A Natural Selection from the Core of a TU Game: The Core-Center*

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Abstract

We present a new allocation rule for the class of games with a nonempty core: the *core-center*. This allocation rule selects a centrally located point within the core of any such game. We provide a deep discussion of its main properties.

Keywords. COOPERATIVE TU GAMES, BALANCED GAMES, CORE, CENTER OF GRAVITY

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Introduction

Given a cooperative game with transferable utility, an allocation rule divides the worth of the grand coalition, $v(N)$, among the players. Several allocation rules have been proposed in the literature. In this paper we introduce a new allocation rule for games with a nonempty core that summarizes the information contained in the latter. Maschler et al. (1979) show that the nucleolus can be characterized as a "lexicographic center" of the core. We study the real center of the core, which we call the *core-center*, and discuss its properties. Assume that we have narrowed attention to the core of the game. If we want only one of all these outcomes as a proposal to divide $v(N)$, how can we do it in a fair way? We suggest selecting the expectation of a uniform distribution defined over the core of the game, in other words, its center of gravity.

This paper deals with the properties of the core-center. The main focus of this axiomatic study is on its continuity, since it is not easy to prove that the core-center is continuous. The problem of continuous selection from multi-functions has been widely studied in mathematics, and Michael (1956) is a central reference. The issue of selection from convex-compact-valued multi-functions (as the core) is discussed in Gautier and Morchadi (1992); they study, as an alternative to the barycentric selection, the Steiner selection, which is continuous. Moreover, they briefly discuss the regularity problems one faces when working with the barycentric selection. In this paper we show that, because of the special structure of the core of a TU game, the barycentric selection from the core, *i.e.*, the core-center, is continuous (Section 4).

We also discuss in detail the monotonicity properties of the core-center. We show that, in this respect, the core-center and the nucleolus behave in parallel ways.

As already said, this is not the first time that a geometric approach is used to motivate an allocation rule. It is widely known that the Shapley value is the center of gravity of the vectors of marginal contributions and, that for convex games, it coincides with the center of gravity of the extreme points of the core taking their multiplicities into account. Also, the τ-value is the center of gravity of the edges of the core-cover, again multiplicities must be considered (González-Díaz et al., 2005).

In Section 1 we introduce the preliminary game theoretical concepts. In Section 2 we define the core-center. In Section 3 we discuss its monotonicity properties. In Section 4, we discuss in depth the issue of the continuity of the core-center.

1 Game Theory Background

A *transferable utility game*, is a pair (N, v), where $N := \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is a function assigning to every coalition $S \subseteq N$ a worth $v(S)$. By convention, $v(\emptyset) = 0$. Thus, a game is a vector in $\mathbb{R}^{2^n-1}$. Let $|S|$ denote the number of elements of coalition S. Let G^n be the set of all n-player games. For the sake of notation, we simply denote a game by v.

Let $x \in \mathbb{R}^n$ be an allocation. Then, x is *efficient* if $\sum_{i=1}^n x_i = v(N)$; x is *individually rational* if, for each $i \in N$, $x_i \geq v(\{i\})$; finally, x is *coalitionally rational* if, for each $S \subseteq N$, $\sum_{i\in S} x_i \geq v(S)$. An *(allocation) rule* is a function which, given a game v in some domain $\Omega \subseteq G^n$, selects an allocation in $\mathbb{R}^n$, *i.e.*,

$$\begin{array}{rlcl} \varphi : & \Omega \subseteq G^n & \longrightarrow & \mathbb{R}^n \\ & v & \longmapsto & \varphi(v). \end{array}$$

Next, we define some properties for rules. Let φ be a rule: φ is *continuous* if it is continuous as a function from $\mathbb{R}^{2^n-1}$ to $\mathbb{R}^n$; φ is *efficient* if it always selects efficient allocations; φ is *individually rational* if it always selects individually rational allocations; φ satisfies *covariance* if, for each pair of games v and w, each $r > 0$, and each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that, for each $S \subseteq N$, $w(S) = rv(S) + \sum_{i\in S} \alpha_i$, then $\varphi(w) = r\varphi(v) + \alpha$; φ satisfies *equal treatment of equals* if for each pair of players i and j such that, for each $S \subseteq N\backslash\{i, j\}$, $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$, we have $\varphi_i(v) = \varphi_j(v)$; φ satisfies the *dummy player property* if, for each $i \in N$ such that, for each $S \subseteq N\backslash\{i\}$, $v(S \cup \{i\}) - v(S) = v(\{i\})$, then $\varphi_i(v) = v(\{i\})$.

The core of a game v (Gillies, 1953), is the set of all the efficient allocations that are coalitionally rational. Formally, $C(v) := \{x\in \mathbb{R}^n : \sum_{i\in N} x_i = v(N)$ and, for each $S \subsetneq N$, $\sum_{i\in S} x_i \geq v(S)\}$.

2 The Core-Center

Suppose now that we regard all core points as equally valuable. This can be formalized by endowing the core with the uniform distribution. The *core-center* is defined as the mathematical expectation of such a probability distribution. Let $U(A)$ denote the uniform distribution defined over the set A and $E(\mathbb{P})$ the expectation of the probability distribution $\mathbb{P}$.

Definition 1. *Let v be a game with nonempty core. The* core-center *of v, $\mu(v)$, is defined by $\mu(v) := E\big(U(C(v))\big)$.*[1]

The core-center satisfies, among others, the following properties: efficiency, individual rationality, coalitional rationality, equal treatment of equals, covariance, and dummy player. The proofs of all of them are straightforward, either because they are inherited from core properties or because they are a consequence of the properties of the mathematical expectation.

2.1 An Example

Consider the following 4-player game taken from Maschler et al. (1979),

$$v(S) = \begin{cases} 2 & S = N \\ 1 & 2 \leq |S| \leq 3 \text{ and } S \neq \{1,3\}, \{2,4\} \\ 0.5 & S = \{1,3\} \\ 0 & \text{otherwise.} \end{cases}$$

Any rule satisfying *equal treatment of equals* has to give players 1 and 3 equal payoffs. The same applies to players 2 and 4. Players 1 and 3 are different from 2 and 4 because coalition $\{1,3\}$ can obtain 0.5 whereas coalition $\{2,4\}$ cannot obtain anything. In this game the Shapley value is $(13/24, 11/24, 13/24, 11/24)$ and the nucleolus is $(1/2, 1/2, 1/2, 1/2)$. The core is the segment joining $(1,0,1,0)$ and $(1/4, 3/4, 1/4, 3/4)$. Hence, the core-center is $(5/8, 3/8, 5/8, 3/8)$. Both the Shapley value and the core-center assign to players 1 and 3 higher payoffs than to 2 and 4. Yet, the nucleolus gives equal payoffs to all four players. Besides, these three allocations are inside the core.

We can use this same example to illustrate the following fact. By definition, games with the same core have the same core-center. But this is not so with the nucleolus. To see

[1] Our definition of the core-center is a particular case of the definition of the *centroid* of a Radon measure (Semadeni, 1971). That is, we might define other allocation rules by changing either the set over which the uniform measure is defined or the measure itself; this general approach is taken in González-Díaz and Sánchez-Rodríguez (2003).

this, consider the game v' obtained from v by changing the worth of coalition $\{1,2,3\}$ to 5/4. The core of v' coincides with the core of v, but now the nucleolus coincides with the core-center, *i.e.*, it is $(5/8, 3/8, 5/8, 3/8)$.

Next, we devote Sections 3 and 4 to the study the monotonicity properties of the core-center and its continuity, respectively.

3 Monotonicity

We define four different monotonicity properties. Let φ be a rule. We say that φ is *strongly monotonic* if, for each pair $v, w \in G^n$ and each $i \in N$ such that, for each $S \subseteq N\backslash\{i\}$, $w(S \cup \{i\}) - w(S) \geq v(S \cup \{i\}) - v(S)$, then $\varphi_i(w) \geq \varphi_i(v)$. Let $v, w \in G^n$ and let $T \subseteq N$ be such that $w(T) > v(T)$ and, for each $S \neq T$, $w(S) = v(S)$: φ satisfies *coalitional monotonicity* if, for each $i \in T$, $\varphi_i(w) \geq \varphi_i(v)$; φ satisfies *aggregate monotonicity* if $T = N$ implies that, for each $i \in N$, $\varphi_i(w) \geq \varphi_i(v)$; φ satisfies *weak coalitional monotonicity* if $\sum_{i \in T} \varphi_i(w) \geq \sum_{i \in T} \varphi_i(v)$.

Young (1985) characterized the Shapley value as the unique strongly monotonic and efficient rule satisfying equal treatment of equals in G^n. Also Young (1985) and Housman and Clark (1998) showed that if a rule always selects an allocation in the core, it does not satisfy coalitional monotonicity when the number of players is greater than three. Although we might try to use the above results to show that, within the class of games with nonempty core, the core-center is neither strongly nor coalitionally monotonic, we prefer to present a direct proof. We show that, indeed, the core-center does not even satisfy the weaker requirement imposed by *aggregate monotonicity.*

Proposition 1. *In the class of games with $n \geq 4$ players and nonempty core, the core-center does not satisfy aggregate monotonicity.*

Proof. The proof is by means of an example when $n = 4$. If $n > 4$ the example can be adapted by adding dummy players. Let $v \in G^n$ be such that $N = \{1,2,3,4\}$ and v is defined as follows:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$v(S)$	0	0	0	0	0	1	1	1	1	0	1	1	1	2	2

Now, $C(v) = \{(0,0,1,1)\}$ and, hence, $\mu(v) = (0,0,1,1)$. Let $\mathrm{co}(A)$ denote the convex hull of the set A. Let w be such that $w(N) = 3$ and, for each $S \neq N$, $w(S) = v(S)$. Then,

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$w(S)$	0	0	0	0	0	1	1	1	1	0	1	1	1	2	3

and $C(w) = \mathrm{co}\{(1,0,1,1),(0,0,2,1),(0,0,1,2),(0,1,1,1),(1,1,1,0),(1,1,0,1),\ (1,2,0,0)\}$. Next, we prove that the core-center does not satisfy *aggregate monotonicity* by showing that $\mu_3(v) > \mu_3(w)$. Let $\hat{w}$ be the game defined as follows:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$\hat{w}(S)$	0	0	0	0	0	1	1	1	1	0	1	1	2	2	3

We have $C(\hat{w}) = \mathrm{co}\{(1,0,1,1),(0,0,2,1),(0,0,1,2),(0,1,1,1),(1,1,1,0),(1,1,0,1)\}$. The game $\hat{w}$ only differs from w in the worth of coalition $\{1,3,4\}$. Figures 1 and 2 show the cores of w and $\hat{w}$, respectively. Since $\hat{w}(\{1,3,4\}) > w(\{1,3,4\})$, $C(\hat{w}) \subsetneq C(w)$. Now, $C(\hat{w})$ is symmetric with respect to the point $(0.5,0.5,1,1)$, *i.e.*, $x \in C(\hat{w}) \Leftrightarrow -\big(x-(0.5,0.5,1,1)\big)+(0.5,0.5,1,1) \in C(\hat{w})$. Hence, $\mu(\hat{w}) = (0.5,0.5,1,1)$.

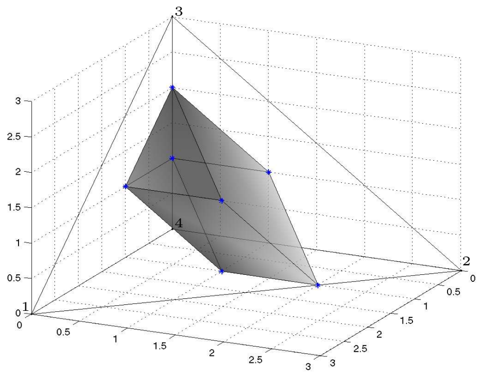

Figure 1: The core of w

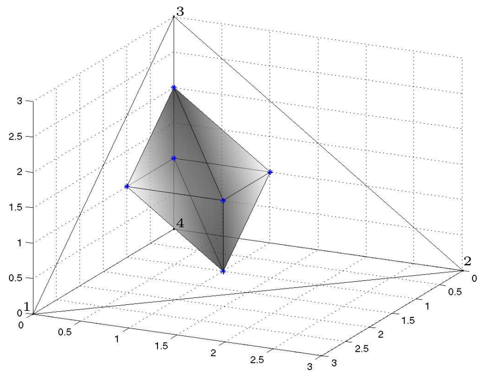

Figure 2: The core of $\hat{w}$

Now, $C(w)\backslash C(\hat{w}) \subsetneq \mathrm{co}\{(1,1,1,0),(0,1,1,1),(1,1,0,1),(1,2,0,0)\}$. Hence, for each $x \in C(w)\backslash C(\hat{w})$, $x_3 \leq 1$. Moreover, the volume of the points in $C(w)\backslash C(\hat{w})$ whose third coordinate is smaller than 1 is positive. By the definition of the core-center, since $\mu_3(\hat{w}) = 1$, we have $\mu_3(w) < 1 = \mu_3(v)$. Hence, the core-center does not satisfy *aggregate monotonicity*. $\square$

The nucleolus (Schmeidler, 1969) also violates the first three monotonicity properties defined above. Zhou (1991) defined weak coalitional monotonicity and showed that the nucleolus satisfies it. So does the core-center:

Proposition 2. *The core-center satisfies weak coalitional monotonicity.*

Proof. Let v and w be two games with a nonempty core satisfying the hypotheses of *weak coalitional monotonicity*, *i.e.*, they only differ in the fact that $w(T) > v(T)$ for some coalition T. If $T = N$ the result follows from efficiency. Hence, we can assume $T \subsetneq N$. If $C(w) = C(v)$, then $\mu(w) = \mu(v)$ and $\sum_{i\in T} \mu_i(w) \geq \sum_{i\in T} \mu_i(v)$. Hence, we can assume that $C(w) \subsetneq C(v)$. Let $x \in C(v)\backslash C(w)$ and $y \in C(w)$, then $\sum_{i\in T} y_i \geq w(T) > \sum_{i\in T} x_i$. Hence, since the core-center is the expectation of the uniform distribution over the core, $\sum_{i\in T} \mu_i(w) \geq \sum_{i\in T} \mu_i(v)$. $\square$

Therefore, the core-center and the nucleolus behave analogously with respect to all monotonicity properties discussed in this paper.

4 Continuity

If two games with a nonempty core are close enough (as vectors of $\mathbb{R}^{2^n-1}$), then their cores are also close to each other. We are computing the center of gravity of these sets when they are endowed with the uniform distribution. Hence, the question is: are the corresponding measures (associated with the uniform distribution) also close to each other? In general the answer is no, as shown by the following example.

Example 1. *Let* $a \geq 0$ *and define the triangle* $T(a) := \text{co}\{(a,0),(-a,0),(0,1)\}$. *For each* $a > 0$, *the center of gravity of* $T(a)$ *is* $(0,1/3)$. *Yet, when* $a = 0$, $T(0)$ *is the segment joining the points* $(0,0)$ *and* $(0,1)$, *whose center of gravity is* $(0,1/2)$, *which is not the limit of the centers of gravity of the* $T(a)$ *sets as* $a \to 0$.[2]

The problem with the continuity arises when the dimension of the polytope[3] under consideration is not fixed, *i.e.*, an $(n-2)$-polytope can be expressed (as a set) as the limit

[2]We would like to thank William Thomson for pointing out this example and the consequent intricacy for the continuity property.

[3]Refer to Section 4.2 for the formal definition of polytope.

of $(n-1)$-polytopes. As shown in the previous example, the continuity property is quite sensitive to this kind of degeneracies: the center of gravity of a convex polytope does not necessarily vary continuously if degeneracies are possible. Even so, the following statement is true.

Theorem 1. *The core-center is a continuous allocation rule within the class of games with a nonempty core.*

4.1 The Problem

Note that, in order to prove Theorem 1, it is enough to show that, for each game v with a nonempty core and each sequence of games with nonempty cores converging to v (under the usual convergence of vectors in $\mathbb{R}^{2^n-1}$), the associated sequence of their core-centers converges to the core-center of v. Formally,

Theorem 2. *Let $\bar{v}$ be a game with a nonempty core and $\{v^t\}$ a sequence of games with nonempty cores such that $\lim_{t\to\infty} v^t = \bar{v}$. Then, $\lim_{t\to\infty} \mu(v^t) = \mu(\bar{v})$.*

Clearly, Theorems 1 and 2 are equivalent. The next Proposition, which is a weaker version of Theorem 2 contains the difficult part of the proof. Theorem 2 and hence, Theorem 1, are easy consequences.

Proposition 3. *Let $\bar{v}$ be a game with a nonempty core and $\{v^t\}$ a sequence of games with nonempty cores such that*

(i) *for each $t \in \mathbb{N}$, we have $\bar{v}(N) = v^t(N)$,*

(ii) $\lim_{t\to\infty} v^t = \bar{v}$.

Then, $\lim_{t\to\infty} \mu(v^t) = \mu(\bar{v})$.

In contrast with Theorem 2, where every possible sequence of games is considered, Proposition 3 only concerns specific sequences. Next, we prepare the ground for Proposition 3. We do so by means of a general geometric result.

4.2 A Geometric Result

Below we state two general geometric results (Theorem 3 and Corollary 1). The proofs are technical and require extra notation and some preliminary results. Hence, we have relegated them to the Appendix. Essentially, Proposition 3 is a particular case of Corollary 1. The idea can be summarized as follows: whenever we consider a game with a nonempty core and its core-center, we can just think of a polytope and its center of gravity. Similarly, whenever we have a polytope and its center of gravity, we can just think of the uniform measure defined over the polytope and the integral of the identity function with respect to it. Thus, to prove that the core-center of a sequence of games converges to the core-center of the limit game (Theorem 2), it is enough to prove that the integrals over the corresponding uniform measures also converge.

A (convex) *polyhedron* is the intersection of a finite number of closed halfspaces. A polyhedron P is an *m-polyhedron* if its dimension is m, *i.e.*, m is the smallest integer such that P is contained in an m-dimensional space. A (convex) *polytope* is a bounded polyhedron. Let M_λ^m stand for Lebesgue measure on $\mathbb{R}^m$. Let $A \subseteq \mathbb{R}^m$ be a Lebesgue measurable set and let $m' \geq m$; we denote $M_\lambda^{m'}(A)$ by $\mathrm{Vol}_{m'}(A)$, *i.e.*, the m'-dimensional volume of A; hence, if $A \subseteq \mathbb{R}^m$ and $m' > m$, then, $\mathrm{Vol}_{m'}(A) = 0$ (for the sake of convenience, we define for each $x \in \mathbb{R}^m$, $\mathrm{Vol}_0(x) := 1$). Let P be an m-polytope and $\mathcal{X}_P$ its characteristic function. Let M_P be the Borel measure such that $M_P := \frac{1}{\mathrm{Vol}_m(P)}\mathcal{X}_P M_\lambda^m$, *i.e.*, the uniform measure over polytope P. Let $u \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. Let H_α^u be the hyperplane normal to u, $H_\alpha^u := \{x \in \mathbb{R}^m : \sum_{j=1}^m u_j x_j = \alpha\}$. For each hyperplane $H = \{x \in \mathbb{R}^m : \sum_{j=1}^m u_j x_j = \alpha\}$, let BH be the (closed) halfspace below H, *i.e.*, $BH := \{x \in \mathbb{R}^m : \sum_{j=1}^m u_j x_j \leq \alpha\}$.

Theorem 3. *Let $P \subsetneq \mathbb{R}^m$ be an $(m-l)$-polytope, $0 \leq l \leq m$. Let $u \in \mathbb{R}^m$. Let $\{\alpha_t\}$ be a sequence of real numbers with limit $\bar{\alpha}$. Let $P_t := P \cap BH_{\alpha_t}^u$ and $\bar{P} := P \cap BH_{\bar{\alpha}}^u$ be nonempty polytopes. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous function. Then, $\lim_{t\to\infty} \int f\, dM_{P_t} = \int f\, dM_{\bar{P}}$.*

Proof. Refer to the Appendix. □

Corollary 1. *Let $P \subsetneq \mathbb{R}^m$ be an $(m-l)$-polytope, $0 \leq l \leq m$. Let $k \in \mathbb{N}$ and $u^1, \ldots, u^k \in \mathbb{R}^m$ be k distinct vectors. For each $i \in \{1, \ldots, k\}$, let $\{\alpha_t^i\}$ be a sequence of real numbers with limit $\bar{\alpha}^i$. Let $P_t := P \cap \left(\bigcap_{i=1}^k BH_{\alpha_t^i}^{u^i}\right)$ and $\bar{P} := P \cap \left(\bigcap_{i=1}^k BH_{\bar{\alpha}^i}^{u^i}\right)$ be nonempty polytopes. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous function. Then, $\lim_{t\to\infty} \int f\, dM_{P_t} = \int f\, dM_{\bar{P}}$.*

Proof. Refer to the Appendix. □

Remark. Corollary 1 is essential to our proof of the continuity of the core-center. Now, if we go back to Example 1, we can see why Corollary 1 does not apply. The reason is that the sequences of polytopes that our results allow for are defined using hyperplanes whose normal vectors belong to the finite set $\{u^1,\ldots,u^k\}$. On the contrary, in Example 1, we would need an infinite number of different vectors to construct the corresponding sequence of polytopes.

4.3 Proofs of the Main Results

Proof of Proposition 3. We distinguish two cases. In Case 1, only the worth of a fixed coalition $\bar{S} \subsetneq N$ varies throughout the sequence $\{v^t\}$. In Case 2, the worths of all coalitions but coalition N can vary.

Case 1: There is $\bar{S} \subsetneq N$ such that, for each $T \neq S$ and each $t \in \mathbb{N}$, $\bar{v}(T) = v^t(T)$.

First, we define a new game whose core contains the cores of all v^t's and of $\bar{v}$. Let $\mathcal{G} := \{v^t : t \in \mathbb{N}\} \cup \{\bar{v}\}$. Let v be defined, for each $S \subseteq N$, by $v(S) := \min_{w\in\mathcal{G}} w(S)$. The game v is well-defined because, for each coalition S, the set $\bar{v}(S) \cup \{v^t(S) : t \in \mathbb{N}\}$ is compact (otherwise the sequence $\{v^t\}$ would not converge). Let $P := C(v)$, $P_t := C(v^t)$ and $\bar{P} := C(\bar{v})$. Clearly, by definition of v, P contains polytopes P_t and $\bar{P}$. Let $e^S \in \mathbb{R}^n$ be such that $e^S_i = 1$ if $i \in S$ and $e^S_i = 0$ if $i \notin S$. Now, since the efficiency condition is met by all the core allocations, $H^{e^{\bar{S}}}_{v^t(\bar{S})} \cap C(v) = H^{e^{N\setminus\bar{S}}}_{v(N)-v^t(\bar{S})} \cap C(v)$ (although the orientations of the two hyperplanes are different). Consider the sequence $\{v(N) - v^t(\bar{S})\}$. It converges and its limit is $v(N) - \bar{v}(\bar{S})$. Now, $P_t = P \cap BH^{e^{N\setminus\bar{S}}}_{v(N)-v^t(\bar{S})}$ and $\bar{P} = P \cap BH^{e^{N\setminus\bar{S}}}_{v(N)-\bar{v}(\bar{S})}$. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $h(x) := x$. Then, $\mu(v^t) = \int h\, dM_{P_t}$ and $\mu(\bar{v}) = \int h\, dM_{\bar{P}}$. For each $i \in N$, the function $h_i : \mathbb{R}^n \to \mathbb{R}$ defined by $h_i(x) = (h(x))_i = x_i$ is continuous. Hence, applying Theorem 3 to each h_i, we have $\lim_{t\to\infty} \mu(v^t) = \mu(\bar{v})$.

Case 2: For each $t \in \mathbb{N}$, $\bar{v}(N) = v^t(N)$. Only $v(N)$ is fixed. Now, the proof parallels the one for Case 1, but, since we have more that one type of halfspaces, Corollary 1 has to be used instead of Theorem 3. Since there are finitely many coalitions, we can apply Corollary 1 for some $k \leq 2^n - 2$. □

Proof of Theorem 2. Now we consider the general case, when the worth of every coalition can vary along the sequence $\{v^t\}$.

Let $\varepsilon_t := \bar{v}(N) - v^t(N)$ (note that ε_t need not be positive). For each $t \in \mathbb{N}$, let $\hat{v}^t$ be the auxiliary game defined, for each $S \subseteq N$, by $\hat{v}^t(S) = v^t(S) + \frac{|S|}{n}\varepsilon_t$. For each

$t \in \mathbb{N}$, we have (i) $\hat{v}^t(N) = \bar{v}(N)$, and (ii) $\mathrm{C}(\hat{v}^t)$ is obtained by translation of $C(v^t)$ by the vector $\frac{1}{n}(\varepsilon_t, \ldots, \varepsilon_t)$. Since $\varepsilon_t \to 0$, $\lim_{t\to\infty} v^t = \bar{v}$ implies that $\lim_{t\to\infty} \hat{v}^t = \bar{v}$. Hence, by Proposition 3, $\lim_{t\to\infty} \mu(\hat{v}^t) = \mu(\bar{v})$. Since the core-center satisfies *covariance*, $\mu(v^t) = \mu(\hat{v}^t) - \frac{1}{n}(\varepsilon_t, \ldots, \varepsilon_t)$. Now, using again the fact that $\varepsilon_t \to 0$, we have

$$\lim_{t\to\infty} \mu(v^t) = \lim_{t\to\infty} \big(\mu(\hat{v}^t) - \frac{1}{n}(\varepsilon_t, \ldots, \varepsilon_t)\big) = \lim_{t\to\infty} \mu(\hat{v}^t) - \lim_{t\to\infty} \frac{1}{n}(\varepsilon_t, \ldots, \varepsilon_t) = \mu(\bar{v}). \qquad \square$$

A Appendix

A.1 Notation

Our geometric results rely on measure theory and functional analysis. For each $i \in \{1, \ldots, m\}$, let $e^i \in \mathbb{R}^m$ be such that $e^i_j = 1$ if $j = i$ and $e^i_j = 0$ if $j \neq i$. For each $x \in \mathbb{R}^m$ and each $l \in \mathbb{N}$, $l \leq m$, let $x_{-l} := (x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_m) \in \mathbb{R}^{m-1}$, $x_L := (x_1, \ldots, x_l) \in \mathbb{R}^l$, and $x_{\bar{L}} := (x_{l+1}, \ldots, x_m) \in \mathbb{R}^{m-l}$. For each $A \subseteq \mathbb{R}^m$, let $\mathrm{Cl}(A)$ denote the closure of set A with the euclidean topology.

Let P be an m-polytope. H is a *supporting hyperplane for* P if $H \cap P \neq \emptyset$ and BH contains P. A *face* of a polytope P is (i) P itself, (ii) the empty set, or (iii) the intersection of P with some supporting hyperplane. Faces of dimension i are called i-faces (with the convention that $\dim(\emptyset) = -1$). The 0-faces, 1-faces, and $(m-1)$-faces of P are respectively its vertices, edges, and facets. Let $\mathcal{F}(P)$ be the set of all facets of P and F an arbitrary facet. The finite set of polytopes $\{P_1, \ldots, P_k\}$ is a *dissection* of P if (i) $P = \bigcup_{j=1}^k P_j$ and (ii) for each pair $\{j, j'\} \subseteq \{1, \ldots, k\}$, with $j \neq j'$, $\mathrm{Vol}_m(P_j \cap P_{j'}) = 0$.

Let P be an m-polytope and $r > 0$ be such that $P \subsetneq (-r, r)^m \subsetneq \mathbb{R}^m$. Let $R := [-r, r]^m$. The pair $(R, \mathcal{B})$, where $\mathcal{B}$ stands for the collection of Borel sets of R, is a measure space. Let $\mathcal{M}(R)$ be the set of all complex-valued regular Borel measures defined on $(R, \mathcal{B})$ and $\mathcal{M}^+(R)$ the subset of real-valued and positive Borel measures. Also, let $C(R)$ and $C^{\mathbb{R}}(R)$ be the sets of all continuous functions $f : R \to \mathbb{C}$ and $f : R \to \mathbb{R}$ respectively.

By the Riesz Representation Theorem, $C(R)^* = \mathcal{M}(R)$, *i.e.*, $\mathcal{M}(R)$ is the dual of $C(R)$. Hence, we can use the weak* topology (henceforth w^*) in $\mathcal{M}(R)$. According to this topology, a sequence of measures $\{M_t\}$ converges to a measure M if and only if, for each $f \in C^{\mathbb{R}}(R)$, $\lim_{t\to\infty} \int f dM_t = \int f dM$.[4] For notational convenience, for each $f \in C(R)$, and

[4] Indeed, we should check the previous convergences for all the functions in $C(R)$. But, since, for each

each measure $M \in \mathcal{M}(R)$, $\langle f, M\rangle$ denotes $\int f dM$. Therefore, convergence of a sequence of measures $\{M_t\}$ to a measure M under w^* just means that, for each continuous function f, the sequence obtained by integration of f under the M_t's converges to the integral under M.

A.2 The Results

We state, without proof, two elementary results.

Lemma 1. *Let P and P' be two m-polytopes such that $P' \subseteq P$. Then, P' belongs to some dissection of P.*

Lemma 2. *Let P be an m-polyhedron, let $u \in \mathbb{R}^m$, and let $\alpha, \beta \in \mathbb{R}$. Let $P \cap H^u_\alpha \neq \emptyset$ and $P \cap H^u_\beta \neq \emptyset$. Then, $P \cap H^u_\alpha$ is bounded if and only if $P \cap H^u_\beta$ is bounded.*

Now, we present a construction that is crucial for the proofs below. Let $l \in \mathbb{N}$, $l \leq m$, and let the family of hyperplanes $\mathcal{H} := \{H^1, \ldots, H^l\}$ be such that $\bigcap_{j=1}^{l} H^j = V^{\mathcal{H}}$, where $V^{\mathcal{H}}$ is a linear variety of dimension $m-l$. Assume, for simplicity, that $x \in V^{\mathcal{H}}$ if and only if, for each $i \in \{1, \ldots, l\}$, $x_i = 0$. Now, let $\bar{Q} \subseteq V^{\mathcal{H}}$ be an $(m-l)$-polytope.[5] A pair $(V^{\mathcal{H}}, \bar{Q})$ so defined is a *pre-polyhedron.*

Lemma 3. *Let $\mathcal{H} := \{H^1, \ldots, H^l\}$ be a set of hyperplanes and $\bar{Q}$ an $(m-l)$-polytope such that $(V^{\mathcal{H}}, \bar{Q})$ is a pre-polyhedron. Then, there is a set of hyperplanes $\bar{\mathcal{H}} := \{\bar{H}^1, \ldots, \bar{H}^k\}$ such that*

(i) $\bar{Q} = V^{\mathcal{H}} \cap (\bigcap_{j=1}^{k} B\bar{H}^j)$*, and*

(ii) *for each $j \in \{1, \ldots, k\}$, there are $u \in \mathbb{R}^m$ and $\alpha > 0$ such that $\bar{H}^j = \{x \in \mathbb{R}^m : \sum_{i=l+1}^{m} u_i x_i = \alpha\}$.*

Proof. Since $V^{\mathcal{H}}$ can be expressed as the intersection of finitely many halfspaces, property (i) says that polytope $\bar{Q}$ can be expressed as the intersection of finitely many halfspaces too, which is true by definition of a polytope. Since the restriction to the halfspaces defining $V^{\mathcal{H}}$ pins down the first l-coordinates of a point in $\bar{Q}$, the remaining halfspaces can be chosen to be such that no extra restriction on the first l coordinates is added. □

$f \in C(R)$, there are functions $f_1, f_2 \in C^{\mathbb{R}}(R)$ such that, for each $x \in R$, $f(x) = f_1(x) + f_2(x)i$, both things are equivalent.

[5]If $V^{\mathcal{H}}$ is a point, then $\bar{Q} = V^{\mathcal{H}}$.

Now, given a pre-polyhedron $(\mathcal{V}^{\mathcal{H}}, \bar{Q})$, and a set of hyperplanes $\bar{\mathcal{H}} := \{\bar{H}^1, \ldots, \bar{H}^k\}$ satisfying the two properties in Lemma 3, we define the polyhedron $Q(V^{\mathcal{H}}, \bar{Q})$ as follows,

$$Q(V^{\mathcal{H}}, \bar{Q}) := \left\{ y \in \mathbb{R}^m \ : \ \begin{array}{l} \text{for each } j \in \{1, \ldots, l\},\, y \in BH^j \text{ and,} \\ \text{for each } j \in \{1, \ldots, k\},\, y \in B\bar{H}^j \end{array} \right\}.$$

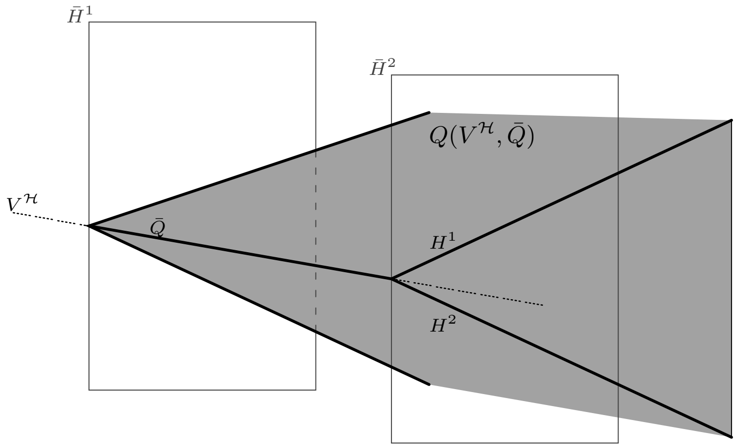

Figure 3: The polyhedron $Q(V^{\mathcal{H}}, \bar{Q})$.

In words, first we weaken the restrictions that pinned down the first l coordinates of the points in $\bar{Q}$ and, second, we still restrict $Q(V^{\mathcal{H}}, \bar{Q})$ to the halfspaces obtained from hyperplanes in $\bar{\mathcal{H}}$, *i.e.*, the last $m-l$ coordinates of the points in $Q(V^{\mathcal{H}}, \bar{Q})$ are subject to the same restrictions we had before for the same coordinates in the points of $\bar{Q}$. Note that the projection of $Q(V^{\mathcal{H}}, \bar{Q})$ into $V^{\mathcal{H}}$ coincides with $\bar{Q}$. Hence, it is easy to check that the previous definition is equivalent to the following,

$$Q(V^{\mathcal{H}}, \bar{Q}) = \left\{ \begin{array}{l} y \in \mathbb{R}^m \ : \text{ for each } j \in \{1, \ldots, l\},\, y \in BH^j \text{ and} \\ y = x + \sum_{i=1}^{l} \gamma_i e^i, \text{ where } x \in \bar{Q} \text{ and, for each } i \in \{1, \ldots, l\}, \gamma_i \in \mathbb{R} \end{array} \right\}.$$

Hence, we do not even need to find the set $\bar{\mathcal{H}}$ to define $Q(V^{\mathcal{H}}, \bar{Q})$. Moreover, all the vertices of $Q(V^{\mathcal{H}}, \bar{Q})$ are those of $\bar{Q}$. Now, for simplicity, denote $Q(V^{\mathcal{H}}, \bar{Q})$ by Q. Then, for each $x \in \bar{Q}$, we define the l-polyhedron $Q(x) := \{y \in Q \ : \ y_{\bar{L}} = x_{\bar{L}}\}$ (a similar construction is depicted in Figure 6). Now, $Q = \bigcup_{x \in \bar{Q}} Q(x)$ and for each pair $x, x' \in \bar{Q}$, $x \neq x'$, we have (i) $Q(x) \cap Q(x') = \emptyset$ and (ii) $Q(x) = \{y \in Q \ : \text{ there is } y' \in Q(x') \text{ such that } y = y' + (x - x')\}$.

In order to state the next result, we introduce some notation related to the asymptotic behavior of functions. Let $f, g : \mathbb{R} \to \mathbb{R}$. We write that $f(t) = \mathrm{O}(g(t))$ as $t \to 0$ if there are

$a > 0$ and $\delta > 0$ such that, for each $|t| < \delta$, $|f(t)| \leq a|g(t)|$. We write that $f(t) = \Omega(g(t))$ as $t \to 0$ if there are $a > 0$ and $\delta > 0$ such that for each $|t| < \delta$, $|f(t)| \geq a|g(t)|$. Finally, $f(t) = \Theta(g(t))$ as $t \to 0$ if $f(t) = \mathrm{O}(g(t))$ and $f(t) = \Omega(g(t))$.

Proposition 4. *Let $P \subsetneq \mathbb{R}^m$ be an m-polytope. Let $u \in \mathbb{R}^m$; $\bar{\alpha} > 0$; $l \in \mathbb{N}$, $l \leq m$; and $f : \mathbb{R} \to \mathbb{R}_+$ with $\lim_{t\to 0} f(t) = 0$ be such that i) for each $t > 0$, $P_t := P \cap BH^u_{\bar{\alpha}+f(t)}$ is an m-polytope and ii) $\bar{P} := P \cap BH^u_{\bar{\alpha}}$ is an $(m-l)$-polytope. Then,* $\mathrm{Vol}_m(P_t) = \Theta(f(t)^l)$.

Proof of Proposition 4. The idea of the proof consists of defining, for each $t > 0$, two polytopes Q_t and Q'_t such that $Q'_t \subseteq P_t \subseteq Q_t$, $\mathrm{Vol}_m(Q'_t) = \Omega(f(t)^l)$, and $\mathrm{Vol}_m(Q_t) = \mathrm{O}(f(t)^l)$. Note that, since the number of vertices of P is finite, there is $t_0 > 0$ such that for each $t < t_0$, all the vertices of P_t are either in $\bar{P}$ or in $H^u_{\bar{\alpha}+f(t)}$. Hence, since we are concerned with the asymptotic behavior of $\mathrm{Vol}(P_t)$ when $t \to 0$, from now on we only consider $t \in \mathbb{R}_+$ such that $t < t_0$. For simplicity, we assume that $\bar{\alpha} = 0$; $u = e^1$; for each $t \in \mathbb{R}$, $f(t) = t$; and, for each $x \in \bar{P}$ and each $i \in \{1, \ldots, l\}$, $x_i = 0$.

Now, since $\bar{P}$ is an $(m-l)$-face of P, $\bar{P}$ lies in the intersection of l facets of P. Let $\mathcal{H} := \{H^1, \ldots, H^l\}$ be the set whose elements are the l hyperplanes containing those facets and let $V^{\mathcal{H}}$ be their intersection. Let $\bar{Q}$ be the projection of P into $V^{\mathcal{H}}$, *i.e.*, $x \in \bar{Q}$ if and only if (i) $x \in V^{\mathcal{H}}$ and (ii) there is $y \in P$ such that for each $i > l$, $x_i = y_i$. Clearly, $\bar{P} \subseteq \bar{Q}$. Now, we have the pre-polyhedron $(V^{\mathcal{H}}, \bar{Q})$. Let $Q := Q(V^{\mathcal{H}}, \bar{Q})$. Since $Q \cap H^{e^1}_0 = \bar{Q}$ is bounded, by Lemma 2, for each $t > 0$, $Q \cap H^{e^1}_t$ is bounded. Hence, for each $t > 0$, $Q_t := Q \cap BH^{e^1}_t$ is a polytope. Moreover, by construction, for each $t > 0$, $P_t \subseteq Q_t$. Also, since, for each $\alpha < 0$, $Q \cap BH^{e^1}_\alpha = \emptyset$, then, for each $x, y \in Q$, $x_1 y_1 \geq 0$, *i.e.*, x_1 and y_1, if different from 0, have the same sign. Hence, we assume, without loss of generality, that, for each $x \in Q$, $x_1 \geq 0$.

We prove that $\mathrm{Vol}_m(Q_t) = \mathrm{O}(t^l)$. For each $x \in \bar{Q}$, and each $t > 0$, we define the l-polytope $Q_t(x) := Q(x) \cap BH^{e^1}_t$. Now, by the properties of the polyhedra $Q(x)$, we have (i) $Q_t = \bigcup_{x\in\bar{Q}} Q_t(x)$ and (ii) for each pair $x, x' \in \bar{Q}$ such that $x \neq x'$, $Q_t(x) \cap Q_t(x') = \emptyset$ and $\mathrm{Vol}_l(Q_t(x)) = \mathrm{Vol}_l(Q_t(x'))$. Hence, given $x \in \bar{Q}$, $\mathrm{Vol}_m(Q_t) = \mathrm{Vol}_{m-l}(\bar{Q})\,\mathrm{Vol}_l(Q_t(x))$. Hence, we just need to find an appropriate upper bound for the volume of each $Q_t(x)$. Recall that, by definition, for each pair $x, x' \in Q_t(x)$, $x_{\bar{L}} = x'_{\bar{L}}$. Now, to study the volume of $Q_t(x)$, we look for bounds on both the maximum and the minimum values that can be achieved at each of the first l coordinates of a point in $Q_t(x)$. Since $Q_t(x)$ is an l-polytope, the extreme values for the different coordinates are achieved at the vertices. All the vertices of polytope Q_t lie either in $\bar{Q}$ or in $H_t^{e^1}$. Hence, if y is a vertex of $Q_t(x)$, either $y_1 = t$ or $y = x$, and recall that for each $i \in \{1, \ldots, l\}$, $x_i = 0$. Let $v \neq x$ be a vertex of $Q_t(x)$. Then, $v_1 = t$ and $v_{\bar{L}} = x_{\bar{L}}$. Hence, $v - x = (t, v_2, \ldots, v_l, 0, \ldots, 0)$. Now, for each pair $u, u' \in \mathbb{R}^m$, let $\sphericalangle(uu')$ denote the interior angle between vectors u and u'. Hence, for each $i \in \{1, \ldots, l\}$, $\cos(\sphericalangle((v - x)e^i)) = v_i/\|(v - x)\|$. Now, since $v_1 = t > 0$, we have $\cos(\sphericalangle((v - x)e^1)) \neq 0$ and $\|(v - x)\| = t/\cos(\sphericalangle((v - x)e^1))$. Now, if $l > 1$, we have that, for each $i \in \{2, \ldots, l\}$, $v_i = t\cos(\sphericalangle((v - x)e^i))/\cos(\sphericalangle((v - x)e^1))$. Hence, if $l > 1$, we can define $v_{\max}$ as follows,

$$v_{\max} := \max_{\substack{i \in \{2,\ldots,l\} \\ v \neq x \text{ vertex of } Q_t(x)}} \{|\cos(\sphericalangle\big((v - x)e^i\big))/\cos(\sphericalangle\big((v - x)e^1\big))|\}.$$

Let B be the l-dimensional polytope such that $y \in B$ if and only if $y_{\bar{L}} = x_{\bar{L}}$, $x_1 \in [0, t]$, and, if $l > 1$, for each $i \in \{2, \ldots, l\}$, $y_i \in [-tv_{\max}, tv_{\max}]$, *i.e.*, B is an l-dimensional box. Now, $Q_t(x) \subseteq B$ and $\mathrm{Vol}_l(B) = t(2tv_{\max})^{l-1}$. Hence, $\mathrm{Vol}_l(Q_t(x)) = \mathrm{O}(t^l)$. Since $\mathrm{Vol}_m(Q_t) = \mathrm{Vol}_{m-l}(\bar{Q})\,\mathrm{Vol}_l(Q_t(x))$, we have $\mathrm{Vol}_m(Q_t) = \mathrm{O}(t^l)$. Hence, since, for each $t > 0$, $P_t \subseteq Q_t$, we have $\mathrm{Vol}_m(P_t) = \mathrm{O}(t^l)$.

Finally, the construction of the polytopes Q'_t relies on similar arguments and we omit it. □

Now, we state, without proof, a fairly standard result.

Lemma 4. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous function and $K \subsetneq \mathbb{R}^l$, $1 < l < m$, a compact set. Then, the functions $h_1, h_2 : \mathbb{R}^{m-l} \to \mathbb{R}$ defined by $h_1(x) := \max_{y \in K} f(x, y)$ and $h_2(x) := \min_{y \in K} f(x, y)$ are continuous.*

We are ready to prove the main result of this Appendix. Theorem 3 is easily derived from it.

Proposition 5. *Let P be an m-polytope and R an m-dimensional cube $[-r, r]^m$ containing P in its interior. Let $u \in \mathbb{R}^m$. Let $\{\alpha_t\}$ be a sequence of real numbers with limit $\bar{\alpha}$. Let $P_t := P \cap BH^u_{\alpha_t}$ and $\bar{P} := P \cap BH^u_{\bar{\alpha}}$ be nonempty polytopes. Then, $M_{P_t} \xrightarrow{w^*} M_{\bar{P}}$.*

Proof. Without loss of generality, we assume that $u = e^1 = (1, 0, \ldots, 0)$ (otherwise, a change of coordinates can be carried out). If $\bar{P}$ is an m-polytope, then there is $t_0 \in \mathbb{N}$ such that, for each $t \geq t_0$, P_t is an m-polytope. Hence, there is no degeneracy in the limit and the result is straightforward. So we assume that $\bar{P}$ is not an m-polytope and hence, since all the P_t and $\bar{P}$ are nonempty, $\bar{\alpha} = \min_{x\in P} x_1$. Moreover, we assume that, for each $t \in \mathbb{N}$, $\alpha_t > \bar{\alpha}$.[6] We distinguish two cases:

Case 1: $\bar{P}$ is an $(m-1)$-polytope.

Let Q be the polyhedron defined as follows,

$$Q := \{y \in \mathbb{R}^m \ : \ y = x + \gamma e^1, \text{ where } x \in \bar{P} \text{ and } \gamma > 0\}.$$

For each $t \in \mathbb{N}$, we define the auxiliary polytopes $Q_t := Q \cap BH^{e^1}_{\alpha_t}$. Also, let $\bar{Q} := Q \cap BH^{e^1}_{\bar{\alpha}}$ (see Figure 4). By definition, $\bar{Q} = \bar{P}$.

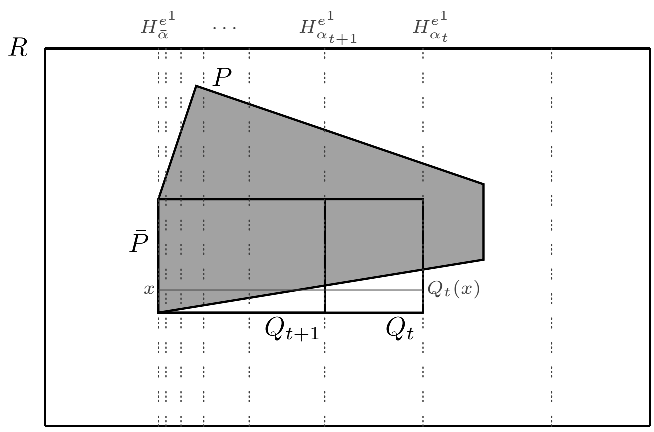

Figure 4: The Q_t polytopes

The proof is in three steps.

Step 1: $M_{Q_t} \xrightarrow{w^*} M_{\bar{Q}}$.

We prove that for each $f \in C^{\mathbb{R}}(R)$, $\lim_{t\to\infty}\langle f, M_{Q_t}\rangle = \langle f, M_{\bar{Q}}\rangle$.

Step 1.a: Let $f \in C^{\mathbb{R}}(R)$ be such that there exists $c : [-r,r]^{m-1} \to \mathbb{R}$ with the following property: for each $x \in [-r,r]^m$, $f(x) = c(x_{-1})$. Let dx_{-1} stand for $dx_2 \dots dx_m$. Also, for each $x \in \bar{Q}$ and each $t \in \mathbb{N}$, we define the 1-polytopes $Q_t(x) := \{y \in Q_t \ : \ y_{-1} = x_{-1}\}$.

[6]If $\{\alpha_t\}$ only has a finite number of elements strictly above $\bar{\alpha}$, then, since the sequence becomes constant for t big enough, the result is straightforward. If $\{\alpha_t\}$ has an infinite number of elements strictly above $\bar{\alpha}$, then we take the subsequence in which all the elements that equal $\bar{\alpha}$ have been removed. If the desired convergence result is true for this subsequence, then it is also true for the original sequence.

Note that, if $x \neq x'$, then $Q_t(x) \cap Q_t(x') = \emptyset$ and $\mathrm{Vol}_1(Q_t(x)) = \mathrm{Vol}_1(Q_t(x')) = \alpha_t - \bar{\alpha}$. Moreover, for each $x \in \bar{Q}$, f is constant in $Q_t(x)$. Then,

$$\begin{aligned}
\langle f, M_{Q_t} \rangle &= \frac{1}{\mathrm{Vol}_m\,(Q_t)} \int_{\bar{Q}} \int_{Q_t(x)} c(x_{-1}) dx_{-1} dx_1 \\
&= \frac{\alpha_t - \bar{\alpha}}{\mathrm{Vol}_m\,(Q_t)} \int_{\bar{Q}} c(x_{-1}) dx_{-1} \\
&= \frac{1}{\mathrm{Vol}_{m-1}\,(\bar{Q})} \int_{\bar{Q}} c(x_{-1}) dx_{-1} \\
&= \langle f, M_{\bar{Q}} \rangle.
\end{aligned}$$

Step 1.b: Let $f \in C^{\mathbb{R}}(R)$. Define the three auxiliary functions

$$\begin{aligned}
f^*(x_1, x_{-1}) &:= f(\bar{\alpha}, x_{-1}), \\
\overline{c}_t(x_1, x_{-1}) &:= \max_{z \in [\bar{\alpha}, \alpha_t]} f(z, x_{-1}), \text{ and} \\
\underline{c}_t(x_1, x_{-1}) &:= \min_{z \in [\bar{\alpha}, \alpha_t]} f(z, x_{-1}).
\end{aligned}$$

By Lemma 4, $\overline{c}_t$ and $\underline{c}_t$ are continuous. Hence, by Step 1.a, we have $\langle \overline{c}_t, M_{Q_t} \rangle = \langle \overline{c}_t, M_{\bar{Q}} \rangle$ and $\langle \underline{c}_t, M_{Q_t} \rangle = \langle \underline{c}_t, M_{\bar{Q}} \rangle$. By the continuity of f, for each $x \in R$, $\lim_{t \to \infty} \underline{c}_t(x) = f^*(x) = \lim_{t \to \infty} \overline{c}_t(x)$. Since $M_{Q_t} \in \mathcal{M}^+(R)$, then $\langle \underline{c}_t, M_{Q_t} \rangle \leq \langle f, M_{Q_t} \rangle \leq \langle \overline{c}_t, M_{Q_t} \rangle$. The Lebesgue's Dominated Convergence Theorem completes Step 1:[7]

$$\begin{array}{ccccc}
\langle \underline{c}_t, M_{Q_t} \rangle & \leq & \langle f, M_{Q_t} \rangle & \leq & \langle \overline{c}_t, M_{Q_t} \rangle \\
\| \ \scriptstyle\text{Step 1.a} & & & & \| \ \scriptstyle\text{Step 1.a} \\
\langle \underline{c}_t, M_{\bar{Q}} \rangle & & & & \langle \overline{c}_t, M_{\bar{Q}} \rangle \\
{\scriptstyle t \to \infty} \downarrow \ \scriptstyle\text{Dom Conv} & & & & {\scriptstyle t \to \infty} \downarrow \ \scriptstyle\text{Dom Conv} \\
\langle f^*, M_{\bar{Q}} \rangle & & & & \langle f^*, M_{\bar{Q}} \rangle \\
\| \ \scriptstyle f^*(x) = f(x),\ x \in \bar{Q} & & & & \| \ \scriptstyle f^*(x) = f(x),\ x \in \bar{Q} \\
\langle f, M_{\bar{Q}} \rangle & & & & \langle f, M_{\bar{Q}} \rangle.
\end{array}$$

Hence, for each $f \in C^{\mathbb{R}}(R)$, $\lim_{t \to \infty} \langle f, M_{Q_t} \rangle = \langle f, M_{\bar{Q}} \rangle$.

Step 2: $\displaystyle\lim_{t \to \infty} \frac{\mathrm{Vol}_m(P_t \backslash Q_t)}{\mathrm{Vol}_m(Q_t)} = \lim_{t \to \infty} \frac{\mathrm{Vol}_m(Q_t \backslash P_t)}{\mathrm{Vol}_m(Q_t)} = 0$ and $\displaystyle\lim_{t \to \infty} \frac{\mathrm{Vol}_m(P_t)}{\mathrm{Vol}_m(Q_t)} = 1$.

We show that $\lim_{t \to \infty} \frac{\mathrm{Vol}_m(P_t \backslash Q_t)}{\mathrm{Vol}_m(Q_t)} = 0$, the proof for $Q_t \backslash P_t$ being analogous. By Lemma 1, there are polytopes $P^1, \ldots, P^k$, $k \geq 1$, such that $\{P^1, \ldots, P^k, Q \cap P\}$ is a dissection of P. Hence, $P \backslash Q \subsetneq \cup_{j=1}^k P^j = \mathrm{Cl}(P \backslash Q)$. Now, for each $t \in \mathbb{N}$ and each $j \in \{1, \ldots, k\}$,

[7] The constant function g defined, for each $x \in R$, by $g(x) := \max_{x \in R} |f(x)|$ dominates the two sequences at hand and meets the requirements needed for Lebesgue's Dominated Convergence Theorem to be applicable.

let $P_t^j := P^j \cap BH_{\alpha_t}^{e^1}$. Then, for each $t \in \mathbb{N}$, $\{P_t^1, \ldots, P_t^k, Q_t \cap P_t\}$ is a dissection of P_t and $P_t \backslash Q_t \subsetneq \cup_{j=1}^k P_t^j = \mathrm{Cl}(P_t \backslash Q_t)$. Hence, $\mathrm{Vol}_m(P_t \backslash Q_t) \leq \sum_{j=1}^k \mathrm{Vol}_m(P_t^j)$ (actually, they are equal).

Now, since $\mathrm{Vol}_m(Q_t) = \mathrm{Vol}_{m-1}(\bar{P})(\alpha_t - \bar{\alpha})$, $\mathrm{Vol}_m(Q_t) = \Theta(\alpha_t - \bar{\alpha})$, *i.e.*, $\mathrm{Vol}_m(Q_t)$ is a linear function of $(\alpha_t - \bar{\alpha})$. Let $j \in \{1, \ldots, k\}$. Since $\bar{Q} = \bar{P}$, $P_1^j \cap BH_{\bar{\alpha}}^{e^1}$ is contained in some facet of $\bar{P}$, *i.e.*, it is in the boundary of $\bar{P}$. Hence, $P_1^j \cap BH_{\bar{\alpha}}^{e^1}$ is, at most, an $(m-2)$-polytope. Hence, if for each $t \in N$, P_t^j is an m-polytope, we have that, in the limit, there is, at least, a 2-dimensional degeneracy. Hence, by Proposition 4, $\mathrm{Vol}_m(P_t^j) = \mathrm{O}((\alpha_t - \bar{\alpha})^2)$. Now, since the number of polytopes in the dissection is finite, $\mathrm{Vol}_m(P_t \backslash Q_t) = \mathrm{O}((\alpha_t - \bar{\alpha})^2)$.

Finally, $\lim_{t\to\infty} \dfrac{\mathrm{Vol}_m(P_t \backslash Q_t)}{\mathrm{Vol}_m(Q_t)} = \lim_{t\to\infty} \dfrac{\mathrm{O}((\alpha_t - \bar{\alpha})^2)}{\Theta(\alpha_t - \bar{\alpha})} = 0.$

We turn now to $\frac{\mathrm{Vol}_m(P_t)}{\mathrm{Vol}_m(Q_t)}$. Since $P_t = Q_t \backslash (Q_t \backslash P_t) \cup (P_t \backslash Q_t)$, and $Q_t \backslash (Q_t \backslash P_t)$ and $P_t \backslash Q_t$ are disjoint sets, then $\mathrm{Vol}_m(P_t) = \mathrm{Vol}_m(Q_t) - \mathrm{Vol}_m(Q_t \backslash P_t) + \mathrm{Vol}_m(P_t \backslash Q_t)$. Hence,

$$\lim_{t\to\infty} \frac{\mathrm{Vol}_m(P_t)}{\mathrm{Vol}_m(Q_t)} = \lim_{t\to\infty} \Big(1 - \frac{\mathrm{Vol}_m(Q_t \backslash P_t)}{\mathrm{Vol}_m(Q_t)} + \frac{\mathrm{Vol}_m(P_t \backslash Q_t)}{\mathrm{Vol}_m(Q_t)}\Big) = 1.$$

Step 3: $M_{P_t} \xrightarrow{w^*} M_{\bar{P}}$.

$$\begin{aligned}
\int f \, dM_{P_t} &= \int \frac{f \mathcal{X}_{P_t}}{\mathrm{Vol}_m(P_t)} dM_\lambda^m \\
&= \frac{1}{\mathrm{Vol}_m(P_t)} \int f (\mathcal{X}_{Q_t} - \mathcal{X}_{Q_t \backslash P_t} + \mathcal{X}_{P_t \backslash Q_t}) \, dM_\lambda^m \\
&= \int \frac{f \mathcal{X}_{Q_t}}{\mathrm{Vol}_m(P_t)} \, dM_\lambda^m - \int \frac{f \mathcal{X}_{Q_t \backslash P_t}}{\mathrm{Vol}_m(P_t)} \, dM_\lambda^m + \int \frac{f \mathcal{X}_{P_t \backslash Q_t}}{\mathrm{Vol}_m(P_t)} \, dM_\lambda^m.
\end{aligned}$$

We want to show that both the second and the third addends tend to 0. We can assume that $\mathrm{Vol}_m(Q_t \backslash P_t) \neq 0$, since, otherwise, $\int f \mathcal{X}_{Q_t \backslash P_t} \, dM_\lambda^m = 0$ and we are done with the corresponding addend. Similarly, we assume that $\mathrm{Vol}_m(P_t \backslash Q_t) \neq 0$. Now,

$$\int f \, dM_{P_t} \quad = \quad A_1 - A_2 + A_3,$$

where,

$$\begin{aligned}
A_1 &= \frac{\mathrm{Vol}_m(Q_t)}{\mathrm{Vol}_m(P_t)} \int \frac{f \mathcal{X}_{Q_t}}{\mathrm{Vol}_m(Q_t)} dM_\lambda^m = \frac{\mathrm{Vol}_m(Q_t)}{\mathrm{Vol}_m(P_t)} \int f \, dM_{Q_t}, \\
A_2 &= \frac{\mathrm{Vol}_m(Q_t \backslash P_t)}{\mathrm{Vol}_m(P_t)} \int \frac{f \mathcal{X}_{Q_t \backslash P_t}}{\mathrm{Vol}_m(Q_t \backslash P_t)} dM_\lambda^m = \frac{\mathrm{Vol}_m(Q_t \backslash P_t)}{\mathrm{Vol}_m(P_t)} \int f dM_{Q_t \backslash P_t}, \text{ and} \\
A_3 &= \frac{\mathrm{Vol}_m(P_t \backslash Q_t)}{\mathrm{Vol}_m(P_t)} \int \frac{f \mathcal{X}_{P_t \backslash Q_t}}{\mathrm{Vol}_m(P_t \backslash Q_t)} dM_\lambda^m = \frac{\mathrm{Vol}_m(P_t \backslash Q_t)}{\mathrm{Vol}_m(P_t)} \int f \, dM_{P_t \backslash Q_t}.
\end{aligned}$$

Since $\int f dM_{Q_t \setminus P_t} \leq \max_{x\in R} f(x)$ and $\int f dM_{P_t \setminus Q_t} \leq \max_{x\in R} f(x)$, then, by Step 2, both A_2 and A_3 tend to 0. We move now to A_1. By Step 2, $\lim_{t\to\infty} \frac{\mathrm{Vol}_m(Q_t)}{\mathrm{Vol}_m(P_t)} = 1$. Since, by Step 1, $\lim_{t\to\infty} \int f\, dM_{Q_t} = \int f\, dM_{\bar{P}}$, we have $\lim_{t\to\infty} A_1 = \int f\, dM_{\bar{P}}$. Hence, $\lim_{t\to\infty} \int f\, dM_{P_t} = \lim_{t\to\infty} A_1 = \int f\, dM_{\bar{P}}$.

Case 2: $\bar{P}$ is an $(m-l)$-polytope, $l > 1$.

We have multiple degeneracies. To study this case, new auxiliary polytopes Q_t and $\bar{Q}$ have to be defined, but the idea of the proof is the same. Assume that the degeneracies are in the first l components. Then, there exist $a_1, \ldots, a_l \in \mathbb{R}$ such that for each $x \in \bar{P}$, $x_1 = a_1, \ldots, x_l = a_l$. Let $\{F_1, \ldots, F_l\} \subseteq \mathcal{F}(P)$ be the set of the facets of P containing $\bar{P}$ and let $\mathcal{H} = \{H^1, \ldots, H^l\}$ be the hyperplanes containing the previous facets. Assume, without loss of generality, that $P \subsetneq BH^j$. Let Q be the polyhedron defined as follows,

$$Q := \left\{ y \in \mathbb{R}^m \;:\; \begin{array}{l} \text{for each } j \in \{1, \ldots, k\},\ y \in BH^j \text{ and} \\ y = x + \sum_{i=1}^{l} \gamma_i e^i,\ \text{where } x \in \bar{P} \text{ and, for each } i \in \{1, \ldots, l\}, \gamma_i \in \mathbb{R} \end{array} \right\}.$$

Now, for each $t \in \mathbb{N}$, we define the auxiliary polytopes $Q_t := Q \cap BH^{e^1}_{\alpha_t}$. Also, let $\bar{Q} := Q \cap BH^{e^1}_{\bar{\alpha}}$ (see Figure 5). Note that, by definition, $\bar{Q} = \bar{P}$. Since $Q_t \cap H^{e^1}_{\bar{\alpha}} = \bar{Q}$ is bounded, applying Lemma 2, we have that Q_t is bounded. Hence, each Q_t is indeed a polytope (Refer to the Appendix for a deeper discussion on the construction and properties of the polyhedron Q and the Q_t polytopes).

Now, all the steps in Case 1 can be adapted for the Q_t's. Only some minor (and natural) changes have to be made. Next, we go through these steps, stressing where modifications are needed.

Step 1: $M_{Q_t} \xrightarrow{w^*} M_{\bar{Q}}$.

Step 1.a: Let $f \in C^{\mathbb{R}}(R)$ be such that there exists $c : [-r, r]^{m-l} \to \mathbb{R}$ with the following property: $f(x_L, x_{\bar{L}}) = c(x_L)$. Also, for each $x \in \bar{Q}$ and each $t \in \mathbb{N}$, we define the l-polytope $Q_t(x) := \{y \in Q_t \;:\; y_{\bar{L}} = x_{\bar{L}}\}$ (Figure 6). Again, if $x \neq x'$, then $Q_t(x) \cap Q_t(x') = \emptyset$ and $\mathrm{Vol}_l(Q_t(x)) = \mathrm{Vol}_l(Q_t(x')) = \frac{\mathrm{Vol}_m(Q_t)}{\mathrm{Vol}_{m-l}(\bar{Q})}$. Moreover, for each $x \in \bar{Q}$, f is constant in $Q_t(x)$. The rest is analogous to Case 1.

Step 1.b: Let $f \in C^{\mathbb{R}}(R)$. Let $\hat{x} \in \bar{Q}$. For each $t \in \mathbb{N}$, we define the compact set $K_t := \{z \in \mathbb{R}^l \;:\; z = y_L,\ \text{where } (y_L, y_{\bar{L}}) = y \in Q_t(\hat{x})\}$, *i.e.*, K_t is the projection of $Q_t(x)$ into $\mathbb{R}^l$. Note that the definition of K_t is independent of the selected $\hat{x} \in \bar{Q}$. Define the

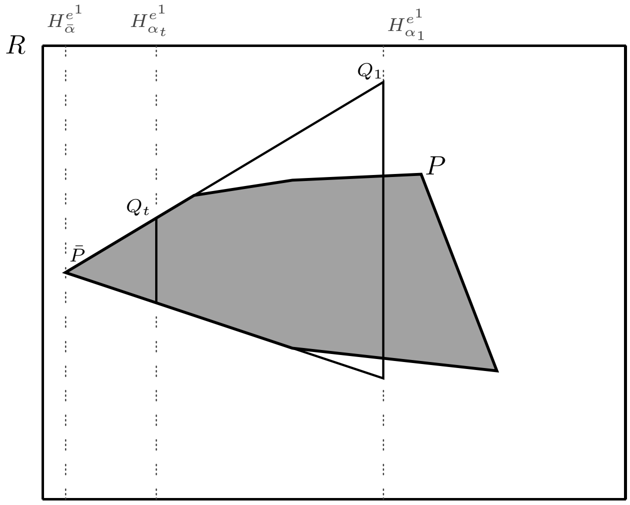

Figure 5: Defining the polytopes Q_t (P is a 2-polytope and $\bar{P}$ a 0-polytope)

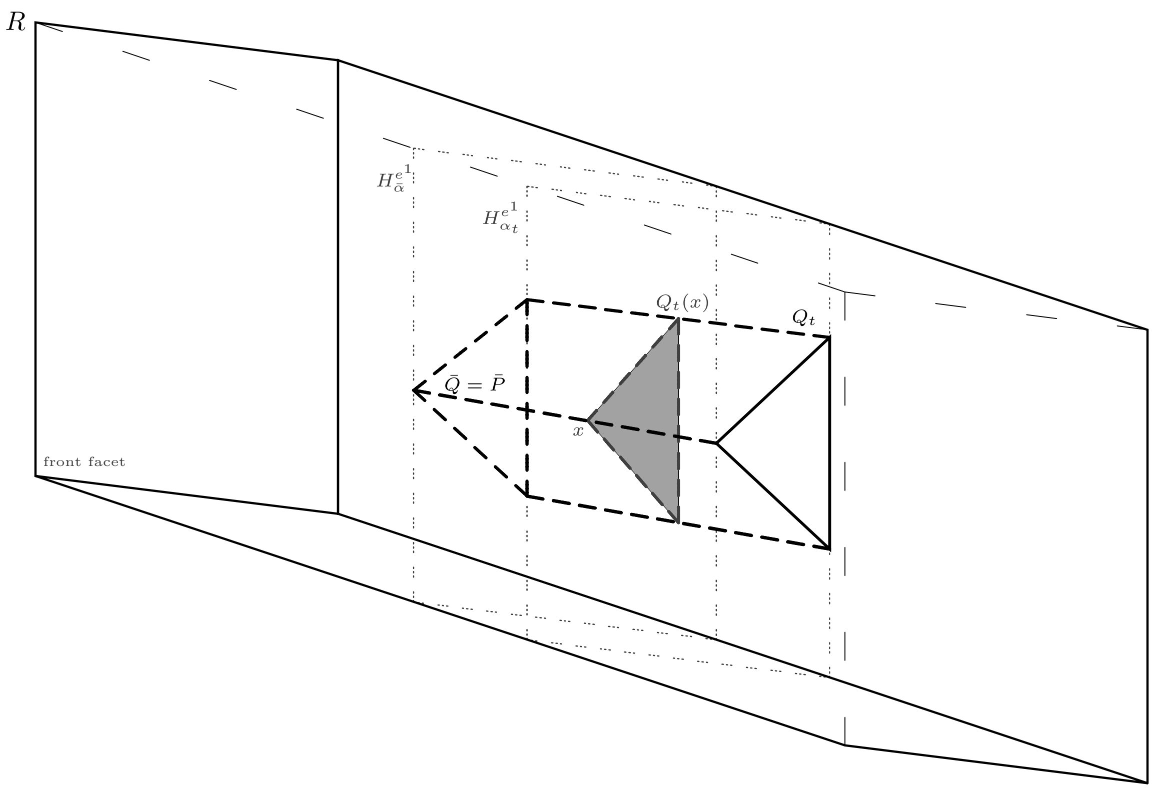

Figure 6: The set $Q_t(x)$ (P is a 3-polytope and $\bar{P}$ a 1-polytope)

three auxiliary functions

$$\begin{aligned} f^*(x_L, x_{\bar{L}}) &:= f(a_1, \ldots, a_l, x_{\bar{L}}), \\ \overline{c}_t(x_L, x_{\bar{L}}) &:= \max_{z \in K_t} f(z, x_{\bar{L}}), \text{ and} \\ \underline{c}_t(x_L, x_{\bar{L}}) &:= \min_{z \in K_t} f(z, x_{\bar{L}}). \end{aligned}$$

With these definitions Lemma 4 still applies. The rest is analogous to Case 1.

Step 2: $\lim_{t\to\infty} \frac{\mathrm{Vol}_m(P_t \backslash Q_t)}{\mathrm{Vol}_m(Q_t)} = \lim_{t\to\infty} \frac{\mathrm{Vol}_m(Q_t \backslash P_t)}{\mathrm{Vol}_m(Q_t)} = 0$ and $\lim_{t\to\infty} \frac{\mathrm{Vol}_m(P_t)}{\mathrm{Vol}_m(Q_t)} = 1.$

Now, $P_1^j \cap BH_{\bar{\alpha}}^{e^1}$ is, at most, an $(m-(l+1))$-polytope. Then, by Proposition 4, $\mathrm{Vol}_m(P_t \backslash Q_t) = \mathrm{O}((\alpha_t - \bar{\alpha})^{l+1})$ and $\mathrm{Vol}_m(Q_t) = \Theta(\alpha_t - \bar{\alpha})^l$. The rest is analogous to Case 1.

Step 3: $M_{P_t} \xrightarrow{w^*} M_{\bar{P}}$. Analogous to Case 1. □

So far, measures M_P have belonged to $\mathcal{M}(R)$. These measures can be extended to (Borel) measures in $\mathbb{R}^m$ by letting the measure of each $A \subseteq \mathbb{R}^m$ be $M_P(A \cap R)$. With a slight abuse of notation, we also denote these extensions by M_P.

Proof of Theorem 3. We distinguish two cases:

Case 1: $l = 0$**.** Let $r > 0$ be such that P is contained in the interior of $R = [-r, r]^m$. Let $f^R : R \to \mathbb{R}$ be the restriction of f to R. Then,

$$\int f \, dM_{P_t} = \int f^R \, dM_{P_t} \xrightarrow{\text{Pr. 5}} \int f^R \, dM_{\bar{P}} = \int f \, dM_{\bar{P}}.$$

Case 2: $l > 0$**.** There exist $a_1, \ldots, a_l \in \mathbb{R}$ such that $x \in P$ if and only if $x_1 = a_1, \ldots, x_l = a_l$. Let $R = a_1 \times \ldots \times a_l \times [-r, r]^{m-l}$ be such that P belongs to its interior. Now, Proposition 5 can be adapted for the M_{P_t}'s, $M_{\bar{P}}$ and this new R. Hence, the same argument of Case 1 leads to the result. □

Proof of Corollary 1. Let $R \subsetneq \mathbb{R}^m$ be defined as in Case 2 of the proof of Theorem 3 above and let f^R be the restriction of f to R. If $k = 1$, we are back to Theorem 3. Now, we show the result for $k = 2$, the proof for an arbitrary $k \geq 2$ being completely analogous. For each pair $(a, b) \in \mathbb{R}^2$, let $P(a, b) := P \cap BH_a^{u^1} \cap BH_b^{u^2}$. Note that $P(\alpha_t^1, \alpha_t^2) = P_t$ and $P(\bar{\alpha}^1, \bar{\alpha}^2) = \bar{P}$. Hence, we want to show the following convergence of real numbers: $\lim_{t\to\infty} \int f^R \, dM_{P(\alpha_t^1, \alpha_t^2)} = \int f^R \, dM_{P(\bar{\alpha}^1, \bar{\alpha}^2)}$. For each $i \in \{1, 2\}$, let $X^i := \{\alpha_t^i \ : \ t \in$

$\mathbb{N}\} \cup \{\bar{\alpha}^i\}, \subsetneq \mathbb{R}$. Let $F : X^1 \times X^2 \to \mathbb{R}$ be defined, for each $(a,b) \in X^1 \times X^2$ by $F(a,b) := \int f^R \, dM_{P(a,b)}$. By Theorem 3, F is continuous in each of its two components at $(\bar{\alpha}^1, \bar{\alpha}^2)$. Hence, since $\lim_{t\to\infty}(\alpha_t^1, \alpha_t^2) = (\bar{\alpha}^1, \bar{\alpha}^2)$, we have $\lim_{t\to\infty} F(\alpha_t^1, \alpha_t^2) = F(\bar{\alpha}^1, \bar{\alpha}^2)$. $\square$

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