Cores of Convex and Strictly Convex Games*

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Abstract: We follow the path initiated in Shapley (1971) and study the geometry of the core of convex and strictly convex games. We define what we call *face games* and use them to study the combinatorial complexity of the core of a strictly convex game. Remarkably, we present a picture that summarizes our results with the aid of Pascal's triangle.

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1 Preliminaries

A cooperative n-player game with transferable utility, shortly, a TU game, is a pair (N, v), where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function assigning, to each coalition $S \in 2^N$, its worth $v(S)$; by convention $v(\emptyset) := 0$. Let G^n be the set of n-player games. Given $S \subseteq N$, let $|S|$ be the number of players in S. For the sake of notation, we denote $\{i\}$ by i.

Let $(N, v) \in G^n$. The *core* (Gillies, 1953), is defined by $C(N, v) := \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N) \text{ and, for each } S \subseteq N, \sum_{i \in S} x_i \geq v(S)\}$. Let BG^n be the set of n-player games with nonempty core. We say $C(N, v)$ is *full dimensional* if it has dimension $n - 1$.

Let $S \subseteq N$ and let $\Pi(S)$ be the set of orderings (permutations) of the elements in S. For each $\sigma_S \in \Pi(S)$ and each $i \in S$, $\sigma_S(i)$ denotes i's position. We denote σ_N by σ. For each $i \in N$ and each $\sigma \in \Pi(N)$, let $P_\sigma(i) := \{j \in N : \sigma(j) < \sigma(i)\}$ be the set of *predecessors* of i with respect to σ. Let $(N, v) \in G^n$ and $\sigma \in \Pi(N)$. The *marginal vector* associated with (N, v) and σ, $m^\sigma(N, v)$, is defined, for each $i \in N$, by $m_i^\sigma(N, v) := v(P_\sigma(i) \cup i) - v(P_\sigma(i))$.

A game (N, v) is *convex* if, for each $i \in N$ and each S and T such that $S \subseteq T \subseteq N \setminus \{i\}$, $v(S \cup i) - v(S) \le v(T \cup i) - v(T)$. Let CG^n be the set of n-player convex games. A game (N, v) is *strictly convex* if, for each $i \in N$ and each S and T such that $S \subsetneq T \subseteq N \setminus \{i\}$, $v(S \cup i) - v(S) < v(T \cup i) - v(T)$.

A (convex) polytope P is the convex hull of a finite set of points. A polytope P is an *m-polytope* if its dimension is m. A hyperplane H is a *supporting hyperplane for P* if $H \cap P \ne \emptyset$ and the halfspace below H contains P. A *face* of a polytope P is defined as (i) P itself, (ii) the empty set, or (iii) the intersection of P with some supporting hyperplane; faces of dimension m are called *m-faces* (with the convention that $\dim(\emptyset) = -1$). The 0-faces, 1-faces, and $(m-1)$-faces of an m-polytope P are respectively its vertices, edges, and facets. Let $\mathcal{F}(P)$ denote the set of all faces of P.

2 Core Complexity and Pascal's Triangle

We discuss now the geometry of the cores of convex and strictly convex games. In our exposition we mainly use the terminology in Shapley (1971) and, also, some of the results included there. Let $(N, v) \in BG^n$. For each $\emptyset \ne T \subseteq N$, let H_T be the hyperplane $H_T := \{x \in \mathbb{R}^n : \sum_{i \in T} x_i = v(T)\}$ and let $F_T := C(N, v) \cap H_{N \setminus T}$. Clearly, $F_\emptyset = C(N, v)$; also, let $F_N := C(N, v)$.[1] Shapley (1971) and Ichiishi (1981) showed that a game is convex if and only if the vertices of the core are the marginal vectors, i.e., $C(N, v) = \text{co}\{m^\sigma(N, v) : \sigma \in \Pi(N)\}$, where $\text{co}(A)$ denotes the convex hull of A. Thus, for convex games, each F_T is a nonempty face of $C(N, v)$ and we refer to F_T as a *T-face of $C(N, v)$*. By definition, in each allocation in F_T, coalition T receives $v(N) - v(N \setminus T)$. Clearly, for each $\emptyset \ne T \subsetneq N$, since both F_T and $F_{N \setminus T}$ lie in H_N, they are parallel to each other. Now we define, for each coalition $T \subseteq N$, a game (N, v_{F_T}) that is closely related to F_T.

The idea of the definition is the following. Given a convex game (N, v) and a face F_T of its core, if we restrict attention to $C(N, v)$, then the allocations in F_T are the best ones for T and the worst ones for $N \setminus T$. Coalition T always receives $v(N) - v(N \setminus T)$ and coalition $N \setminus T$ gets $v(N \setminus T)$. Yet, there is still freedom for $v(N) - v(N \setminus T)$ to be shared among the players in T and for $v(N \setminus T)$ to be shared among the ones in $N \setminus T$. Next definition captures this idea.

Definition 1. *Let $(N, v) \in BG^n$ and $T \subseteq N$. The T-face game (N, v_{F_T}) is defined, for each $S \subseteq N$, by $v_{F_T}(S) := v((S \cap T) \cup (N \setminus T)) - v(N \setminus T) + v(S \cap (N \setminus T))$.*

Note that, if $T = \emptyset$ or $T = N$, then $(N, v_{F_T}) = (N, v)$. Besides, if $S \cap T = \emptyset$, then $v_{F_T}(S) = v(S)$. If $\emptyset \ne T \subsetneq N$ then, in the game (N, v_{F_T}), the worths of coalitions T and $N \setminus T$ are $v_{F_T}(T) = v(N) - v(N \setminus T)$ and $v_{F_T}(N \setminus T) = v(N \setminus T)$, respectively. Moreover, if $(N, v) \in CG^n$, then $v_{F_T}(T)$ is also the maximum payoff that coalition T can get in $C(N, v)$; similarly, $v_{F_T}(N \setminus T)$ is the minimum for $N \setminus T$.

Lemma 1. *Let $(N, v) \in BG^n$ and $T \subseteq N$. Let $\sigma = (\sigma_{N \setminus T}, \sigma_T)$ and let $\bar{\sigma} \in \Pi(N)$ be such that it induces the orders σ_T and $\sigma_{N \setminus T}$ in T and $N \setminus T$, respectively. Then,*

(i) $m^\sigma(N, v_{F_T}) = m^\sigma(N, v)$.

(ii) $m^\sigma(N, v_{F_T}) = m^{\bar{\sigma}}(N, v_{F_T})$.

Moreover, if $(N, v) \in CG^n$ then,

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[1] Shapley (1971) defines F_T as $C(N, v) \cap H_T$. Although Shapley's definition might seem more natural, ours is more convenient for the exposition below.

(iii) $(N, v_{F_T}) \in CG^n$.

(iv) $m^{\bar{\sigma}}(N, v) \in F_T$ *if and only if* $m^\sigma(N, v) = m^{\bar{\sigma}}(N, v)$. *Hence,* $F_T = \mathrm{co}\{m^\sigma(N, v) : \sigma = (\sigma_{N \setminus T}, \sigma_T)\}$.

Proof. Since $(N, v_{F_\emptyset}) = (N, v_{F_N}) = (N, v)$, the result is trivial for $T = \emptyset$ and $T = N$. Hence, let $\emptyset \neq T \subsetneq N$.

(i) Let $\sigma = (\sigma_{N \setminus T}, \sigma_T)$. We show that, for each $i \in N$, $m_i^\sigma(N, v) = m_i^\sigma(N, v_{F_T})$. Suppose that $i \in N \setminus T$. Since $P_\sigma(i) \subset P_\sigma(i) \cup i \subseteq N \setminus T$, then $v(P_\sigma(i) \cup i) = v_{F_T}(P_\sigma(i) \cup i)$ and $v(P_\sigma(i)) = v_{F_T}(P_\sigma(i))$. Suppose that $i \in T$. In this case, $N \setminus T \subseteq P_\sigma(i)$ and it is easy to check that, again, $v_{F_T}(P_\sigma(i) \cup i) = v(P_\sigma(i) \cup i)$ and $v_{F_T}(P_\sigma(i)) = v(P_\sigma(i))$.

(ii) If $\bar{\sigma} = (\sigma_{N \setminus T}, \sigma_T)$ the result is trivial. Let $\bar{\sigma} \neq (\sigma_{N \setminus T}, \sigma_T)$. Then, $\bar{\sigma}$ can be written as $(\sigma_{R_1}, \sigma_{T_1}, \sigma_{R_2}, \sigma_{T_2}, \ldots, \sigma_{R_p}, \sigma_{T_q})$, where $T_1, \ldots, T_q \subset T$, $R_1, \ldots, R_p \subset N \setminus T$, and T_1 and R_2 are nonempty. Let $\sigma^* := (\sigma_{R_1}, \sigma_{R_2}, \sigma_{T_1}, \sigma_{T_2}, \ldots, \sigma_{R_p}, \sigma_{T_q})$, i.e., R_2 and T_1 are swapped. We show that $m^{\bar{\sigma}}(N, v_{F_T}) = m^{\sigma^*}(N, v_{F_T})$. Once the latter is proved, we get, after a finite number of swaps, that $m^{\bar{\sigma}}(N, v_{F_T}) = m^\sigma(N, v_{F_T})$. Clearly, the marginal vectors associated with $\bar{\sigma}$ and σ^* can only differ for the players in T_1 or R_2. We distinguish two cases. **Case 1:** $i \in T_1$. Clearly, $P_{\bar{\sigma}}(i) = R_1 \cup P_{\sigma_{T_1}}(i)$ and $P_{\sigma^*}(i) = R_1 \cup R_2 \cup P_{\sigma_{T_1}}(i)$. By the definition of v_{F_T}, $v_{F_T}(P_{\bar{\sigma}}(i) \cup i) - v_{F_T}(P_{\bar{\sigma}}(i)) = v(P_{\sigma_{T_1}}(i) \cup (N \setminus T) \cup i) - v(P_{\sigma_{T_1}}(i) \cup (N \setminus T)) = v_{F_T}(P_{\sigma^*}(i) \cup i) - v_{F_T}(P_{\sigma^*}(i))$. **Case 2:** $i \in R_2$. Clearly, $P_{\bar{\sigma}}(i) = R_1 \cup T_1 \cup P_{\sigma_{R_2}}(i)$ and $P_{\sigma^*}(i) = R_1 \cup P_{\sigma_{R_2}}(i)$. By the definition of v_{F_T}, $v_{F_T}(P_{\bar{\sigma}}(i) \cup i) - v_{F_T}(P_{\bar{\sigma}}(i)) = v(R_1 \cup P_{\sigma_{R_2}}(i) \cup i) - v(R_1 \cup P_{\sigma_{R_2}}(i)) = v_{F_T}(P_{\sigma^*}(i) \cup i) - v_{F_T}(P_{\sigma^*}(i))$.

(iii) We show that, for each $R \subseteq S \subseteq N \setminus i$, $v_{F_T}(R \cup i) - v_{F_T}(R) \leq v_{F_T}(S \cup i) - v_{F_T}(S)$. Suppose that $i \in N \setminus T$. Since $v_{F_T}(S \cup i) - v_{F_T}(S) = v((S \cup i) \cap (N \setminus T)) - v(S \cap (N \setminus T))$, $v_{F_T}(R \cup i) - v_{F_T}(R) = v((R \cup i) \cap (N \setminus T)) - v(R \cap (N \setminus T))$, and (N, v) is convex, the desired inequality holds. Suppose that $i \in T$. Since $v_{F_T}(S \cup i) - v_{F_T}(S) = v((S \cap T) \cup (N \setminus T) \cup i) - v((S \cap T) \cup (N \setminus T))$, $v_{F_T}(R \cup i) - v_{F_T}(R) = v((R \cap T) \cup (N \setminus T) \cup i) - v((R \cap T) \cup (N \setminus T))$, and (N, v) is convex, the desired inequality holds.

(iv) Since $v \in CG^n$, $m^\sigma(N, v) \in F_T$ and the necessity is trivial. We prove the sufficiency. Since $m^\sigma(N, v) \in F_T$, then $\sum_{i \in T} m_i^{\bar{\sigma}} = v(N) - v(N \setminus T) = \sum_{i \in T} m_i^\sigma$. By convexity, for each $i \in T$, $m_i^{\bar{\sigma}}(N, v) \leq m_i^\sigma(N, v)$ and, since $\sum_{i \in T} m_i^{\bar{\sigma}} = \sum_{i \in T} m_i^\sigma$, we have that, for each $i \in T$, $m_i^{\bar{\sigma}}(N, v) = m_i^\sigma(N, v)$. Similarly, for each $i \in N \setminus T$, $m_i^{\bar{\sigma}}(N, v) = m_i^\sigma(N, v)$. $\qquad \square$

Proposition 1. *Let* $(N, v) \in CG^n$ *and* $T \subseteq N$. *Then,* $C(N, v_{F_T}) = F_T$. *Therefore,* $C(N, v) = \mathrm{co}\{C(N, v_{F_T}) : \emptyset \neq T \subsetneq N\}$.

Proof. The equality $C(N, v_{F_T}) = F_T$ is trivial for $T = \emptyset$ and $T = N$. Let $\emptyset \neq T \subsetneq N$. By Lemma 1 (i), for each $\sigma \in \Pi(N)$, $m^\sigma(N, v_{F_T})$ is a vertex of $C(N, v_{F_T})$ and $C(N, v_{F_T}) = \mathrm{co}\{m^\sigma(N, v_{F_T}) : \sigma \in \Pi(N)\}$. Now, by Lemma 1 (ii) and (iii), $C(N, v_{F_T}) \subseteq F_T$ and, by Lemma 1 (ii) and (iv), $F_T \subseteq C(N, v_{F_T})$. $\qquad \square$

The following result is a compilation of different results in Shapley (1971).

Lemma 2. *Let* (N, v) *be a strictly convex game. Then,*

(i) $m^\sigma(N, v) = m^{\bar{\sigma}}(N, v)$ *if and only if* $\sigma = \bar{\sigma}$. *Hence,* $C(N, v)$ *has* $n!$ *vertices.*

(ii) $C(N, v)$ *is full dimensional and has* $2^n - 2$ *facets, one for each* $\emptyset \neq T \subsetneq N$.

(iii) *Let* $\emptyset \neq T \subsetneq N$. *Then,* $m^\sigma(N, v) \in F_T$ *if and only if* σ *is of the form* $(\sigma_{N \setminus T}, \sigma_T)$.

Remark. From the previous result, for each strictly convex game and each $\emptyset \neq T \subsetneq N$, F_T is a facet of $C(N, v)$, *i.e.*, an $(n - 2)$-polytope. Moreover, (i) and (iii) imply that F_T has $|T|! \, (n - |T|)!$ vertices and, hence, (N, v_{F_T}) is not strictly convex. Recall that, for each $t \in \{0, \ldots, n\}$, the number of t-player coalitions is $\binom{n}{t}$. Hence, $\binom{n}{t}$ is also the number of faces of $C(N, v)$ that are associated with a coalition of size t.

Now, we introduce one more concept from Shapley (1971). Let $\mathcal{P} = \{N_1, \ldots, N_p\}$ be a partition of N, with $p \geq 2$. The game (N, v) is *decomposable with respect to* $\mathcal{P}$ if, for each $S \subseteq N$, $v(S) = v(S \cap N_1) + \ldots + v(S \cap N_p)$. That is, v is the addition of p smaller games; each of them is referred to as a *component*. The following result is also a compilation of different results in Shapley (1971).

Lemma 3. (i) *A strictly convex game is indecomposable.*

(ii) *A decomposable game is convex if and only if each component is convex.*

(iii) *The core of a decomposable convex game is the cartesian product of the cores of the components of any decomposition.*

Proposition 2. *Let* $(N, v) \in BG^n$ *and* $\emptyset \neq T \subsetneq N$*. Then, the game* (N, v_{F_T}) *is decomposable with respect to* $\mathcal{P} = \{T, N \backslash T\}$*.*

Proof. Let $S \subseteq N$. Then, $v_{F_T}(S \cap T) = v((S \cap T) \cup (N \backslash T)) - v(N \backslash T) + v(\emptyset)$ and $v_{F_T}(S \cap (N \backslash T)) = v(N \backslash T) - v(N \backslash T) + v(S \cap (N \backslash T))$. Hence, $v_{F_T}(S \cap T) + v_{F_T}(S \cap (N \backslash T)) = v_{F_T}(S)$. $\qquad\square$

Let (T, v^T) and $(N \backslash T, v^{N \backslash T})$ denote the two components of the decomposition in Proposition 2. Next result is now completely straightforward.

Corollary 1. *Let* (N, v) *be a strictly convex game and* $\emptyset \neq T \subsetneq N$*. Then,* (T, v^T) *and* $(N \backslash T, v^{N \backslash T})$ *are strictly convex games such that* $C(N, v_{F_T}) = C(T, v^T) \times C(N \backslash T, v^{N \backslash T})$*. The cores in the cartesian product have dimensions* $|T| - 1$ *and* $|N \backslash T| - 1$*, respectively.*

We move now to core complexity. Two polytopes P and P' are *combinatorially equivalent* if there is a one-to-one map $f : \mathcal{F}(P) \rightarrow \mathcal{F}(P')$ that is inclusion preserving, *i.e.*, $F \subseteq F'$ if and only if $f(F) \subseteq f(F')$. We define the *combinatorial complexity* of the core of a game as the number of different equivalence classes there are among its facets according to the above relation. Given a strictly convex game and a coalition $\emptyset \neq T \subsetneq N$, all the $|T|$-faces are combinatorially equivalent and, moreover, the faces F_T and $F_{N \backslash T}$ are also combinatorially equivalent. Let $\lfloor \cdot \rfloor$ be the floor function, *i.e.*, for each $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the largest integer not larger than r. The following corollaries are immediate from Lemma 2 and the remark below it.

Corollary 2. *Let* (N, v) *be a strictly convex game. Then, for each* $t \in \{1, \ldots, n - 1\}$*,* $C(N, v)$ *has* $2\binom{n}{t}$ *combinatorially equivalent facets and each of them can be decomposed as the cartesian product of the cores of two strictly convex games with* t *and* $n - t$ *players, respectively.*

Corollary 3. *Let* (N, v) *and* (N, w) *be two strictly convex games. Then,* $C(N, v)$ *and* $C(N, w)$ *are combinatorially equivalent. Indeed, given* $t \in \{0, \ldots, n\}$*, for each* $T, S \subseteq N$ *such that* $|T| = |S| = t$*,* $C(N, v_{F_T})$ *and* $C(N, w_{F_S})$ *are combinatorially equivalent.*

Corollary 4. *Let* (N, v) *be a strictly convex game. Then, the combinatorial complexity of* $C(N, v)$ *is* $\lfloor \frac{n}{2} \rfloor$*.*

In Figure 1 we illustrate, with the aid of Pascal's triangle, the above corollaries. Given n, below $\binom{n}{0}$ and $\binom{n}{n}$, we draw the $F_\emptyset$ and F_N faces, *i.e.*, $C(N, v)$. Then, for each $t \in \{1, \ldots, n-1\}$, the polytope below $\binom{n}{t}$ represents one of the $\binom{n}{t}$ combinatorially equivalent T-facets (with $|T| = t$) of $C(N, v)$ and, since $F_T = C(N, v_{F_T})$, it also represents the cores of the T-face games. Remarkably, Figure 1 contains a lot of information about the geometry of the cores of strictly convex games and, moreover, does it in a noteworthy visual way.

References

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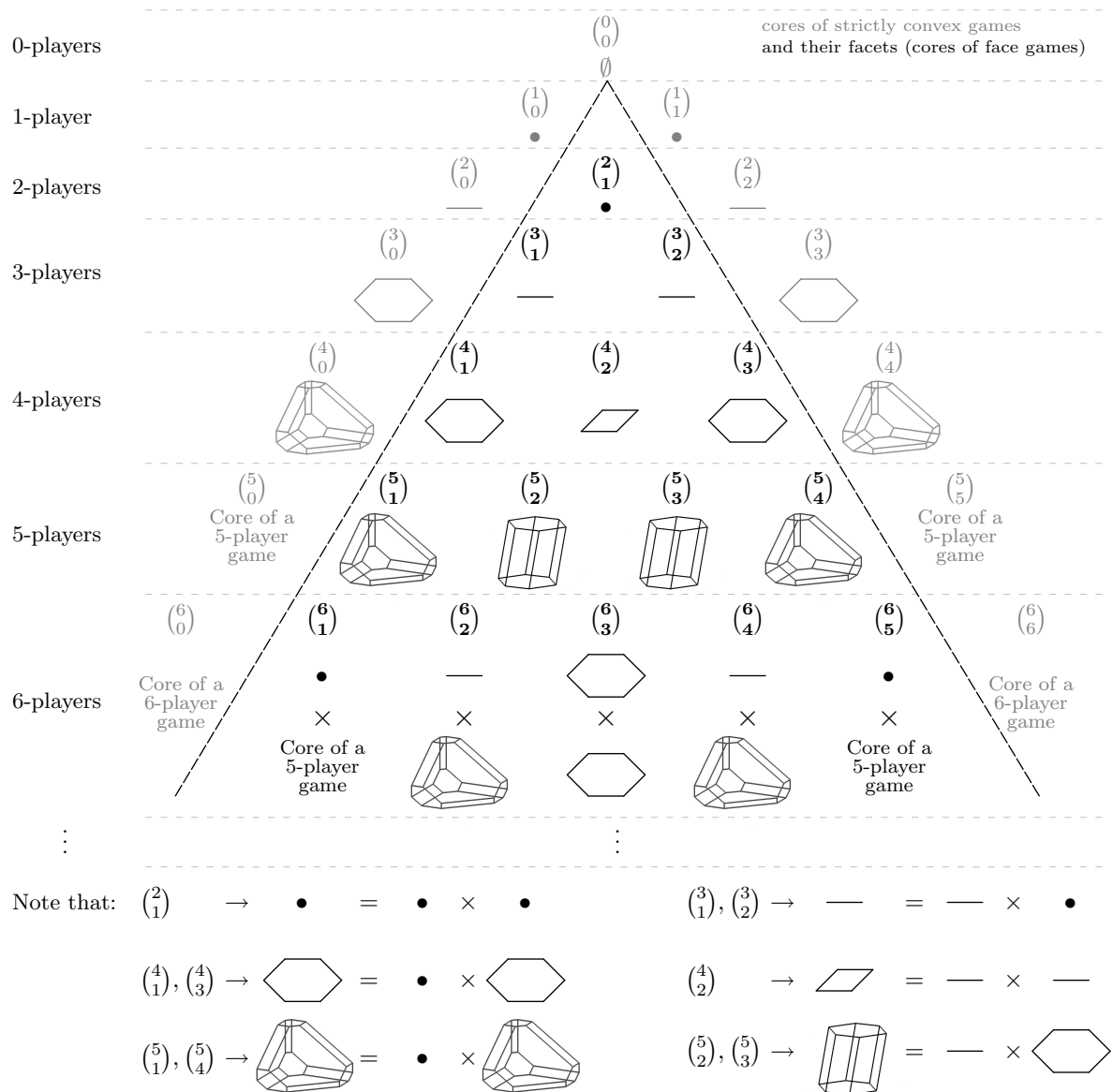SHAPLEY, L. S. (1971): "Cores of Convex Games," *International Journal of Game Theory*, 1, 11–26.

0-players $\binom{0}{0}$ $\emptyset$ cores of strictly convex games and their facets (cores of face games)

1-player $\binom{1}{0}$ $\binom{1}{1}$

2-players $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

3-players $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

4-players $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

5-players $\binom{5}{0}$ Core of a 5-player game $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$ Core of a 5-player game

6-players $\binom{6}{0}$ Core of a 6-player game $\binom{6}{1}$ $\binom{6}{2}$ $\binom{6}{3}$ $\binom{6}{4}$ $\binom{6}{5}$ $\binom{6}{6}$ Core of a 6-player game

Core of a 5-player game   Core of a 5-player game

Note that: $\binom{2}{1} \rightarrow \bullet = \bullet \times \bullet$ $\binom{3}{1}, \binom{3}{2} \rightarrow \text{—} = \text{—} \times \bullet$

$\binom{4}{1}, \binom{4}{3} \rightarrow \bigcirc = \bullet \times \bigcirc$ $\binom{4}{2} \rightarrow \diagup\diagdown = \text{—} \times \text{—}$

$\binom{5}{1}, \binom{5}{4} \rightarrow = \bullet \times$ $\binom{5}{2}, \binom{5}{3} \rightarrow = \text{—} \times \bigcirc$

Figure 1: Pascal's triangle and core complexity

- For each n, below $\binom{n}{n}$ and $\binom{n}{0}$ we depict the core of an n-player strictly convex game. The core of any other n-player strictly convex game being combinatorially equivalent to it (Corollary 3).

- For each n and each $t \in \{1, \ldots, n-1\}$, below each $\binom{n}{t}$ we depict one of the $\binom{n}{t}$ combinatorially equivalent T-facets (with $|T| = t$) of $C(N, v)$ (Corollary 2).

- For each n and each $t \in \{1, \ldots, n-1\}$, the facets below $\binom{n}{t}$ and $\binom{n}{n-t}$ are also combinatorially equivalent to each other (Corollary 2).

- Differently form Pascal's triangle, for each n and each $t \in \{1, \ldots, n-1\}$, the facets below $\binom{n}{t}$ are not obtained through $\binom{n-1}{t-1}$ and $\binom{n-1}{t}$. Instead, they are the product of two cores: the one below $\binom{t}{t}$ and the one below $\binom{n-t}{n-t}$ (Corollary 1). For $n \leq 5$, this products are represented in the three lines below the triangle.

- Finally, Corollary 4 is readily verified from the picture.