FINITELY REPEATED GAMES: A GENERALIZED NASH FOLK THEOREM

By Julio González-Díaz*

Published in Games and Economic Behavior (2006) 55, 100-111
Published version available at http://www.sciencedirect.com
DOI 10.1016/j.geb.2005.03.003

Julio González-Díaz
Department of Statistics and OR
Faculty of Mathematics. Universidade de Santiago de Compostela
15782 Santiago de Compostela, SPAIN
Phone: +34 981 563100 (ext. 13378). Cellular phone: +34 626018019
e-mail: julkin@usc.es

*The Author is indebted to Ignacio García-Jurado, William Thomson, Ian Charles Coleman, and an anonymous referee for their helpful comments. Supported by the Spanish Ministry for Education and Science and FEDER through project BEC2002-04102-C02-02 and from the *Xunta de Galicia* under project PGIDT03PXIC20701PN.

Abstract

This paper characterizes the set of feasible payoffs of finitely repeated games with complete information that can be approximated arbitrarily closely by Nash equilibria.

KEYWORDS: Repeated Games, Nash Equilibrium, Folk Theorem, Finite Horizon, Complete Information.

JEL classification: C72

1 Introduction

Over the past thirty years, necessary and sufficient conditions have been published for numerous "folk theorems", asserting that the individually rational feasible payoffs of finitely or infinitely repeated games with complete information can be achieved by Nash or subgame perfect equilibria.[1] The original folk theorem was concerned about the Nash Equilibria of infinitely repeated games. This folk theorem stated that every individually rational feasible payoff of the original game can be obtained as a Nash Equilibrium of the repeated game; no assumption was needed for this result (a statement and proof of this result can be found in Fudenberg and Maskin (1986)). Then, the theorists turned to study subgame perfection in infinite horizon models and they found a counterpart of the previous result for undiscounted repeated games; again, no assumptions were needed (Aumann and Shapley, 1976; Rubinstein, 1979). A few years later, discount parameters were incorporated again into the model; in this case, some conditions were needed to get the perfect folk theorem (Fudenberg and Maskin, 1986). These conditions were refined in the mid-nineties (Abreu et al., 1994; Wen, 1994).

Together with the previous results, also grew the literature on finitely repeated games. The main results for finite horizon models obtained conditions for the Nash folk Theorem (Benoît and Krishna, 1987), and also for the perfect one (Benoît and Krishna, 1985). This perfect folk theorem relied on the fact that mixed strategies were observable; the same result but without that assumption was obtained in the mid-nineties (Gossner, 1995). Assuming again observable mixed strategies, Smith (1995) obtained a *necessary and sufficient* condition for the arbitrarily close approximation of strictly rational feasible payoffs by subgame perfect equilibria with finite horizon: that the game have "recursively distinct Nash payoffs", a premise that relaxes the assumption in Benoît and Krishna (1985) that each player have multiple Nash payoffs in the stage game.

Smith claimed that this condition was also necessary for approximation of the individually rational feasible payoffs of finitely repeated games by Nash equilibria. In this paper we show that this is not so by establishing a similar but distinct sufficient condition that is weaker than both Smith's condition and the assumptions made by Benoît and Krishna (1987). Moreover, our condition is also necessary.

[1] The survey by Benoît and Krishna (1999) includes many of these results.

Essentially, the difference between the subgame perfect and Nash cases hinges on the weakness of the Nash solution concept: in the Nash case it is not necessary for threats of punitive action against players who deviate from the equilibrium not to involve loss to the punishing players themselves, *i.e.*, threats need not be credible. The kind of equilibrium we define in this paper require for its corresponding path ρ, to finish, for each player i, with a series Q_i of rounds in which i cannot unilaterally improve his stage payoff by deviation from ρ_i, and for this terminal phase to start with a series Q_i^0 of rounds in which the other players, regardless of the cost to themselves, can punish him effectively for any prior deviation by imposing a loss that wipes out any gains he may have made in deviating.

Many of the results mentioned above concern the approximability of the entire set of individually rational feasible payoffs. The main theorem in this paper is more general in that, for any game, it characterizes the set of feasible payoffs that are approximable.

Although subgame perfect equilibrium is a desirable refinement of Nash equilibrium, results for the latter are still needed for games in which the perfect folk theorem does not apply. Game G in Figure 1 shows that, indeed, this is the case for a generic class of games. The assumptions for the perfect folk theorem do not hold for game G. Moreover, Theorem 2 in Smith (1995) implies that $(3,3)$ is the unique payoff achievable via subgame perfect equilibrium in any repeated game such that G is its stage game. However, every feasible and individually rational payoff, (*e.g.*, (4,4)) can be approximated in Nash equilibrium in many of those repeated games (for small enough discount and big enough number of repetitions).

	L	M	R
T	3,3	6,2	1,0
M	2,6	0,0	0,0
B	0,1	0,0	0,0

Figure 1: A game for which the Nash folk theorem is needed.

We have structured the paper as follows. We introduce notation and concepts in Section 2. In Section 3 we state and prove the main result. Next, in Section 2.5 we are concerned about unobservable mixed strategies. Finally, we conclude in Section 5.

2 Basic Notation, Definitions and an Example

2.1 The Stage Game

A game G in strategic form is a triplet (N, A, φ), where:

- $N := \{1, \ldots, n\}$ is the set of players,
- $A := \prod_{i \in N} A_i$ and A_i is the set of player i's strategies,

- $\varphi := (\varphi_1, \ldots, \varphi_n)$ and $\varphi_i : A \to \mathbb{R}$ is the payoff function of player i.

Let $\mathcal{G}^N$ be the set of games with set of players N.

We assume that, for each $i \in N$, the sets A_i are compact and the functions φ_i are continuous. Let a_{-i} be a strategy profile for players in $N \backslash \{i\}$ and A_{-i} the set of such profiles. For each $i \in N$ and each $a_{-i} \in A_{-i}$, let $\mu(a_{-i}) := \max_{a_i \in A_i} \{\varphi_i(a_{-i}, a_i)\}$. Also, for each $i \in N$, let $v_i := \min_{a_{-i} \in A_{-i}} \{\mu(a_{-i})\}$. The vector $v := \{v_1, \ldots, v_n\}$ is the *minimax payoff vector*. Let F be the set of *feasible* payoffs: $F := \mathrm{co}\{\varphi(a) : a \in A\}$. Let $\bar{F}$ be the set of all *feasible and individually rational* payoffs:

$$\bar{F} := F \cap \{u \in \mathbb{R}^n : u \geq v\}.$$

To avoid confusion with the strategies of the repeated game, in what follows we refer to the strategies $a_i \in A_i$ and the strategy profiles $a \in A$ of the stage game as actions and action profiles, respectively.

2.2 The Repeated Game

Let $G(\delta, T)$ be the game consisting in the T-fold repetition of G with payoff discount parameter $\delta \in (0, 1]$. In this game we assume *perfect monitoring*, *i.e.*, each player can choose his action in the current stage in the light of all actions taken by all players in all previous stages. Let σ be a strategy profile of $G(\delta, T)$, and the action profile sequence $\rho = \{\rho^1, \ldots, \rho^T\}$ its corresponding *path*. Let $\varphi_i^t(\rho)$ be the stage payoff of player i at stage t when all players play in accordance with ρ. Then, player i's payoff in $G(\delta, T)$ when σ is played is his average discounted stage payoff: $\psi_i(\sigma) \equiv \psi_i(\rho) := ((1-\delta)/(1-\delta^T)) \sum_{t=1}^{T} \delta^{t-1} \varphi_i^t(\rho)$.[2]

2.3 Minimax-Bettering Ladders

Let M be an m-player subset of N. Let $A_M := \prod_{i \in M} A_i$ and let $G(a_M)$ be the game induced for the $n-m$ players in $N \backslash M$ when the actions of the players in M are fixed at $a_M \in A_M$. By abuse of language, if $i \in N \backslash M$, $a_M \in A_M$, and $\sigma \in A_{N \backslash M}$ we write $\varphi_i(\sigma)$ for i's payoff at σ in $G(a_M)$. A *minimax-bettering ladder* of a game G is a triplet $\{\mathcal{N}, \mathcal{A}, \Sigma\}$, where $\mathcal{N}$ is a strictly increasing chain $\{\emptyset = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ of $h+1$ subsets of N ($h \geq 1$), $\mathcal{A}$ is a chain of action profiles $\{a_{N_1} \in A_{N_1}, \ldots, a_{N_{h-1}} \in A_{N_{h-1}}\}$ and Σ is a chain $\{\sigma^1, \ldots, \sigma^h\}$ of Nash equilibria of $G = G(a_{N_0}), G(a_{N_1}), \ldots, G(a_{N_{h-1}})$, respectively, such that at σ^l the players of $G(a_{N_{l-1}})$ receiving payoffs strictly greater than their minimax payoff are exactly those in $N_l \backslash N_{l-1}$: for each $i \in N_l \backslash N_{l-1}$, $\varphi_i(\sigma^l) > v_i$, and for each $i \in N \backslash N_l$, $\varphi_i(\sigma^l) \leq v_i$.

Let the sets in $\mathcal{N}$ be the rungs of the ladder. In algorithmic terms, if the first $l-1$ rungs of the ladder have been constructed, then, for the l-th rung to exist, there must be $a_{N_{l-1}} \in A_{N_{l-1}}$ such that the game $G(a_{N_{l-1}})$ has an equilibrium σ^l. Moreover, σ^l has to be such that there are players $i \in N \backslash N_{l-1}$ for whom

[2] Or, $\psi_i(\sigma) \equiv \psi_i(\rho) := (1/T) \sum_{t=1}^{T} \varphi_i^t(\rho)$ if there are no discounts ($\delta = 1$).

$\varphi_i(\sigma^l) > v_i$. Let $N_l \backslash N_{l-1}$ be this subset of players of $G(a_{N_{l-1}})$. The game played in the next step is defined by some action profile a_{N_l}. The set N_h is the *top rung of the ladder.* A ladder with top rung N_h is *maximal* if there is no ladder with top rung $N_{h'}$ such that $N_h \subsetneq N_{h'}$. A game G is *decomposable as a complete minimax-bettering ladder* if it has a minimax-bettering ladder with N as its top rung. We show below that being decomposable as a complete minimax-bettering ladder is a necessary and sufficient condition for it to be possible to approximate all payoff vectors in $\bar{F}$ by Nash equilibria of $G(\delta, T)$ for some δ and T. Clearly, being decomposable as a complete minimax-bettering ladder is a weaker property than the requirement in Smith (1995), that at each step $l-1$ of a similar kind of ladder there be action profiles $a_{N_{l-1}}$, $b_{N_{l-1}}$ such that the games $G(a_{N_{l-1}})$ and $G(b_{N_{l-1}})$ have Nash equilibria σ_a^l and σ_b^l with $\varphi_i(\sigma_a^l) \neq \varphi_i(\sigma_b^l)$ for a nonempty set of players (those in $N_l \backslash N_{l-1}$).

2.4 An Example

Let $G \in \mathcal{G}^N$, let L be a maximal ladder of G, and $N_{\max}$ its top rung. For each $i \in N$, let l_i be the unique integer such that $i \in N_{l_i} \backslash N_{l_i - 1}$. In the equilibrium strategy profile constructed in Theorem 1 below, the action profile sequence in the terminal phase Q_i referred to in the Introduction, consists of repetitions of $(a_{N_{l_i-1}}, \sigma^{l_i}), (a_{N_{l_i-2}}, \sigma^{l_i-1}), \ldots, (a_{N_2}, \sigma^2)$ and σ; and the σ^j are Nash equilibria of the corresponding games $G(a_{N_{j-1}})$. Since player i is a player in all these games, he can indeed gain nothing by unilateral deviation during this phase. In the potentially punishing series of rounds Q_i^0, the action profile sequence consists of repetitions of $(a_{N_{l_i-1}}, \sigma^{l_i})$, in which i obtains more than his minimax payoff, with the accompanying threat of punishing a prior unilateral deviation by i by minimaxing him instead.

	l	m	r
T	0, 0, 3	0,-1, 0	0,-1, 0
M	-1, 0, 0	0,-1, 0	0,-1, 0
B	-1, 0, 0	0,-1, 0	0,-1, 0

L

l	m	r
0, 3,-1	0,-1,-1	1,-1,-1
-1, 0,-1	-1,-1,-1	0,-1,-1
-1, 0,-1	-1,-1,-1	0,-1,-1

R

Figure 2: A game that is decomposable as a complete minimax-bettering ladder

As an illustration of the above ideas, consider the three-player game G shown in Figure 2. Its minimax payoff vector is $(0, 0, 0)$, and its unique Nash equilibrium is the action profile $\sigma^1 = (T, l, L)$, with associated payoff vector $(0, 0, 3)$. Hence, $N_1 = \{3\}$; player 3 can be punished by 1 and 2 by playing one of his minimax profiles instead of playing $(T, l, \cdot)$. If player 3 now plays R ($a_{N_1} = R$), the resulting game $G(a_{N_1}) = G(R)$ has an equilibrium $\sigma^2 = (T, l)$ with payoff vector $(0, 3)$. Hence, $N_2 = \{2, 3\}$ and player 2 can be punished by 1 and 3 by playing one of his minimax profiles instead of playing $(T, \cdot, R)$. Finally if players 2 and 3 now play r and R ($a_{N_2} = (r, R)$), the resulting game $G(a_{N_2}) = G(r, R)$ has the trivial equilibrium $\sigma^3 = (T)$ with payoff 1 for player 1. Hence, player 1 can be punished by 2 and 3 if they play one of his minimax profiles instead of playing $(\cdot, r, R)$.

2.5 Further Preliminaries

As a consequence of the next Lemma we can unambiguously refer to the *top rung of a game G*.

Lemma 1. *Let $G \in \mathcal{G}^N$. Then, all its maximal ladders have the same top rung.*

Proof. Suppose there are maximal ladders $L = \{\mathcal{N}, \mathcal{A}, \Sigma\}$, $L' = \{\mathcal{N}', \mathcal{A}', \Sigma'\}$ with $\mathcal{N} = \{\mathcal{N}_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h\}$ and $\mathcal{N}' = \{\mathcal{N}'_0 \subsetneq N'_1 \subsetneq \cdots \subsetneq N'_k\}$ such that, $N_h \neq N'_k$. Assume, without loss of generality, that $N'_k \backslash N_h \neq \emptyset$. For each $j \in N'_k$, let l_j be the unique integer such that $j \in N'_{l_j} \backslash N'_{l_j - 1}$. Let $i \in \operatorname{argmin}_{j \in N'_k \backslash N_h} l_j$. Then, $N'_{l_i - 1} \subseteq N_h$. Let a_{N_h} be the action profile defined as follows:

$$\text{for each } j \in N, \quad (a_{N_h})_j = \begin{cases} (a'_{N'_{l_i-1}})_j & j \in N'_{l_i - 1} \\ (\sigma'^{l_i})_j & j \in N_h \backslash N'_{l_i - 1} \end{cases},$$

where $\sigma'^{l_i} \in \Sigma'$ is an equilibrium of the game $G(a'_{N'_{l_i-1}})$ induced by the action profile $a'_{N'_{l_i-1}} \in \mathcal{A}'$.

Now, let σ^{h+1} be the restriction of σ'^{l_i} to $N \backslash N_h$. Since σ'^{l_i} is an equilibrium of $G(a'_{N'_{l_i-1}})$, and $N \backslash N_h \subseteq N \backslash N'_{l_i-1}$, σ^{h+1} is an equilibrium of $G(a_{N_h})$. Moreover, the set of players $j \in N \backslash N_h$ for whom $\varphi_j(\sigma^{h+1}) > v_j$ is $N'_{l_i} \backslash N_h$. Let $N_{h+1} := N'_{l_i} \backslash N_h$. Since N_{h+1} contains i, it is nonempty. Let $L'' = \{\mathcal{N}'', \mathcal{A}'', \Sigma''\}$ be the ladder defined by

- $\mathcal{N}'' = \{\mathcal{N}_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_h \subsetneq N_{h+1}\}$,
- $\mathcal{A}'' = \{a_{N_1}, \ldots, a_{N_{h-1}}, a_{N_h}\}$,
- $\Sigma'' = \{\sigma^1, \ldots, \sigma^h, \sigma^{h+1}\}$.

The top rung of L'' strictly contains that of L. Hence, L is not maximal, which proves the Lemma. □

Let G be a game with set of players N and let $N' \subseteq N$. We say that $G \in \mathrm{TR}_{N'}(\mathcal{G}^N)$ if the top rung of any maximal ladder of G is N'. Hence, a game G is decomposable as a complete minimax-bettering ladder if and only if $G \in \mathrm{TR}_N(\mathcal{G}^N)$.

Let $G \in \mathrm{TR}_{N_{\max}}(\mathcal{G}^N)$ and $\hat{a} \in A_{N_{\max}}$. Let $\Lambda(\hat{a}) := \{\lambda = (\hat{a}, \sigma) \in A : \sigma \text{ Nash equilibrium of } G(\hat{a})\}$ and $\Lambda := \bigcup_{\hat{a} \in A_{N_{\max}}} \Lambda(\hat{a})$. Let $\varphi(\Lambda) := \{\varphi(\lambda) : \lambda \in \Lambda\}$. Let $\bar{F}_{N_{\max}}$ be the set of N_{max}*-attainable* payoffs of G: $\bar{F}_{N_{\max}} := \bar{F} \cap \operatorname{co} \varphi(\Lambda)$. Note that, by the definition of $N_{\max}$, for each $u \in \bar{F}_{N_{\max}}$ and each $i \in N \backslash N_{\max}$, $u_i = v_i$. Moreover, when $N_{\max} = N$ we have $\Lambda = A$ and $\bar{F}_{N_{\max}} = \bar{F}$.

Lemma 2. *Let $G \in \mathrm{TR}_{N_{max}}(\mathcal{G}^N)$. Then, the set $\bar{F}_{N_{max}}$ is closed.*

Proof. First, we show that Λ is closed. Let $\{(a_n, \sigma_n)\}$ be a sequence of action profiles in Λ with limit (a, σ). Since $A_{N_{\max}}$ is compact, $a \in A_{N_{\max}}$. Since φ is continuous, σ is a Nash equilibrium of $G(a)$. Hence, $(a, \sigma) \in \Lambda$.

The set $\varphi(\Lambda)$ is the image of a closed set under a continuous function. Since φ has a compact domain, $\varphi(\Lambda)$ is closed. Hence, $\bar{F} \cap \operatorname{co} \varphi(\Lambda)$ is closed. □

The promised result concerning the approximability of all payoffs in $\bar{F}$ by Nash equilibrium payoffs is obtained below as an immediate corollary of a more general theorem concerning the approximability of all payoffs in $\bar{F}_{N_{\max}}$. In this more general case, the collaboration of the players in $N_{\max}$ is secured by a strategy analogous to that sketched in the Example of Section 2.4, while the collaboration of the players in $N \backslash N_{\max}$ is also ensured because none of them is able to obtain any advantage by unilateral deviation from any action profile in Λ.

3 The Theorem

In the theorem that follows, the set of action profiles A may consist either of pure or mixed action profiles; in the latter case, we assume that all players are cognizant not only of the pure actions actually put into effect at each stage, but also of the mixed actions of which they are realizations. We discuss unobservable mixed actions in Section 2.5. Also, we assume *public randomization*: at each stage of the repeated game, players can let their actions depend on the realization of an exogenous continuous random variable. The assumption of public randomization is without loss of generality. Given a correlated mixed action, its payoff can be approximated by alternating pure actions with the appropriate frequencies. More precisely, for each $u \in \bar{F}$ and each $\varepsilon > 0$, there are pure actions $a_1, \ldots, a_l$ such that $||u - (a_1 + \ldots + a_l)/l|| < \varepsilon$. Hence, if the discount parameter δ is close enough to 1, the same inequality is still true if we consider discounted payoffs. Then, since we state Theorem 1 in terms of approximated payoffs, public randomization assumption can be dispensed with.[3]

Theorem 1. *Let $G \in \mathrm{TR}_{N_{max}}(\mathcal{G}^N)$. Let $u \in F$. Then, a necessary and sufficient condition for there to be for each $\varepsilon > 0$, an integer T_0 and a positive real number $\delta_0 < 1$ such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, $G(\delta, T)$ has a Nash equilibrium payoff w such that $\|w - u\| < \varepsilon$ is that u be N_{max}-attainable (i.e., $u \in \bar{F}_{N_{max}}$).*

Proof. $\overset{\text{suffic}}{\Longleftarrow}$ Let $a \in \Lambda$ be an action profile of G such that $\varphi(a) = u$, and let $L = \{\mathcal{N}, \mathcal{A}, \Sigma\}$ be a maximal minimax-bettering ladder of G. By the definition of

[3]For further discussion on public randomization refer to Fudenberg and Maskin (1991) and Olszewski (1997). Also, refer to Gossner (1995) for a paper in which public randomization is not assumed and the approximation procedure we described above is explicitly made (though discounts are not considered).

Λ, players in $N \backslash N_{\max}$ have no incentive for unilateral deviation from a. Let ρ be the following action profile sequence:

$$\rho := \{\underbrace{a, \ldots, a}_{T-T_0+q_0}, \underbrace{\lambda^h, \ldots, \lambda^h}_{q_h}, \underbrace{\lambda^{h-1}, \ldots, \lambda^{h-1}}_{q_{h-1}}, \ldots, \underbrace{\lambda^1, \ldots, \lambda^1}_{q_1}\},$$

where for each $l \in \{1, \ldots h\}$, $\lambda^l = (a_{N_{l-1}}, \sigma^l)$ with $a_{N_{l-1}} \in \mathcal{A}$ and $\sigma^l \in \Sigma$. Let $\varepsilon > 0$. Next, we obtain (in this order) values for $q_h, \ldots, q_1$, the discount δ_0, q_0, and T_0 to ensure that for each $T \geq T_0$ and each $\delta \in (\delta_0, 1]$, there is a Nash equilibrium of $G(\delta, T)$ whose path is ρ and such that $||\varphi(\rho) - u|| < \varepsilon$.

First, we calculate how many repetitions of $G(a_{N_{l_i-1}})$ are necessary for the players in $N \backslash \{i\}$ to be able to punish a player $i \in N_{\max}$ for prior deviation. For each action profile $\hat{a} \in A$, let $\bar{\mu}(\hat{a}) := \mu(\hat{a}_{-i}) - \varphi_i(\hat{a})$, *i.e.*, the maximum "illicit" profit that player i can obtain by unilateral deviation from $\hat{a}$. Let $\bar{\mu}_i = \max\{\bar{\mu}(a), \bar{\mu}((a_{N_{h-1}}, \sigma^h)), \ldots, \bar{\mu}(\sigma^1)\}$ and $m_i = \min\{\varphi_i(a) \,:\, a \in A\}$. Let $l_i \in \mathbb{N}$ be such that $i \in N_{l_i} \backslash N_{l_i-1}$. Let $\delta_0 \in (0,1)$ and let $q_h, \ldots, q_1$ be the natural numbers defined through the following iterative procedure:

Step 0:
For each $i \in N_h \backslash N_{h-1}$, let $r_i \in \mathbb{N}$ and $\delta_i \in (0,1)$ be
$r_i := \min\{r \in \mathbb{N} \,:\, r(\varphi_i(\sigma^{l_i}) - v_i) > \bar{\mu}_i\}$,[4]
$\delta_i := \min\{\delta_i \in (0,1) \,:\, \bar{\mu}_i - \sum_{t=1}^{r_i} \delta_i^t(\varphi_i(\sigma^{l_i}) - v_i) < 0\}$.
Let $q_h \in \mathbb{N}$ be
$q_h := \max\{r_i \,:\, i \in N_h \backslash N_{h-1}\}$.

Step k ($k < h$):
Let $T_k := \sum_{l=0}^{k-1} q_{h-l}$.
For each $i \in N_{h-k} \backslash N_{h-k-1}$, let $r_i \in \mathbb{N}$ and $\delta_i \in (0,1)$ be
$r_i := \min\{r \in \mathbb{N} \,:\, r(\varphi_i(\sigma^{l_i}) - v_i) > \bar{\mu}_i + T_k(v_i - m_i)\}$,
$\delta_i := \min\{\delta_i \in (0,1) \,:\, \bar{\mu}_i + \sum_{t=1}^{T_k} \delta_i^t(v_i - m_i) - \sum_{t=T_k+1}^{T_k+r_i} \delta_i^t(\varphi_i(\sigma^{l_i}) - v_i) < 0\}$.
Let $q_{h-k} \in \mathbb{N}$ be
$q_{h-k} := \max\{r_i \,:\, i \in N_{h-k} \backslash N_{h-k-1}\}$.

Step h:
$\delta_0 := \max_{i \in N} \delta_i$.

The natural numbers $q_h, \ldots, q_1$ and the discount δ_0 are such that for each $l \in \{1, \ldots, h\}$, q_l repetitions of $G(a_{N_{l-1}})$ suffice to allow any player in $N_l \backslash N_{l-1}$ to be punished. Next, we obtain the values for q_0 and T_0. Let q_0 be the smallest integer such that:

$$\left\| \frac{q_0\, \varphi(a) + q_h\, \varphi(\lambda^h) + \cdots + q_1\, \varphi(\lambda^1)}{q_0 + q_h + \cdots + q_1} - \varphi(a) \right\| < \varepsilon \tag{1}$$

Let $T_0 := q_0 + q_1 + \cdots + q_h$. Let $T \geq T_0$ and $\delta \in [\delta_0, 1]$. We prescribe for $G(\delta, T)$ the strategy profile in which all players play according to ρ unless and until there is a

[4] The natural number r_i is such that, at each step, punishing player i during r_i stages suffices to wipe out any stage gain he could get by deviating from ρ when the discount is $\delta = 1$.

unilateral deviation. In such a deviation occurs, the deviating player is minimaxed by all the others in the remaining stages of the game. It is straightforward to check that this profile is a Nash equilibrium of $G(\delta, T)$. Moreover, by inequality (1), its associated payoff vector w differs from u by less than $\frac{T_0}{T}\varepsilon$ if $\delta = 1$. Hence, the same observation is certainly true if $\delta < 1$, in which case payoff vectors of the early stages, $\varphi(a)$, receive greater weight than the payoff vectors of the endgame.

$\overset{\text{necess}}{\Longrightarrow}$ Let $u \notin \bar{F}_{N_{\max}}$. Suppose that $N_{\max} = N$. Then, $\bar{F}_{N_{\max}} = \bar{F}$. Hence, u is not individually rational. Hence, it can not be the payoff associated to any Nash equilibrium. Then, we can assume $N_{\max} \subsetneq N$. Since $\bar{F}_{N_{\max}}$ is a closed set, there is $\varepsilon > 0$ such that $\|w - u\| < \varepsilon$ implies $w \notin \bar{F}_{N_{\max}}$. Hence, if for some T and δ there is a strategy profile σ of $G(\delta, T)$ such that $\|\varphi(\sigma) - u\| < \varepsilon$, then $\varphi(\sigma) \notin \bar{F}_{N_{\max}}$. Hence, by the definition of $\bar{F}_{N_{\max}}$, there is at least one stage of $G(\delta, T)$ in which, with positive probability, σ prescribes an action profile not belonging to Λ . Let q be the last such stage and $\bar{a} = (\bar{a}_{N_{\max}}, \bar{a}_{N \setminus N_{\max}})$ the corresponding action profile. By the definition of $\bar{F}_{N_{\max}}$, $\bar{a}_{N \setminus N_{\max}}$ cannot be a Nash equilibrium of $G(\bar{a}_{N_{\max}})$. Hence, there is a player $j \in N \setminus N_{\max}$ who can increase his payoff in round q by deviating unilaterally from $\bar{a}$. Since, by the definition of q, σ assigns j a stage payoff of v_j in all subsequent rounds, this deviation cannot subsequently be punished. Hence, σ is not an equilibrium of $G(\delta, T)$. □

Corollary 1. *Let $G \in \mathcal{G}^N$ be decomposable as a complete minimax-bettering ladder,* (i.e., $G \in \mathrm{TR}_N(\mathcal{G}^N)$). *Then, for each $u \in \bar{F}$ and each $\varepsilon > 0$, there is $T_0 \in \mathbb{N}$ and $\delta_0 < 1$ such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, there is a Nash equilibrium payoff w of $G(\delta, T)$ with $\|w - u\| < \varepsilon$.*

Proof. $N = N_{\max} \Rightarrow \bar{F} = \bar{F}_{N_{\max}}$. Hence, this result is a consequence of Theorem 1. □

Corollary 2. *Let $G \in \mathcal{G}^N$ be not decomposable as a complete minimax-bettering ladder* (i.e., $G \notin \mathrm{TR}_N(\mathcal{G}^N)$). *Then, for each $T \in \mathbb{N}$, each $\delta \in (0, 1]$, each $i \in N \setminus N_{max}$, and each Nash equilibrium σ of $G(\delta, T)$ we have $\varphi_i(\sigma) = v_i$.*

Proof. For each $u \in \bar{F}_{N_{\max}}$ and for each $i \in N \setminus N_{\max}$, $u_i = v_i$. Hence, this result follows by an argument paralleling the proof of necessity in Theorem 1. □

4 Unobservable Mixed Actions

In what follows, we drop the assumption that mixed actions are observable. Hence, if a mixed action is chosen by one player, the others can only observe its realization. To avoid confusion, for each game G, let G^u be the corresponding game with unobservable mixed actions. We need to introduce one additional piece of notation to distinguish between pure and mixed actions. Let A_i and S_i be the sets of player i's pure and mixed actions respectively (with generic elements a_i and s_i). Similarly, let A and S be the sets of pure and mixed action profiles. Hence, a game is now a triplet (N, S, φ).

The game G (or G^u) in Figure 3 illustrates some of the differences between the two frameworks. Although it is not entirely straightforward, it is not difficult to check that the minimax payoff of G is $v = (0, 0, 0)$. Let $s_3 = (0, 0.5, 0.5)$ be the mixed action of player 3 in which he plays L with probability 0, and M and R with probability 0.5. Let $\sigma^2 \in A_{\{1,2\}}$. Let $\mathcal{N} = \{\emptyset, \{3\}, N\}$, $\mathcal{S} = \{s_3\}$ and $\Sigma = \{(T, l, L), \sigma^2\}$. Then, $L = \{\mathcal{N}, \mathcal{S}, \Sigma\}$ is a complete minimax-bettering ladder of G regardless of σ^2 (note that in the game $G(s_3)$, for each $\sigma^2 \in A_{\{1,2\}}$, both players 1 and 2 receive the constant payoff 0.5). Hence, G satisfies the assumptions of Corollary 1, so every payoff in $\bar{F}$ can be approximated in Nash equilibrium.

	l	r
T	0, 0, 2	0, 0, 0
B	0, 0, 0	0, 0, 0

L

	l	r
T	0, 0,-1	2,-1,-1
B	-1, 2,-1	1, 1,-1

M

	l	r
T	1, 1,-8	-1, 2,-8
B	2,-1,-8	0, 0,-8

R

Figure 3: A game where unobservable mixed actions make a difference

Consider now the game G^u. Let $u \in \bar{F}$, and let a be such that $\varphi(a) = u$ (recall that we assumed public randomization). If we follow the path ρ constructed in the proof of Theorem 1, there are natural numbers q_0, q_1, and q_2 such that ρ leads to play i) a during the first q_0 stages, ii) (σ^2, s_3) during the following q_2 stages, and iii) (T,l,L) during the last q_1 stages. Let Q be the phase described in ii). Since player 3 is not indifferent between the two actions in the support of s_3, we need a device to detect possible deviations from that support. But, once such a device has been chosen, it is not clear whether we can ensure that there are not realizations for the first $q_2 - 1$ stages of Q that would allow player three to play L in the last stage of Q without being detected.[5]

Next, we revisit the results of Section 3 to understand the extent to which their counterparts hold. Unfortunately, we have not found a necessary and sufficient condition for the Folk Theorem under unobservable mixed actions, *i.e.*, we have not found an exact counterpart for Theorem 1. More precisely, as the previous example shows, unobservable mixed actions invalidate the proofs related to sufficiency conditions. On the other hand, proofs related to necessary conditions still carry over.

For the next result, we need to introduce a restriction on the ladders. The objective is to rule out situations as the one illustrated with Figure 3. Let $L = \{\mathcal{N}, \mathcal{S}, \Sigma\}$ be a ladder with $\mathcal{S} = \{s_{N_1}, \ldots, s_{N_{h-1}}\}$. L is a *p-ladder* if, for each $l \in \{1, \ldots, h-1\}$, $s_{N_l} \in A_{N_l}$. That is, at each rung of the ladder we only look at

[5]Game G^u partially illustrates why the arguments in Gossner (1995) cannot be easily adapted to our case. First, *mutatis mutandi*, he applies an existence of equilibrium theorem to the subgame in Q. If we want to do so, we need to ensure that players 1 and 2 get more than 0 in Q. Second, Gossner also uses the assumption of full-dimensionality of F to punish all the players who deviate during Q. We do not have that assumption and hence, it could be the case that we could not punish more than one player at the end of the game.

subgames obtained by fixing pure action profiles.[6]

Lemma 3. *Let $G \in \mathcal{G}^N$. Then, all its maximal p-ladders have the same top rung.*

Proof. Analogous to the proof of Lemma 1. □

Let G (or G^u) be a game with set of players N and let $N' \subseteq N$. We say that $G \in \mathrm{TR}^P_{N'}(\mathcal{G}^N)$ if the top rung of any maximal p-ladder of G is N'. Clearly, if $G \in \mathrm{TR}^P_{N'}(\mathcal{G}^N)$, then $G \in \mathrm{TR}_{N''}(\mathcal{G}^N)$ with $N' \subseteq N''$. The game G in Figure 3 provides an example in which the converse fails: $G \in \mathrm{TR}^P_{\{3\}}(\mathcal{G}^N)$ and $G \in \mathrm{TR}_{\{N\}}(\mathcal{G}^N)$. Let $G \in \mathrm{TR}^P_{N_{\max}}(\mathcal{G}^N)$ and a pure strategy $\hat{a} \in A_{N_{\max}}$. We can define $\bar{F}^P_{N_{\max}}$ paralleling the definition of $\bar{F}_{N_{\max}}$ in Section 2.

Next, we state the results. Note that the sets TR and $\bar{F}_{N_{\max}}$ are used for the necessity results and the sets TR^P and $\bar{F}^P_{N_{\max}}$ for the sufficiency ones.

Proposition 1 (Sufficient condition)**.** *Let $G^u \in \mathrm{TR}^P_{N_{max}}(\mathcal{G}^N)$. Then, for each $u \in \bar{F}^P_{N_{max}}$ and each $\varepsilon > 0$, there are $T_0 \in \mathbb{N}$ and $\delta_0 < 1$ such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, there is a Nash equilibrium payoff w of $G(\delta, T)$, with $\|w - u\| < \varepsilon$.*

Proof. Analogous to the proof of the sufficiency condition in Theorem 1. This is because, as far as a p-ladder is used to define the path ρ, whenever a player plays a mixed action, all the pure actions in its support are best replies to the actions of the others. □

Corollary 3. *Let $G^u \in \mathrm{TR}^P_N(\mathcal{G}^N)$. Then, for each $u \in \bar{F}$ and each $\varepsilon > 0$, there are $T_0 \in \mathbb{N}$ and $\delta_0 < 1$ such that for each $T \geq T_0$ and each $\delta \in [\delta_0, 1]$, there is a Nash equilibrium payoff w of $G(\delta, T)$, with $\|w - u\| < \varepsilon$.*

Proof. $N = N_{\max} \Rightarrow \bar{F} = \bar{F}_{N_{\max}} = \bar{F}^P_{N_{\max}}$. Hence, this result is an immediate consequence of Proposition 1. □

Note that the folk theorem in Benoît and Krishna (1987) is a particular case of this corollary. Next two results show that the exact counterparts of the necessity results in Section 3 carry over.

Proposition 2 (Necessary condition)**.** *Let $G^u \in \mathrm{TR}_{N_{max}}(\mathcal{G}^N)$. If $N_{max} \subsetneq N$ then, for each $u \notin \bar{F}_{N_{max}}$ there is $\varepsilon > 0$ such that for each $T \in \mathbb{N}$ and each $\delta \in (0, 1]$, $G(\delta, T)$ does not have a Nash equilibrium payoff w such that $\|w - u\| < \varepsilon$.*

Proof. Analogous to the proof of the necessity condition in Theorem 1. □

Corollary 4. *Let $G^u \notin \mathrm{TR}_N(\mathcal{G}^N)$. Then, for each $T \in \mathbb{N}$, each $\delta \in (0, 1]$, each $i \in N \backslash N_{max}$, and each Nash equilibrium σ of $G(\delta, T)$ we have $\varphi_i(\sigma) = v_i$.*

Proof. Analogous to the proof of Corollary 2. □

[6]Note that the games G and G^u have the same ladders and the same p-ladders.

5 Final Remarks

- **Remark 1.** Theorem 1 requires no use of the concept of effective minimax payoff, because non-equivalent utilities are irrelevant to the approximation of $N_{\max}$-attainable payoffs by Nash equilibria, in which there is no need for threats to be credible.[7]

- **Remark 2.** Corollaries 1 and 3 hold for a wider class of games than the result obtained by Benoît and Krishna (1987).

- **Remark 3.** Theorem 1 raises the question whether a similarly general result on the approximability of payoffs by equilibria also holds for subgame perfect equilibria. The main problem is to determine the subgame perfect equilibrium payoffs of players with "recursively distinct Nash payoffs" (Smith (1995)) when the game is not completely decomposable.

- **Remark 4.** Results in Section 2.5 raise the question whether a necessary and sufficient condition exists for the Nash folk theorem under unobservable mixed strategies.

- **Remark 5.** The results of this paper can be easily extended to the case in which each player has a different discount δ.

Dept. of Statistics and OR. Faculty of Mathematics, Santiago de Compostela University. 15782 Santiago de Compostela. Spain. e-mail: julkin@usc.es

References

Abreu, D., P. K. Dutta, L. Smith. 1994. The Folk Theorem for Repeated Games: A NEU Condition. *Econometrica* **62** 939–948.

Aumann, R. J., L. S. Shapley. 1976. Long Term Competition, A Game Theoretic Analysis. Tech. rep., Hebrew University.

Benoît, J.-P., V. Krishna. 1985. Finitely Repeated Games. *Econometrica* **53** 905–922.

———. 1987. Nash Equilibria of Finitely Repeated Games. *International Journal of Game Theory* **16** 197–204.

———. 1999. The Folk Theorems for Repeated Games: A Synthesis. *EconWPA-9902001* .

[7] See Wen (1994) and Abreu et al. (1994) for details on the effective minimax payoff and non-equivalent utilities respectively.

Fudenberg, D., E. Maskin. 1986. The Folk Theorem in Repeated Games with Discounting or with Incomplete Information. *Econometrica* **54** 533–554.

———. 1991. On the Dispensability of Public Randomization in Discounted Repeated Games. *Journal of Economic Theory* **53** 428–438.

Gossner, O. 1995. The Folk Theorem for Finitely Repeated Games with Mixed Strategies. *International Journal of Game Theory* **24** 95–107.

Olszewski, W. 1997. Perfect Folk Theorems. Does Public Randomization Matter? *International Journal of Game Theory* **27** 147–156.

Rubinstein, A. 1979. Strong Perfect Equilibrium in SuperGames. *International Journal of Game Theory* **9** 1–12.

Smith, L. 1995. Necessary and Sufficient Conditions for the Perfect Finite Horizon Folk Theorem. *Econometrica* **63** 425–430.

Wen, Q. 1994. The "Folk Theorem" for Repeated Games with Complete Information. *Econometrica* **62** 949–954.