, by Proposition 10 the latter is equivalent to

$$\mu_j(F_i) = \frac{\frac{\partial U^j_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})}{\frac{\partial V_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})} \geq \frac{U^j_{n-1}(c_1,\ldots,c_{n-1})}{V_{n-1}(c_1,\ldots,c_{n-1})} = \mu_j(c).$$ □

We have established that the variation of what the core-center assigns to player j with respect to the cost parameter c_i depends on the relative position of the core-center of the game and the centroid of the no-subsidy face F_i. Therefore, the monotonicity properties (see Thomson (2013) for a complete description) of the core-center of an airport game can be studied by comparing the core-center assignments given to a player in the original game and those in the corresponding no-subsidy face games. González-Díaz et al. (2015) follow this approach to prove that the core-center satisfies individual cost monotonicity, downstream-cost monotonicity, weak cost monotonicity, and population monotonicity.

To end the section, we will further explore the relationship of the core-centers of the no-subsidy faces of the core and the core-center of the core itself. This analysis will lead to a recursive algorithm to compute the core-center of an airport game in terms of the core-centers of the reduced games of Proposition 7.

Let $c \in \mathcal{C}^N$ be an airport problem. For each $i \in N\backslash\{n\}$, let $K_i = \{\lambda z : 0 \leq \lambda \leq 1, z \in F_i\}$ denote the no-subsidy cone rooted at the origin and generated by the i-th no-subsidy face F_i of $\hat{C}(c)$. The no-subsidy cone K_i contains all the line segments for which one end point is the origin and the other end point belongs to the no-subsidy face F_i. Since the i-th no-subsidy face F_i contains the worst core allocations for the coalition $\{1,\ldots,i\}$ and the origin is the best core allocation for any coalition for which the highest cost airline does not belong to, the no-subsidy cone K_i can be interpreted as the set of all stable allocations that are a trade-off between the most desirable payment and one of the worst payments for the coalition of all airlines with cost less or equal than c_i.

Certainly, any two no-subsidy cones have negligible intersection and $\hat{C}(c)$ is the union of all the no-subsidy cones. Moreover, the $(n-1)$-volume and centroid of the no-subsidy cone K_i are proportional to the $(n-2)$-volume and centroid of the no-subsidy face F_i, respectively.

Proposition 12. *Let $c \in \mathcal{C}^N$ be an airport problem and $i \in N\backslash\{n\}$. Then:*

1. $\hat{C}(c) = \bigcup_{i=1}^{n-1} K_i$ *and* $m_{n-1}(K_i \cap K_j) = 0$ *if* $i \neq j$.

2. $m_{n-1}(K_i) = \frac{c_i}{(n-1)\sqrt{i}} m_{n-2}(F_i)$ *and* $\mu(K_i) = \frac{n-1}{n}\mu(F_i)$.

3. $m_{n-1}\big(\hat{C}(c)\big) = \frac{1}{n-1}\sum_{i=1}^{n-1}\omega_i$, *where* $\omega_i = \frac{c_i}{\sqrt{i}} m_{n-2}(F_i)$ *for all* $i \in N\backslash\{n\}$.

4. $\mu_j(c) = \frac{n-1}{n}\sum_{i=1}^{n-1}\frac{\omega_i}{\sum_{k=1}^{n-1}\omega_k}\mu_j(F_i)$, *for all* $j \in N\backslash\{n\}$.

Proof. In general, let $K \subset \mathbb{R}^n$ be a finite cone rooted at the origin whose base K_h belongs to the hyperplane $x_n = h$. For every $t \in [0,h]$ denote by K_t the intersection of K with the hyperplane $x_n = t$, that is $K_t = K \cap \{x \in \mathbb{R}^n : x_n = t\}$. Then $K_t = \frac{t}{h}K_h$, $m_{n-1}(K_t) = \left(\frac{t}{h}\right)^{n-1} m_{n-1}(K_h)$ and $\mu(K_t) = \frac{t}{h}\mu(K_h)$. Now, integrating

$$m_n(K) = \int_0^h m_{n-1}(K_t)dt = \int_0^h \left(\tfrac{t}{h}\right)^{n-1} m_{n-1}(K_h)dt = \tfrac{h}{n} m_{n-1}(K_h).$$

$$m_n(K)\mu_j(K) = \int_0^h m_{n-1}(K_t)\mu_j(K_t)dt = \int_0^h \left(\tfrac{t}{h}\right)^{n} m_{n-1}(K_h)\mu_j(K_h)dt = \tfrac{h}{n+1} m_{n-1}(K_h)\mu_j(K_h).$$

Therefore, $m_n(K) = \frac{h}{n}m_{n-1}(K_h)$ and $\mu(K) = \frac{n}{n+1}\mu(K_h)$. Applying these general equalities, all the statements are straightforward.[5] □

Naturally, Proposition 12 reduces the computation of the core-center of an airport game to the computation of the core-centers of the no-subsidy faces. But, by Proposition 7, the no-subsidy faces are indeed cores of airport games. Then, we have outlined a recursive algorithm to compute the core-center of an airport problem. Observe that, in the final step, the core-center is obtained as a weighted sum of the midpoints of all the edges of the no-subsidy faces of the core, that is, the core-centers of all the two-agent airport problems obtained by recursively applying Proposition 7.

6 Concluding remarks

Following a game-theoretic approach, we have thoroughly studied the behavior in airport problems of a rule defined for more general cooperative games, namely, the core-center. The definition of the core-center is very intuitive, it is the average of all the stable allocations. Though the exact calculation of the core-center is always a difficult task, we have provided two alternative methods to compute the core-center of an airport problem that have a distinctive economic flavor: one based on the representation of the core-center as a ratio of the volume of the core of an airport game for which a player is cloned over the volume of the original core, the other that gives the core-center as a weighted average of the core-center of some basic two-agent airport problems obtained recursively from the no-subsidy faces of the core.

We have seen that the core-center is a well behaved rule. It satisfies the following properties: non-negativity, cost-boundedness, efficiency, no-subsidy, homogeneity, equal treatment of equals, continuity, order preservation for contributions, order preservations for benefits, last two agents equal benefits, and last-agent cost additivity. Moreover, using the necessary and sufficient condition for the monotonicity of the core-center with respect

[5] The equality for $m_{n-1}\big(\hat{C}(c)\big)$ can be obtained directly from the decomposition of $\hat{C}(c)$ or by applying the general formula given in Lasserre (1983) to compute the volume of a polyhedron $P = \{x : Ax \leq b\}$ in terms of the volumes of its facets.

to the cost parameters developed in this paper, González-Díaz et al. (2015) show that the core-center also satisfies individual cost monotonicity, downstream-cost monotonicity, weak cost monotonicity, and population monotonicity.[6]

The two more notable game-theoretic solutions for the airport problem, the nucleolus and the Shapley value satisfy all the properties listed above (see Thomson (2013)). Therefore, we would like to point out some instances in which the behavior of the core-center and that of the Shapley value and the nucleolus differ. Loosely speaking, a variation, no matter how small, in any cost parameter of an airport problem, produces a change in the shape of the core and therefore in its volume. The core-center is highly sensitive to such changes. We have already seen that the core-center satisfies unequal treatment of unequals (agents with different costs pay different amounts) a property that the nucleolus violates. In order to study the monotonicity of the core-center, González-Díaz et al. (2015) introduced two new properties that reflect whether or not a variation in a particular agent's cost is beneficial to the other agents.[7] It is easy to check that given an agent i, a variation on the cost parameter of any airline with cost higher than c_i has no effect on the Shapley value of agent i. As for the core-center, if a single cost c_i increases, then the contributions requested by the rule for the agents with cost lower than c_i strictly increase (while the contributions for the agents with cost higher than c_i strictly decrease).

Besides, an examination of these three rules based on how differentially they treat relatively larger airlines as compared to relatively smaller airlines has been carried out in Mirás-Calvo et al. (2016). The Lorenz order is commonly used as an egalitarian criteria. In order to compare a pair of allocations, x and y, with the Lorenz ordering, first one has to rearrange the coordinates of each allocation in a non-decreasing order. Then, we say that x Lorenz dominates y, and write $x \succeq y$, if all the cumulative sums of the rearranged coordinates are greater with x than with y. It is shown that the core-center is a rule that Lorenz dominates the Shapley value, Sh, and is Lorenz dominated by the nucleolus, η, that is, $\eta \succeq \mu \succeq \text{Sh}$.

The question of finding characterizations of the core-center rule goes beyond the scope of this paper. Nevertheless, we would like to point out some directions that could lead to an axiomatic characterization of this value and the difficulties that might be encountered. One possible approach is to adapt the characterization of the core-center given in González-Díaz and Sánchez-Rodríguez (2009) for the general class of balanced games to the airport problem. The key property for that characterization is a weighted additivity, called the trade-off property, that is based on a principle of fairness with respect to the core. The general idea of the characterization is to decompose the original core in pieces that are "simple" cores of games. The solution in these "simple" cores is described by standard axioms (efficiency and symmetry properties). Then, the trade-off property is used to obtain the core-center as the weighted sum of the solution applied to the "simple" cores of the decomposition. The downside when considering airport games is that the pieces of the core dissection are not cores of airport games themselves but translates of cores of airport games. Therefore, one has to work in a larger class of problems. A different approximation consists in using a consistency type property as the basis for a characterization of the core-center rule. As a first attempt, we say that a rule satisfies first-agent weighted consistency if the payment required to each agent is a weighted average of the corresponding payments in all the downstream-substraction reduced problems of c with respect to $N\backslash\{1\}$ and $x_1 \in [0, c_1]$.

- A rule ψ satisfies *first-agent weighted consistency* if, for each $c \in \mathcal{C}^N$ and each $i \in N\backslash\{1, n\}$,

$$\psi_i(c) = \int_0^{c_1} \psi_i(N\backslash\{1\}, c^{1,x_1}) f(x_1, c) dx_1,$$

  where $f(x_1, c) = \frac{m_{n-2}(\hat{C}(N\backslash\{1\}, c^{1,x_1}))}{m_{n-1}(\hat{C}(N,c))}$.

It is easy to prove that the core-center satisfies first-agent weighted consistency. On the other hand, we have shown that the core-center satisfies last two agents equal benefits. We claim that these two properties characterize

[6] The core-center does not satisfy other properties such as marginalism or the different types of consistency properties given in the literature. The reader can construct easy three-agent examples to observe this point.

[7] These properties are called higher-cost decreasing monotonicity and lower-cost increasing monotonicity.

the core-center rule. Indeed, if a rule satisfies last two agents equal benefits then we know how the rule chooses for any two-agent problem. Then, a repeated application of first-agent weighted consistency would allow us to prove that it must coincide with the core-center. As we see it, the weakness of this approach is that the consistency axiom is too strong, so a weaker form of it should be found in order to have a useful characterization.

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