A Silent Battle over a Cake

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Published in European Journal of Operational Research (2007) 177, 591-603
Published version available at http://www.sciencedirect.com
DOI 10.1016/j.ejor.2005.10.062

Abstract: The division of a cake by n players is modeled as a (silent) game of timing. We show that such games admit a unique Nash equilibrium.

Key Words: Game theory, timing games, War of Attrition, Nash equilibrium.

1 Introduction

There are many strategic situations in which some agents face a decision problem in which timing is important. The literature on timing games has been devoted to analyze these situations and provide theoretical models to study the underlying strategic problem. A first approach to timing games appears in Karlin (1959) in the zero sum context. More recent contributions are Baston and Garnaev (2000) and Laraki et al. (2003). A classic example of timing game is the war of attrition, introduced in Smith (1974) and widely studied, for instance, in Hendricks et al. (1988). More specifically, consider the following war of attrition game. Two rival firms are engaged in a race to make a patentable discovery, and hence, as soon as one firm makes the discovery, all the previous effort made by the other firm turns out to be useless. This patent race model has been widely studied in literature (see, for instance, Fudenberg et al. (1983)). In this model it is assumed that, as soon as one of the firms leaves the race, the game ends. The motivation for this assumption is that, once there is only one firm in the race, the game reduces to a decision problem in which the remaining firm has to optimize its resources. Hence, the strategy of each firm consists of deciding, for each time t, whether to leave the race or not. Most of the literature in timing games models what we call *non-silent* timing games, that is, as soon as one player acts, the others are informed and the game ends.[1] In this paper, on the contrary, we provide a formal model for the *silent* situation. We use again the patent race to motivate our approach. Consider a situation in which two firms are engaged in a patent race and also in an advertising campaign. Suppose that one of the two rival firms, say firm 1, decides to leave the patent race. Then, it will probably be the case that firm 1 does not want firm 2 to realize that 1 not in the race anymore; and therefore, firm 1 can get a more advantageous position for the advertising campaign. Moreover, if firm 2 does not realize about the fact that firm 1 has

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[1]An exception is Reinganum (1981), although her model is very different from ours.

already left the race, it can also be the case that, having already firm 1 left the race, firm 2 leaves the race before making the discovery, benefiting again firm 1.

Next, we introduce our silent timing game. We consider the situation that n players have to divide a cake of size S. At time 0 player i has the initial right to receive the amount α_i, where it is assumed that $\sum_{i\in N}\alpha_i < S$. If player i claims his part at time $t > 0$ then he receives the discounted part $\delta^t\alpha_i$ of the cake, unless he is the last claimant in which case he receives the discounted remaining part of the cake $\delta^t(S - \sum_{j\neq i}\alpha_j)$. We refer to this game as a *cake sharing game*.

Hamers (1993) showed that 2-person cake sharing games always admit a unique Nash equilibrium. In this paper we consider cake sharing games that are slightly different from the games introduced in Hamers (1993). We first provide an alternative, but more direct, existence and uniqueness result for 2-player cake sharing games and we generalize this result to cake sharing games with more players.

It is worth to mention the similarities between our results and some well known results in all-pay auctions (Weber, 1985). At first glance, our model seems quite different from that of all-pay auctions, but, it turns out to be the case that they have many similarities. Indeed, in this paper we show that the same kind of results obtained for the all-pay auction (Hilman and Riley, 1989; Baye et al., 1996) can be obtained for our timing game. Anyhow, even when both the results and also the arguments underlying some of the proofs are very similar, the two models are different enough so that our results can not be derived from those in the all-pay auctions literature.

This paper is organized as follows. In Section 2 we introduce cake sharing games. In Sections 3 and 4 we deal with 2-player cake sharing games and more player cake sharing games, respectively.

2 The Model

In this section we formally introduce the cake sharing games.

Let $N = \{1,\ldots,n\}$ be a set of players with $n \geq 2$, let $S > 0$, let $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^N_+$ be such that $\alpha_1 + \cdots + \alpha_n < S$ and let $\delta \in (0,1)$. Throughout this paper we assume that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. The number S is called the *size of the cake*, the vector α the *initial right vector* and δ the *discount factor*.

The *cake sharing game with pure strategies*, associated with S, α and δ, is the triple $\Gamma^{\text{pure}}_{S,\alpha,\delta} = \langle N, \{A_i\}_{i\in N}, \{\pi_i\}_{i\in N}\rangle$, where

- $A_i = [0,\infty)$ is the set of pure strategies of player $i \in N$. A strategy for player $i \in N$ is to choose the time moment at which he claims his part of the cake.
- π_i is the payoff function of player $i \in N$, defined by:

$$\pi_i(t_1,\ldots,t_n) = \begin{cases} (S - \sum\limits_{j\neq i}\alpha_j)\delta^{t_i} & \text{if } t_i > \max\limits_{j\neq i} t_j \\ \alpha_i\delta^{t_i} & \text{otherwise.} \end{cases}$$

So, if there is a unique last claimant this player will receive the discounted value of the cake that remains after that the other players have taken their initial rights. If there is not a unique last claimant all players receive the discounted value of their initial rights. Note that the payoff functions defined above differ slightly from the payoff functions introduced in Hamers (1993), where, in case there is not a unique last claimant, the discounted value of the remaining cake is shared equally between the last claimants.[2]

One easily verifies that $\Gamma^{\text{pure}}_{S,\alpha,\delta}$ has no Nash equilibria: if there is a unique last claimant this player can improve his payoff by claiming a little bit earlier (remaining the last claimant of course), if there is no unique last claimant then one of the last claimants can improve his payoff by claiming a little bit later (becoming the unique last claimant in this way). So, for an appropriate analysis of cake sharing games we need to consider mixed strategies.

Formally, a *mixed strategy* is a function $G : [0, \infty) \to [0, 1]$ satisfying:

- $G(0) = 0$,
- G is a nondecreasing function,
- G is left-continuous,
- $\lim_{x\to\infty} G(x) = 1$.

For a mixed strategy G we can always find a probability measure P on $[0, \infty)$[3] such that

$$G(x) = P\big([0, x)\big) \tag{1}$$

for all $x \in [0, \infty)$. On the other hand, every probability measure P on $[0, \infty)$ defines by formula (1) a mixed strategy G. So the set of mixed strategies coincides with the set of probability measures on $[0, \infty)$.[4] Let $\mathcal{G}$ denote the set of all mixed strategies. We introduce now some other notations related to mixed strategy G:

- for $x \in [0, \infty)$ we denote $\lim_{y\downarrow x} G(y)$, the probability of choosing an element in the closed interval $[0, x]$, by $G(x^+)$.
- if for an $x > 0$ it holds that for every $a, b \in [0, \infty)$ with $a < x < b$ we have $G(b) > G(a^+)$, i.e., the probability of choosing an element in (a, b) is positive, then x is an element of the support of G. If for every $b > 0$ it holds that $G(b) > 0$, i.e., the probability of choosing an element in $[0, b)$ is positive, then 0 is an element of the support of G. The support of the distribution function G will be denoted by $S(G)$. One easily verifies that $S(G)$ is a closed set.
- the set of jumps (discontinuities) of G is $J(G) := \{x \in [0, \infty) \mid G(x^+) > G(x)\}$, i.e., the set of pure strategies which are chosen with positive probability.

[2]Let us make some coments concerning the relation between the cake sharing game (CS) and the all-pay auctions model (AP). For simplicity, we think of the two player case. Setting aside the issue of timing, note the following differences: (i) Initial rights: in CS they depend on the player (α_i), in AP they are 0; (ii) in CS each player wants to get $1 - (\alpha_1 + \alpha_2)$, in AP the valuation of the object depends on the player; and (iii) In CS waiting till time t, each player is "paying" $\alpha_i - (\alpha_i)\delta^t$, *i.e.*, it depends on the player, in AP bidding v, each player is "paying" v. All the other strategic elements are analogous in the two models.

[3]See Rohatgi (1976) for more details.

[4]An alternative way of defining mixed strategies G is as a nondecreasing, right-continuous function from $[0, \infty)$ to $[0, 1]$ with $\lim_{x\to\infty} G(x) = 1$. For such a function we can always find a probability measure P on $[0, \infty)$ such that $G(x) = P\big([0, x]\big)$ for all $x \in [0, \infty)$, i.e., G is the (cumulative) distribution function corresponding to P. Although this equivalent approach seems more natural, this would lead to technical problems when computing Lebesgue-Stieltjes integrals later on.

If player i chooses pure strategy t and all other players choose mixed strategies $\{G_j\}_{j\neq i}$ then the expected payoff for player i is

$$\begin{aligned}&\pi_i(G_1,\ldots,G_{i-1},t,G_{i+1},\ldots,G_n)\\&\quad=\prod_{j\neq i}G_j(t)\cdot\delta^t\cdot(S-\sum_{j\neq i}\alpha_j)+(1-\prod_{j\neq i}G_j(t))\cdot\delta^t\cdot\alpha_i\\&\quad=\delta^t\cdot(\alpha_i+(S-\sum_{j\in N}\alpha_j)\cdot\prod_{j\neq i}G_j(t)).\end{aligned}$$

If player i also chooses a mixed strategy G_i, whereas all other players stick to mixed strategies $\{G_j\}_{j\neq i}$ then the expected payoff for player i can be computed by use of the Lebesgue-Stieltjes integral:

$$\pi_i(G_1,\ldots,G_n)=\int\pi_i(G_1,\ldots,G_{i-1},t,G_{i+1},\ldots,G_n)dG_i(t). \tag{2}$$

Note that, with a slight abuse of notation, the functions π_i do not only denote payoffs to players when pure strategies are played, but also when mixed strategies are used.

The *cake sharing game*, associated with S, α and δ, is defined by the triple $\Gamma_{S,\alpha,\delta}=\langle N,\{X_i\}_{i\in N},\{\pi_i\}_{i\in N}\rangle$, where

- $X_i=\mathcal{G}$ is the set of mixed strategies of player $i\in N$,
- π_i, defined by (2), is the (expected) payoff function of player $i\in N$.

Given a strategy profile $G=(G_1,G_2,\ldots,G_n)\in\mathcal{G}^n$, let $\pi_i^G(t)$ denote the payoff $\pi_i(G_1,\ldots,G_{i-1},t,G_{i+1},\ldots,G_n)$. So $\pi_i^G(t)$ is the expected payoff of player i when he chooses the pure strategy t and all the other players act in accordance with G.

3 Two Players

In this section we provide an alternative proof of the result of Hamers (1993) for 2-player cake sharing games. Our incentives for doing this job are threefold. First of all we want to recall that our model is slightly different from the model of Hamers (1993), so a new proof is required. Secondly, our proof is more direct than Hamers' proof. Finally, our proof will form the basis for the results in Section 4 for cake sharing games with three or more players.

First we derive a number of properties for Nash equilibria of n-player cake sharing games. The following lemma shows that in a Nash equilibrium players do not put positive probability on a pure strategy $t>0$.

Lemma 1. *Let $\Gamma_{S,\alpha,\delta}$ be an n-player cake sharing game and let the profile $G=(G_i)_{i\in N}\in\mathcal{G}^N$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Then $J(G_i)\cap(0,\infty)=\emptyset$ for every $i\in N$.*

Proof. Let $i\in N$. We will show that $J(G_i)\cap(0,\infty)=\emptyset$. Without loss of generality we may assume that $i=1$.

Suppose that $u\in J(G_1)\cap(0,\infty)$. If $G_i(u^+)=0$ for some $i\neq 1$ then $\pi_1^G(t)=\delta^t\alpha_1$ whenever $t\in[0,u]$. Since the function $\pi_1^G(\cdot)$ is strictly decreasing on $[0,u]$, player 1 would be better off moving the probability in u to 0. So $G_i(u^+)>0$ for all $i\in N$. Now consider the functions

$$\pi_i^G(t)=\delta^t(\alpha_i+(S-\sum_{j\in N}\alpha_j)\prod_{j\neq i}G_j(t))$$

for $i \in N\backslash\{1\}$. Since G_1 is discontinuous at u, i.e., $G_1(u^+) > G_1(u)$, there exist $u_1 < u$, $u_2 > u$ and $\varepsilon > 0$ such that for every $i \neq 1$

$$\pi_i^G(u_2) - \pi_i^G(t) \geq \varepsilon \text{ for every } t \in [u_1, u].$$

If player $i \in N\backslash\{1\}$ puts positive probability on $[u_1, u]$, i.e., if $G_i(u^+) > G_i(u_1)$, then he can increase his payoff by at least $\varepsilon(G_i(u^+) - G_i(u_1))$ by moving all this probability to u_2. So for all $i \in N\backslash\{1\}$ we have $G_i(u^+) = G_i(u_1)$ and hence $G_i(t) = G_i(u)$ for every $t \in [u_1, u]$. Therefore the function

$$\pi_1^G(t) = \delta^t(\alpha_1 + (S - \sum_{j\in N} \alpha_j) \prod_{j\neq 1} G_j(t))$$

is strictly decreasing on $[u_1, u]$. So player 1 can improve his payoff by moving some probability from u to u_1. $\square$

Lemma 1 implies that in a Nash equilibrium G players use mixed strategies which are continuous on $(0, \infty)$. Therefore we may write $G_i(t^+) = G_i(t)$ for every $i \in N$ and $t > 0$. Moreover, the functions $\pi_i^G(\cdot)$ are continuous on $(0, \infty)$.

Lemma 2. *Let $\Gamma_{S,\alpha,\delta}$ be an n-player cake sharing game and let the profile $G = (G_i)_{i\in N} \in \mathcal{G}^N$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Let $i \in N$ and $t \in S(G_i)$. There exists $j \in N\backslash\{i\}$ such that $t \in S(G_j)$.*

Proof. Suppose that $t \notin \cup_{j\neq i} S(G_j)$. We will distinguish between two cases:

Case (i): $t > 0$. Since for each $j \in N$, $S(G_j)$ is a closed set, also $\cup_{j\neq i} S(G_j)$ is a closed set. Hence, its complement is an open set. So there exist $t_1 < t < t_2$ such that $[t_1, t_2] \subset [0, \infty) \setminus \cup_{j\neq i} S(G_j)$. Moreover, since the functions G_j are constant outside the support, we have $G_j(t_2) = G_j(t_1)$ for all $j \neq i$. Hence $G_j(u) = G_j(t_2)$ for every $u \in [t_1, t_2]$ and every $j \neq i$. Therefore the function

$$\pi_i^G(u) = \delta^u(\alpha_i + (S - \sum_{j\in N} \alpha_j) \prod_{j\neq i} G_j(u))$$

is strictly decreasing on $[t_1, t_2]$. Since $t \in S(G_i)$ we have $G_i(t_2) > G_i(t_1^+)$, i.e., player i puts positive probability on (t_1, t_2). Now player i can strictly improve his payoff by moving all this probability to t_1.

Case (ii): $t = 0$. Let $b > 0$ be the smallest element in $\cup_{j\neq i} S(G_j)$ (recall that all the $S(G_j)$ are closed). Clearly $G_j(b) = 0$ for all $j \neq i$. Again, if $G_i(b) > G_i(0^+)$, i.e., if player i puts positive probability on $(0, b)$, then similar arguments as in Case (i) can be used to show that player i can strictly improve his payoff by moving this probability to 0. So $G_i(b) = G_i(0^+)$ and hence, since $0 \in S(G_i)$, we have $G_i(0^+) > 0$. Moreover $G_i(t) = G_i(b)$ for every $t \in (0, b]$ (this is relevant only for the case $n = 2$), so

$$\pi_j^G(t) = \delta^t(\alpha_j + (S - \sum_{k\in N} \alpha_k) \prod_{k\neq j} G_k(t))$$

is strictly decreasing on $(0, b]$ for every $j \in N\backslash\{i\}$.

Let $a \in (0, b)$ and let $j \in N\backslash\{i\}$ be a player such that $b \in S(G_j)$. Clearly $\varepsilon := \pi_j^G(a) - \pi_j^G(b) > 0$. Since the function $\pi_j^G(\cdot)$ is continuous on $(0, \infty)$ we have, for $\delta > 0$ sufficiently small,

$$\pi_j^G(a) - \pi_j^G(t) > \frac{1}{2}\varepsilon \text{ for every } t \in [b, b + \delta].$$

Since $b \in S(G_j)$ we have $G_j(b + \delta) > 0 = G_j(b)$. So, player j can improve his payoff by moving the probability he assigns to $[b, b + \delta)$ to a. Contradiction. $\square$

The following lemma shows that if some pure strategy t does not belong to the support of any of the equilibrium strategies, then no pure strategy $t' > t$ belongs to the support of any of the equilibrium strategies either.

Lemma 3. *Let $G = (G_i)_{i\in N}$ be a Nash equilibrium of the n-player cake sharing game $\Gamma_{S,\alpha,\delta}$. Suppose $t \in [0,\infty)$ is such that $t \notin S(G_j)$ for every $j \in N$. Then $(t,\infty)\cap S(G_j) = \emptyset$ for every $j \in N$.*

Proof. Define $K = \cup_{j\in N} S(G_j)$. Clearly K is closed and $t \notin K$. We have to show that $K \cap (t,\infty) = \emptyset$. Suppose that $K \cap (t,\infty) \neq \emptyset$. Define $t^* = \min\{u \in K | u > t\}$. Let $j^* \in N$ be such that $t^* \in S(G_{j^*})$. Because $[t,t^*) \cap S(G_j) = \emptyset$ for every $j \in N$ we have $G_j(t) = G_j(t^*)$ for every $j \in N$. So the functions G_j are constant on $[t,t^*]$. Since

$$\pi^G_{j^*}(u) = \delta^u(\alpha_{j^*} + (S - \sum_{j\in N}\alpha_j)\prod_{j\neq j^*} G_j(u))$$

for every $u \in [0,\infty)$, the function $\pi^G_{j^*}(\cdot)$ is strictly decreasing on $[t,t^*]$. By continuity of $\pi^G_{j^*}(\cdot)$ at t^* we infer that $\pi^G_{j^*}(t) > \pi^G_{j^*}(u)$ for every $u \in [t^*, t^*+\varepsilon]$, with $\varepsilon > 0$ sufficiently small. Hence G_{j^*} is constant on $[t^*, t^*+\varepsilon]$ as well, contradicting the fact that $t^* \in S(G_{j^*})$. □

Now we provide specific results for 2-player cake sharing games. The following lemma shows that in a Nash equilibrium players use mixed strategies of which the supports coincide.

Lemma 4. *Let $\Gamma_{S,\alpha,\delta}$ be a 2-player cake sharing game and let $(G_1,G_2) \in \mathcal{G}\times\mathcal{G}$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Then $S(G_1) = S(G_2)$.*

Proof. This result is just a consequence of Lemma 2. □

In the following lemma we will show that the supports of the strategies in a Nash equilibrium are compact intervals.

Lemma 5. *Let $\Gamma_{S,\alpha,\delta}$ be a 2-player cake sharing game and let $G = (G_1,G_2) \in \mathcal{G}\times\mathcal{G}$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Let $k := \log_\delta \frac{\alpha_2}{S-\alpha_1}$. Then $S(G_1) = S(G_2) = [0,k]$.*

Proof. First we will show that $S(G_1) = S(G_2) \subseteq [0,k]$. For every $t \in (k,\infty)$ we have

$$\begin{aligned}
\pi^G_2(t) &= \delta^t(\alpha_2 + (S-\alpha_1-\alpha_2)G_1(t)) \\
&\leq \delta^t(\alpha_2 + (S-\alpha_1-\alpha_2)) \\
&= \delta^t(S-\alpha_1) \\
&< \delta^k(S-\alpha_1) \\
&= \alpha_2 \\
&= \pi^G_2(0).
\end{aligned}$$

If $G_2(k) = G_2(k^+) < 1$, i.e., if player 2 puts positive probability on (k,∞), he can improve his payoff strictly by moving all this probability to 0. Therefore $G_2(k) = 1$ and hence $S(G_1) = S(G_2) \subseteq [0,k]$.
Let k^* be the largest element in the closed set $S(G_1)$. Clearly $k^* \leq k$. If $k^* = 0$ then (G_1,G_2) would be an equilibrium in pure strategies, a contradiction. So $k^* > 0$. According to Lemma 3 we have $S(G_1) = S(G_2) = [0,k^*]$.

The only thing which remains to be shown is that $k^* = k$. Suppose that $k^* < k$. For $\tau \in (0, k - k^*)$ we have

$$\begin{aligned}
\pi_1^G(k^* + \tau) &= \delta^{k^*+\tau}(\alpha_1 + (S - \alpha_1 - \alpha_2)G_2(k^* + \tau)) \\
&= \delta^{k^*+\tau}(\alpha_1 + (S - \alpha_1 - \alpha_2)) \\
&= \delta^{k^*+\tau}(S - \alpha_2) \\
&> \delta^k(S - \alpha_2) \\
&= \frac{\alpha_2(S - \alpha_2)}{S - \alpha_1} \\
&\geq \alpha_1 \\
&= \pi_1^G(0),
\end{aligned}$$

where at the weak inequality we used that $\alpha_2(S - \alpha_2) \geq \alpha_1(S - \alpha_1)$. So, if $G_1(0^+) > 0$, i.e., if player 1 plays pure strategy 0 with positive probability, then he can improve his payoff by moving some probability from 0 to pure strategy $k^* + \tau$. Therefore $G_1(0^+) = 0$. For some $t \in (k^*, k)$ we have

$$\begin{aligned}
\pi_2^G(t) &= \delta^t(\alpha_2 + (S - \alpha_1 - \alpha_2)G_1(t)) \\
&= \delta^t(\alpha_2 + (S - \alpha_1 - \alpha_2)) \\
&= \delta^t(S - \alpha_1) \\
&> \delta^k(S - \alpha_1) \\
&= \alpha_2 \\
&= \pi_2^G(0).
\end{aligned}$$

Since $0 \in S(G_1)$ and $\pi_2^G(\cdot)$ is continuous at 0 (due to $G_1(0^+) = 0$) player 2 can improve his payoff strictly by moving some probability from the neighborhood of 0 to t. Contradiction, so $k^* = k$. □

Now we are able to prove the main theorem of this section.

Theorem 1. *Let $\Gamma_{S,\alpha,\delta}$ be a 2-player cake sharing game and $k := \log_\delta \frac{\alpha_2}{S-\alpha_1}$. Define $G^* = (G_1^*, G_2^*) \in \mathcal{G} \times \mathcal{G}$ by*

$$\begin{aligned}
G_1^*(t) &= \begin{cases} \dfrac{\alpha_2 - \alpha_2\delta^t}{\delta^t(S - \alpha_1 - \alpha_2)} & \text{if } 0 \leq t \leq k \\ 1 & \text{if } t > k \end{cases} \\
G_2^*(t) &= \begin{cases} 0 & \text{if } t = 0 \\ \dfrac{\alpha_2(S - \alpha_2) - \alpha_1(S - \alpha_1)\delta^t}{\delta^t(S - \alpha_1)(S - \alpha_1 - \alpha_2)} & \text{if } 0 < t \leq k \\ 1 & \text{if } t > k \end{cases}
\end{aligned}$$

Then G^* is the unique Nash equilibrium of $\Gamma_{S,\alpha,\delta}$ with equilibrium payoffs

$$\begin{aligned}
\pi_1(G_1^*, G_2^*) &= \frac{\alpha_2(S - \alpha_2)}{S - \alpha_1} \\
\pi_2(G_1^*, G_2^*) &= \alpha_2.
\end{aligned}$$

Proof. One easily verifies that

$$\pi_1^{G^*}(t) = \begin{cases} \alpha_1 & \text{if } t = 0 \\ \dfrac{\alpha_2(S - \alpha_2)}{S - \alpha_1} & \text{if } 0 < t \leq k \\ \delta^t(S - \alpha_2) & \text{if } t > k \end{cases}$$

and

$$\pi_2^{G^*}(t) = \begin{cases} \alpha_2 & \text{if } 0 \le t \le k \\ \delta^t(S-\alpha_1) & \text{if } t > k. \end{cases}$$

Therefore

$$\pi_1(G_1^*, G_2^*) = \frac{\alpha_2(S-\alpha_2)}{S-\alpha_1} \text{ and } \pi_2(G_1^*, G_2^*) = \alpha_2.$$

Since

$$\pi_1(t, G_2^*) \le \frac{\alpha_2(S-\alpha_2)}{S-\alpha_1} \text{ and } \pi_2(G_1^*, t) \le \alpha_2$$

for every $t \in [0,\infty)$ we infer that G^* is a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$.

In order to show that there are no other Nash equilibria let (G_1, G_2) be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. According to Lemma 1 the strategies G_1 and G_2 are continuous on $(0,\infty)$. In the same way as in the proof of Lemma 5 we can show that $G_1(0^+) = 0$. So, the function $\pi_1^G(\cdot)$ is continuous on $(0,\infty)$ and the function $\pi_2^G(\cdot)$ is continuous on $[0,\infty)$. By Lemma 5 we have $S(G_1) = S(G_2) = [0,k]$. So, there exist constants c and d such that

$$\begin{aligned} c &= \pi_1^G(t) = \delta^t(\alpha_1 + (S-\alpha_1-\alpha_2)G_2(t)) \text{ for every } t \in (0,k] \\ d &= \pi_2^G(t) = \delta^t(\alpha_2 + (S-\alpha_1-\alpha_2)G_1(t)) \text{ for every } t \in [0,k]. \end{aligned}$$

Since $G_1(0) = 0$ we have $d = \pi_2^G(0) = \alpha_2$, from which we derive that

$$G_1(t) = \frac{\alpha_2 - \alpha_2\delta^t}{\delta^t(S-\alpha_1-\alpha_2)} = G_1^*(t)$$

for every $t \in [0,k]$. Clearly, $G_1(t) = 1 = G_1^*(t)$ for every $t > k$, so $G_1 = G_1^*$. Moreover, since $G_2(k) = 1$ we have $c = \pi_1^G(k) = \delta^k(S-\alpha_2) = \frac{\alpha_2(S-\alpha_2)}{S-\alpha_1}$, from which we derive that

$$G_2(t) = \frac{\alpha_2(S-\alpha_2) - \alpha_1(S-\alpha_1)\delta^t}{\delta^t(S-\alpha_1)(S-\alpha_1-\alpha_2)} = G_2^*(t)$$

for every $t \in (0,k]$. Clearly $G_2(0) = 0 = G_2^*(0)$ and $G_2(t) = 1 = G_2^*(t)$ for every $t > k$, so $G_2 = G_2^*$. This finishes the proof. □

4 More Players

In this section we consider cake sharing games with more than two players. Again we will show that such games admit a unique Nash equilibrium.

First we show that mixed strategies in a Nash equilibrium have a bounded support.

Lemma 6. *Let $\Gamma_{S,\alpha,\delta}$ be an n-player cake sharing game with $n \ge 3$ and let $G = (G_i)_{i\in N} \in \mathcal{G}^N$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. For every $i \in N$ let*

$$k_i := \log_\delta \frac{\alpha_i}{S - \sum_{j\neq i}\alpha_j}.$$

Then $k_1 > k_2 > \cdots > k_n$ and $S(G_i) \subset [0,k_i]$ for every $i \in N$. Moreover $S(G_1) \subset [0,k_2]$.

Proof. Let $i,j \in N$ be such that $i > j$. Define $\gamma := \sum_{l\neq i,j} \alpha_l$. Then

$$\begin{aligned} &\alpha_i(S-\gamma-\alpha_i) - \alpha_j(S-\gamma-\alpha_j) \\ &\quad = \alpha_i(S-\gamma-\alpha_i) - \alpha_i\alpha_j + \alpha_i\alpha_j - \alpha_j(S-\gamma-\alpha_j) \\ &\quad = (\alpha_i-\alpha_j)(S - \textstyle\sum_{l\in N}\alpha_l) \\ &\quad > 0. \end{aligned}$$

So

$$\frac{\delta^{k_i}}{\delta^{k_j}} = \frac{\dfrac{\alpha_i}{S-\gamma-\alpha_j}}{\dfrac{\alpha_j}{S-\gamma-\alpha_i}} = \frac{\alpha_i(S-\gamma-\alpha_i)}{\alpha_j(S-\gamma-\alpha_j)} > 1 \tag{3}$$

and hence $k_i < k_j$.
Now, for every $i \in N$ and $t \in (k_i, \infty)$

$$\begin{aligned}
\pi_i^G(t) &= \delta^t(\alpha_i + (S - \textstyle\sum_{j\in N}\alpha_j)\prod_{j\neq i} G_j(t)) \\
&\leq \delta^t(\alpha_i + (S - \textstyle\sum_{j\in N}\alpha_j)) \\
&= \delta^t(S - \textstyle\sum_{j\neq i}\alpha_j) \\
&< \delta^{k_i}(S - \textstyle\sum_{j\neq i}\alpha_j) \\
&= \alpha_i \\
&= \pi_i^G(0).
\end{aligned}$$

Repeating the reasoning in Lemma 5, if $G_i(k_i) = G_i(k_i^+) < 1$, then player i can strictly improve his payoff by moving all the probability in (k_i, ∞) to 0. Therefore $G_i(k_i) = 1$ for all $i \in N$. For any $j \in N\backslash\{1\}$ we have $k_2 \geq k_j$ and hence $G_j(k_2) \geq G_j(k_j) = 1$. So $G_j(k_2) = 1$ and hence $S(G_j) \subset [0, k_2]$. Now, because of Lemma 2 we conclude that $S(G_1) \subset [0, k_2]$ as well. □

In the following lemma we show that pure strategy 0 belongs to the support of every equilibrium strategy. Moreover, players $2, \ldots, n$ play this strategy with positive probability.

Lemma 7. *Let $\Gamma_{S,\alpha,\delta}$ be an n-player cake sharing game and let $G = (G_i)_{i\in N} \in \mathcal{G}^N$ be a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. Then $0 \in S(G_j)$ for every $j \in N$. Moreover $G_j(0^+) > 0$ for every $j \in N\backslash\{1\}$.*

Proof. Suppose $0 \notin S(G_i)$ for some $i \in N$. Let $s > 0$ be the smallest element in the closed set $S(G_i)$. Then $[0, s) \cap S(G_i) = \emptyset$ and hence $G_i(t) = 0$ for every $t \in [0, s]$. Consequently the function

$$\pi_j^G(t) = \delta^t(\alpha_j + (S - \sum_{k\in N}\alpha_k)\prod_{k\neq j} G_k(t)) = \alpha_j\delta^t$$

is strictly decreasing on $[0, s]$ for every $j \in N\backslash\{i\}$. Hence $(0, s) \cap S(G_j) = \emptyset$ for every $j \in N\backslash\{i\}$. Choose $s^* \in (0, s)$. Then $s^* \notin S(G_j)$ for every $j \in N$. According to Lemma 3 we have $(s^*, \infty) \cap S(G_i) = \emptyset$, a contradiction with $s \in S(G_i)$. So, $0 \in S(G_j)$ for every $j \in N$.
Now suppose $i \in N\backslash\{1\}$ is such that $G_i(0^+) = 0$. This implies that the function

$$\pi_1^G(t) = \delta^t(\alpha_1 + (S - \sum_{j\in N}\alpha_j)\prod_{j\neq 1} G_j(t))$$

is continuous at 0. Let $k_2 := \log_\delta \dfrac{\alpha_2}{S - \sum_{j\neq 2}\alpha_j}$. According to Lemma 6 we have $G_j(k_2) = 1$ for every $j \in N$. Hence

$$\pi_1^G(k_2) = \delta^{k_2}(S - \sum_{j\neq 1}\alpha_j) = \frac{\alpha_2}{S - \sum_{j\neq 2}\alpha_j}(S - \sum_{j\neq 1}\alpha_j) > \alpha_1 = \pi_1^G(0).$$

Due to continuity of $\pi_1^G(\cdot)$ at 0 we have $\pi_1^G(k_2) > \pi_1^G(t)$ for every $t \in [0, \varepsilon]$ with $\varepsilon > 0$ sufficiently small. So, $[0, \varepsilon) \cap S(G_1) = \emptyset$, contradicting the fact that $0 \in S(G_1)$. □

The following lemma provides the equilibrium payoffs in a Nash equilibrium.

Lemma 8. *Let $G = (G_i)_{i \in N}$ be a Nash equilibrium of the n-player cake sharing game $\Gamma_{S,\alpha,\delta}$ and let $\eta = (\eta_i)_{i \in N}$ be the corresponding vector of equilibrium payoffs. Then*

$$\eta_1 = \frac{\alpha_2(S - \sum_{j \neq 1} \alpha_j)}{S - \sum_{j \neq 2} \alpha_j},$$

and $\eta_j = \alpha_j$ for every $j \in N \backslash \{1\}$.

Proof. According to Lemma 7 we have $G_j(0^+) > 0$ for every $j \in N \backslash \{1\}$. Hence $\eta_j = \pi_j^G(0) = \alpha_j$ for every $j \in N \backslash \{1\}$. Again, let $k_2 := \log_\delta \frac{\alpha_2}{S - \sum_{j \neq 2} \alpha_j}$. According to Lemma 6 we have $G_j(k_2) = 1$ for all $j \in N$. Hence

$$\eta_1 \geq \pi_1^G(k_2) = \frac{\alpha_2(S - \sum_{j \neq 1} \alpha_j)}{S - \sum_{j \neq 2} \alpha_j}.$$

If $\eta_1 > \pi_1^G(k_2)$ then, by the continuity of $\pi_1^G(\cdot)$ at k_2, we infer that for $\gamma > 0$ sufficiently small we have $\eta_1 > \pi_1^G(t)$ for every $t \in [k_2 - \gamma, k_2]$. Hence $S(G_1) \subseteq [0, k_2 - \gamma]$. Now player 2 can obtain more than α_2 by putting all his probability at $k_2 - \gamma + \epsilon$ for $\epsilon > 0$ small enough. □

In the following lemma we show that in a Nash equilibrium players $3, \ldots, n$ claim their initial right immediately, i.e., they play pure strategy 0.

Lemma 9. *Let $G = (G_i)_{i \in N}$ be a Nash equilibrium of the n-player cake sharing game $\Gamma_{S,\alpha,\delta}$ with $n \geq 3$. Then for every $i \in N \backslash \{1, 2\}$ we have*

$$G_i(t) = \begin{cases} 0 & \textit{if } t = 0 \\ 1 & \textit{if } t > 0 \end{cases}$$

i.e., G_i corresponds to pure strategy 0.

Proof. Let $i \in N \backslash \{1, 2\}$ and suppose that it is not true that

$$G_i(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Define $t^* = \inf\{t \mid G_i(t) = 1\}$. Note that $t^* > 0$ and $t^* \in S(G_i)$. Moreover, by continuity of G_i in t^* we have $G_i(t^*) = 1$. Now we have

$$\pi_2^G(t^*) = \delta^{t^*}(\alpha_2 + (S - \sum_{j \in N} \alpha_j) \prod_{j \neq 2} G_j(t^*)) \leq \alpha_2, \tag{4}$$

since otherwise player 2 could deviate to pure strategy t^* obtaining strictly more than his equilibrium payoff α_2. Moreover, due to $t^* \in S(G_i)$,

$$\pi_i^G(t^*) = \delta^{t^*}(\alpha_i + (S - \sum_{j \in N} \alpha_j) \prod_{j \neq i} G_j(t^*)) = \alpha_i, \tag{5}$$

From (4) and (5) we obtain

$$\delta^{t^*}(S - \sum_{j \in N} \alpha_j) \prod_{j \neq 2} G_j(t^*) \leq \alpha_2(1 - \delta^{t^*}),$$

$$\delta^{t^*}(S - \sum_{j\in N}\alpha_j)\prod_{j\neq i} G_j(t^*) = \alpha_i(1-\delta^{t^*}).$$

According to Lemma 7 we have $G_j(t^*) > 0$ for every $j \in N$. So, dividing these two expressions we get

$$\frac{G_i(t^*)}{G_2(t^*)} \leq \frac{\alpha_2}{\alpha_i} < 1$$

which leads to the conclusion that $G_2(t^*) > G_i(t^*) = 1$, a contradiction. $\square$

As a consequence of the last result, the only possible Nash equilibrium in a cake sharing game is one in which players $3,\ldots,n$ play pure strategy 0 and players 1 and 2 play the game with total cake size $S - \sum_{i=3}^n \alpha_i$.

Theorem 2. *Let $\Gamma_{S,\alpha,\delta}$ be an n-player cake sharing game with $n \geq 3$ and let $k_2 := \log_\delta \frac{\alpha_2}{S-\sum_{j\neq 2}\alpha_j}$. Define $G^* = (G_i^*)_{i\in N} \in \mathcal{G}^N$ by*

$$G_1^*(t) = \begin{cases} \dfrac{\alpha_2 - \alpha_2\delta^t}{\delta^t(S-\sum_{j\in N}\alpha_j)} & \text{if } 0 \leq t \leq k_2 \\ 1 & \text{if } t > k_2 \end{cases}$$

$$G_2^*(t) = \begin{cases} 0 & \text{if } t = 0 \\ \dfrac{\alpha_2(S-\sum_{j\neq 1}\alpha_j) - \alpha_1\delta^t(S-\sum_{j\neq 2}\alpha_j)}{\delta^t(S-\sum_{j\in N}\alpha_j)(S-\sum_{j\neq 2}\alpha_j)} & \text{if } 0 < t \leq k_2 \\ 1 & \text{if } t > k_2 \end{cases}$$

$$G_i^*(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$

for every $i \in \{3,\ldots,n\}$. Then G^* is the unique Nash equilibrium of $\Gamma_{S,\alpha,\delta}$.

Proof. Suppose $G = (G_i)_{i\in N} \in \mathcal{G}^N$ is a Nash equilibrium of $\Gamma_{S,\alpha,\delta}$. As a consequence of Lemma 9 we have $G_i = G_i^*$ for every $i \in \{3,\ldots,n\}$, so players $3,\ldots,n$ claim their initial rights immediately. Now (G_1, G_2) is a Nash equilibrium of the 2-player cake sharing game with cake size $S - \sum_{i=3}^n \alpha_i$ and initial right vector (α_1,α_2). According to Theorem 1 we have $G_1 = G_1^*$ and $G_2 = G_2^*$. So $G = G^*$.

Now we show that G^* is indeed a Nash equilibrium. Since, according to Theorem 1, (G_1^*, G_2^*) is a Nash equilibrium of the 2-player cake sharing game with cake size $S - \sum_{i=3}^n \alpha_i$ and initial right vector (α_1,α_2), players 1 and 2 can not gain by deviating unilaterally. Now we show that players $3,\ldots,n$ are not interested in deviating either. For this it is sufficient to show that $\pi_i^{G^*}(t) \leq \alpha_i$ for every $i \in \{3,\ldots,n\}$ and every $t \in [0,\infty)$. So, let $i \in \{3,\ldots,n\}$. Then for every $t \in [k_2,\infty)$ we have

$$\begin{aligned} \pi_i^{G^*}(t) &= \delta^t(\alpha_i + (S-\textstyle\sum_{j\in N}\alpha_j)\prod_{j\neq i} G_j^*(t)) \\ &= \delta^t(\alpha_i + (S-\textstyle\sum_{j\in N}\alpha_j)) \\ &= \delta^t(S-\textstyle\sum_{j\neq i}\alpha_j) \\ &\leq \delta^{k_2}(S-\textstyle\sum_{j\neq i}\alpha_j) \\ &\leq \delta^{k_i}(S-\textstyle\sum_{j\neq i}\alpha_j) \\ &= \alpha_i, \end{aligned}$$

where $k_i := \log_\delta \frac{\alpha_i}{S-\sum_{j\neq i}\alpha_j}$ and at the last inequality we used the fact that $k_i < k_2$ (see Lemma 6). Hence it is sufficient to show that $\pi_i^{G^*}(t) \leq \alpha_i$ for every $t \in [0, k_2]$. Note that for every $t \in [0, k_2]$ (also for $t = 0$!) we have

$$\begin{aligned}
\pi_i^{G^*}(t) &= \delta^t(\alpha_i + (S - \sum_{j\in N}\alpha_j)\prod_{j\neq i} G_j^*(t)) \\
&= \delta^t(\alpha_i + (S - \sum_{j\in N}\alpha_j)G_1^*(t)G_2^*(t)) \\
&= \delta^t\alpha_i + (\alpha_2 - \alpha_2\delta^t)\frac{\alpha_2(S-\sum_{j\neq 1}\alpha_j) - \alpha_1\delta^t(S - \sum_{j\neq 2}\alpha_j)}{\delta^t(S-\sum_{j\in N}\alpha_j)(S-\sum_{j\neq 2}\alpha_j)} \\
&= \delta^t(\alpha_i + \frac{\alpha_1\alpha_2}{S-\sum_{j\in N}\alpha_j}) + \delta^{-t}\frac{\alpha_2^2(S-\sum_{j\neq 1}\alpha_j)}{(S-\sum_{j\in N}\alpha_j)(S-\sum_{j\neq 2}\alpha_j)} \\
&\quad -(\frac{\alpha_1\alpha_2}{S-\sum_{j\in N}\alpha_j} + \frac{\alpha_2^2(S-\sum_{j\neq 1}\alpha_j)}{(S-\sum_{j\in N}\alpha_j)(S-\sum_{j\neq 2}\alpha_j)}) \\
&= a\delta^t + b\delta^{-t} + c,
\end{aligned}$$

where

$$a = \alpha_i + \frac{\alpha_1\alpha_2}{S-\sum_{j\in N}\alpha_j}$$

$$b = \frac{\alpha_2^2(S-\sum_{j\neq 1}\alpha_j)}{(S-\sum_{j\in N}\alpha_j)(S-\sum_{j\neq 2}\alpha_j)}$$

$$c = -\frac{\alpha_1\alpha_2}{S-\sum_{j\in N}\alpha_j} - \frac{\alpha_2^2(S-\sum_{j\neq 1}\alpha_j)}{(S-\sum_{j\in N}\alpha_j)(S-\sum_{j\neq 2}\alpha_j)}.$$

Making the change of variables $x = \delta^t$ note that it is sufficient to show that for the function $f : (0, \infty) \to \mathbb{R}$, defined by

$$f(x) = ax + \frac{b}{x} + c$$

we have $f(x) \leq \alpha_i$ for every $x \in [\delta^{k_2}, 1]$. Since $f''(x) = \frac{2b}{x^3} > 0$ for every $x \in [\delta^{k_2}, 1]$, the function f is convex on $[\delta^{k_2}, 1]$. Consequently, $f(x) \leq \max\{f(\delta^{k_2}), f(1)\}$. Since $f(1) = a + b + c = \alpha_i$ and $f(\delta^{k_2}) = \pi_i^{G^*}(k_2) \leq \alpha_i$, this finishes the proof. $\square$

Remark 1. *Throughout this paper we assumed that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Scrutinizing the proofs of Lemmas 1-5 and Theorem 1 we may conclude that for 2-player cake sharing games the same result (existence and uniqueness of a Nash equilibrium) also holds in case $\alpha_1 = \alpha_2$. For cake sharing games with at least three players the existence result is still valid in the more general case $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ (and a Nash equilibrium is still provided by the profile described in Theorem 2). With few additional efforts we can show that this Nash equilibrium is unique if and only if $\alpha_2 < \alpha_3$.*[5]

Remark 2. *We think that it is worth to note the following behavior of the equilibrium payoffs. Since $\eta_1 < \alpha_2$ and for each $i \neq 1$, $\eta_i = \alpha_i$, if the surplus $S - \sum \alpha_i$ is large, then $S - \sum \eta_i$ is large too. Hence, in this cases we have that, in the equilibrium, a relevant part of S is not shared among the players.*

[5]Moreover, it would be interesting to study whether similar results to those in the all-pay auctions model hold for the different configurations of the initial right vector.

Remark 3. *In this paper we have studied the structure of the Nash equilibria of a very specific model. We present it as a background tool that should be tested, for instance, in the different settings we have discussed in the Introduction.*

Acknowledgements. Financial support from the Spanish Ministry for Science and Technology and FEDER through project BEC2002-04102-C02-02, and from the *Xunta de Galicia* under project PGIDT03PXIC20701PN is gratefully aknowledged.

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