Affiliation, Positive Dependence and the Revenue Ranking of Private Value Auctions[*] (Preliminary version)

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Abstract

We give necessary and sufficient conditions for existence of a pure strategy equilibrium for first price private value auctions. The signals of the players may have any kind of dependence. The conditions are given for a set of distributions which is dense in the set of all symmetric distributions.

The approach allows numerical simulations, which show that affiliation is a very restrictive assumption, not satisfied in many cases that have pure strategy equilibrium. We also show that neither existence nor the revenue ranking implied by affiliation (superiority of the English auction) generalizes for positively dependent distributions. Nevertheless, the revenue ranking is valid in a weak sense (on average) for the dense set of distributions.

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1 Introduction

Information play a crucial role in auctions. Indeed, one of the most important reasons for using an auction to sell a good is the uncertainty about the market price for the item in question. Also, most conclusions of auction theory depend crucially on information structure. For instance, under independence of the information of bidders (and other standard assumptions, such as symmetry and risk neutrality), all kind of auctions give the same revenue (Revenue

Equivalence Theorem, Vickrey 1961, Myerson 1981). This is no longer true if the independence assumption is relaxed (see below).

The same example also illustrates that the information structure should be considered in real world decisions. For instance, if the information possessed by bidders is correlated, the choice of the auction may influence the expected revenue of the seller.[1] Thus, a natural question arises: if the bidders have correlated types (which is natural to expect), which auction format (among the common ones) provides the highest expected payoff for the seller?

The best contribution to the solution of this problem was made through a remarkable insight of Milgrom and Weber (1982a), which introduced the concept of affiliation in auction theory.[2] Affiliation is a generalization of independence that was explained through the appealing notion that our assessments of values are positively dependent: "Roughly, this [affiliation] means that a high value of one bidder's estimate makes high values of the others' estimates more likely." (Milgrom and Weber (1982a), p. 1096. We give a formal definition of affiliation in section 2.) Under this assumption, Milgrom and Weber (1982a) obtain two main results:[3]

- there exists a symmetric pure strategy equilibrium (this is a generalization of the independent types case);

- the second price auction gives greater revenue than the first price auction (a truly new result that breaks the Revenue Equivalence Theorem and is a consequence of their linkage principle, which states that, on average, revenues are enhanced by always providing the bidders with as much information as possible about the value of the good).

In the face of these results, it is possible to cite at least three reasons for the profound influence of the afore mentioned paper in auction theory: (i) its theoretical depth and elegance; (ii) the plausibility of the hypothesis of affiliation, as explained by a clear economic intuition; (iii) the fact that it implies that English auctions yield higher revenues than first price auctions, which is a good explanation for the fact that English auctions are more common than first price auctions.

Nevertheless, some counterexamples and comments in the literature have suggested that affiliation may be too restrictive and its consequences not valid

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[1] Indeed, under correlation of types, Crémer and McLean (1988) show that it is possible to fully extract the surplus, through an optimal auction (provided that some conditions are met). We will not consider optimal designs, but only the single-object auctions most used in the real world: the English auction and the first price auction.

[2] In two previous papers, Milgrom presented results that use a particular version of the same concept, under the traditional statistical name "monotone likelihood ratio property" (MLRP): Milgrom (1981a, 1981b). Before Milgrom's works, Wilson (1969 and 1977) made important contributions that may be considered the foundations of the affiliated value model. Nevertheless, the concept is fully developed and the term affiliation first appears in Milgrom and Weber (1982a). See also Milgrom and Weber (1982b).

[3] The paper has many other insightful conclusions. We restrict ourselves to these because they seem the most important and relevant for our discussion.

in a more general context (see Perry and Reny 1999 and Klemperer 2002, for instance). Thus, one would like to have an assessment of how restrictive is affiliation and how far its implications can go. In section 2 we show that affiliation is quite restrictive.

From this, it seems important to have a picture of the equilibrium existence problem and the revenue ranking for a bigger set of distributions. Consider the simple setting of symmetric private value auctions with two players, but general dependence of types. While second price auctions always have pure strategy equilibrium in weakly dominant strategies, we only know that first price auctions have a mixed strategy equilibrium (see Jackson and Swinkels 2005). However, pure strategy equilibria have the advantage of providing grounds for revenue comparisons.

We offer necessary and sufficient conditions for pure strategy equilibrium existence for a set of distributions which is dense in the set of all symmetric distributions with bivariate density functions (p.d.f.). The set is as follows. Let $\left[\underline{t}, \overline{t}\right]^2$ be the fixed domain and divide it in n^2 equal squares. Our set is that of symmetric distributions which are constant in each of these squares. It is easy to see that, as n goes to ∞, we can approximate any p.d.f. (including non-continuous ones).

Although the necessary and sufficient conditions are not simple — as one could expect —, they are easy to check numerically. We use this feature to make simulations, which show that the set of distributions with pure strategy equilibrium is much bigger than the set of affiliated distributions. There is no easy relation between positive correlation and equilibrium existence, since we show that distributions with strong positive dependence properties may not have equilibrium. However, distributions with positive dependence are more likely to have pure strategy equilibria.

We also show that auctions with distributions satisfying strong positive dependence properties may fail to maintain the revenue ranking implied by affiliation (that is, the superiority of English auctions over first price auctions).

Nevertheless, since we consider a set which is dense in the set of distributions, we can give a general picture of the average of the distributions. Looking to the problem in this weak sense, that is, in the average of the distributions, we recover the revenue ranking of affiliation. The notion of "average" over distributions is well defined because we work with finite dimensional sets (see subsection 2.2).

The paper is organized as follows. Section 2 compares affiliation and other definitions of positive dependence. Section 3 presents the equilibrium existence results. Section 4 generalizes the revenue ranking and presents a counterexample for further generalizations. It also shows the numerical evaluation of the revenue for each given simple distribution. As a conclusion, section 5 summarizes the contribution.

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2 How restrictive is affiliation?

The introduction of the affiliation concept was made through an appealing economic intuition: "Roughly, this [affiliation] means that a high value of one bidder's estimate makes high values of the others' estimates more likely." Milgrom and Weber (1982a), p. 1096. The formal definition is as follows, for the case where there exists a density function.[4] The random variables $X_1, ..., X_N$ with density function $f : \mathbb{R}^N \to \mathbb{R}_+$ are affiliated if $f(x) f(y) \leqslant f(x \wedge y) f(x \vee y)$, where x and y are realizations of $(X_1, ..., X_N)$ and $x \wedge y = (\min\{x_i, y_i\})_{i=1}^{N}$ and $x \vee y = (\max\{x_i, y_i\})_{i=1}^{N}$.

We illustrate the definition in Figure 1, for $N = 2$. Affiliation requires that the product of the weights at the points (x', y') and (x, y) (where both values are high or both are low) is greater than (x, y') and (x', y) (where they are high and low, alternatively).
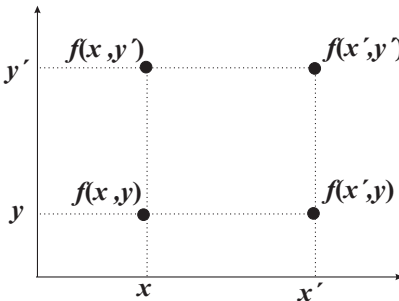

Figure 1: Affiliation requires $f(x', y) f(x, y') \leq f(x, y) f(x', y')$.

Thus, affiliation seems to be a good concept to express positive dependence. Indeed, there is a predominant view in auction theory that understands affiliation as a suitable synonym of positive dependence. This can be seen in the intuitions normally given to affiliation, along the same lines of the previous quote. One can say that the literature seems to mix two different ideas that we would like to state separately: (1) positive dependence is a sensible assumption (an idea that we call *positive dependence intuition*); and (2) affiliation is a suitable mathematical definition for positive dependence (an idea that, for easier future reference, we will call *rough identification*).

The positive dependence intuition seems very reasonable, because many mechanisms may lead us to correlated assessments of values: education, culture and even evolution.

Nevertheless, we argue that the rough identification is misleading because affiliation is too strong to be a suitable definition of positive dependence. In the

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[4]It is possible to give a definition of affiliation for distributions without density functions. See Milgrom and Weber (1982a) or the appendix. We will assume the existence of a density function and use only this definition.

following subsection, we present some theoretical concepts that also correspond to positive dependence and are strictly weaker than affiliation. In subsection 2.2, we evaluate numerically how relevant such restrictiveness is. The numerical evaluation is made through a "canonical" model – in a sense made precise in that subsection – and is surprisingly negative.

Since these results do not confirm the usual understanding (rough identification), it is useful to reassess other arguments and models that lead to affiliation. Subsection 2.3 considers the conditional independence model. Subsection 2.4 discusses the use of affiliation in other sciences.

The results and arguments of this section concern only the restrictiveness of affiliation as a good description of the world. Thus, they do not imply that affiliation is a worthless assumption. Such judgment must take into account the most important of all criteria: whether the resulting theory "yields sufficiently accurate predictions"(Friedman (1953), p. 14). Because of this, we study the main consequences of affiliation – namely, equilibrium existence and the revenue rank – in sections 3 and 4.

2.1 Affiliation and positive dependence

In the statistical literature, various concepts were proposed to correspond to the notion of positive dependence. Let us consider the bivariate case, and assume that the two real random variables X and Y have joint distribution F and strictly positive density function f. The following concepts are formalizations of the notion of positive dependence:[5]

Property I - X and Y are positively correlated (PC) if $cov(X, Y) \geqslant 0$.

Property II - X and Y are said to be positively quadrant dependent (PQD) if $cov(g\left(X\right), h\left(Y\right)) \geqslant 0$, for all g and h non-decreasing.

Property III - The real random variables X and Y are said to be associated (As) if $cov(g\left(X, Y\right), h\left(X, Y\right)) \geqslant 0$, for all g and h non-decreasing.

Property IV - Y is said to be left-tail decreasing in X (denoted LTD$(Y|X)$) if $\Pr[Y \leqslant y | X \leqslant x]$ is non-increasing in x for all y. X and Y satisfy property IV if LTD$(Y|X)$ and LTD$(X|Y)$.

Property V - Y is said to be positively regression dependent on X (denoted PRD$(Y|X)$) if $\Pr[Y \leqslant y | X = x] = F_{Y|X}\left(y|x\right)$ is non-increasing in x for all y. X and Y satisfy property V if PRD$(Y|X)$ and PRD$(X|Y)$.

Property VI - Y is said to be Inverse Hazard Rate Decreasing in X (denoted IHRD$(Y|X)$) if $\frac{F_{Y|X}(y|x)}{f_{Y|X}(y|x)}$ is non-increasing in x for all y, where $f_{Y|X}\left(y|x\right)$

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[5]Most of the concepts can be properly generalized to multivariate distributions. See, e.g., Lehmann (1966) and Esary, Proschan and Walkup (1967). The hypothesis of strictly positive density function is made only for simplicity.

is the p.d.f. of Y conditional to X. X and Y satisfy property VI if IHRD$(Y|X)$ and IHRD$(X|Y)$.

Property VII (Affiliation) - Y and X are said to be affiliated (or that they satisfy property (VII)) if $\frac{f_{Y|X}\left(y'|x\right)}{f_{Y|X}(y|x)}$ is non-increasing in x for all y and y' with $y > y'$.[6,7]

We have the following:

Theorem 1. *The above properties are successively more restrictive and are all different, that is,*

$$(VII) \Rightarrow (VI) \Rightarrow (V) \Rightarrow (IV) \Rightarrow (III) \Rightarrow (II) \Rightarrow (I)$$

and all implications are strict.

This theorem illustrates how strong affiliation is.[8] Some implications of Theorem 1 are trivial and most of them were previously established. Our contribution regards Property VI, that we use later to prove convenient generalizations of equilibrium existence and revenue rank results. We prove that Property VI is strictly weaker than affiliation and that it is sufficient for, but not equivalent to Property V.

Although Theorem 1 says that affiliation is mathematically restrictive, it could be the case that affiliation is satisfied in most of the cases with positive correlation. That is, although there are counterexamples for each of the implications above, such counterexamples could be pathologies and affiliation could be true in many cases where positive correlation (property I) holds. Thus, one should evaluate how typical it is.

Obviously, the best tests for this are the empirical and experimental ones (see section 6). Nevertheless, a theoretical evaluation would be useful. Unfortunately, this is not easy to do, since the objects (distributions) are of infinite dimension. As is well known, there are no natural measures (as the Lebesgue measure) for infinite dimension sets. To overcome this problem, we present a methodology to make such evaluation in the next subsection.

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[6] In statistic literature, affiliation is known as positively likelihood ratio dependent (PLRD). The reason for this name becomes clear from this form of the definition.

[7] Although Property VII seems asymmetric, it is indeed symmetric. To see this and that Property (VII) is equivalent to the previous definition of affiliation, observe that Property (VII) holds if $x \geqslant x'$, $y \geqslant y'$ (in the support of the distribution) imply

$$\frac{f_{Y|X}\left(y'|x\right)}{f_{Y|X}\left(y|x\right)} \leqslant \frac{f_{Y|X}\left(y'|x'\right)}{f_{Y|X}\left(y|x'\right)} \Leftrightarrow f_{Y|X}\left(y'|x\right) f_{Y|X}\left(y|x'\right) \leqslant f_{Y|X}\left(y'|x'\right) f_{Y|X}\left(y|x\right).$$

Multiplying both sides on the right by $f_X\left(x\right) f_X\left(x'\right)$ we obtain the affiliation inequality and dividing by $f_Y\left(y\right) f_Y\left(y'\right)$, we obtain the symmetrical condition for property (VII). Due to the fact that Property VII is equivalent to the monotonicity of $\frac{f_{Y|X}\left(y|x'\right)}{f_{Y|X}(y|x)}$, it is also known as Monotone Likelihood Ratio Property (MLRP).

[8] We defined only seven concepts for simplicity. Yanagimoto (1972) defines more than thirty concepts of positive dependence and, again, affiliation is the most restrictive of all, but one.

2.2 How often are positive dependent variables affiliated?

Consider the density function $f(t_1, t_2)$, with support $[\underline{t}, \overline{t}] \times [\underline{t}, \overline{t}]$. Without loss of generality (by reparametrization), we can assume $[\underline{t}, \overline{t}]^2 = [0, 1]^2$. Let $\mathcal{D}$ be the set of density functions on $[0, 1]^2$. For $n \geq 2$, define the transformation $T^n : \mathcal{D} \to \mathcal{D}$ given by

$$T^n(f)(x, y) = n^2 \int_{\frac{p-1}{n}}^{\frac{p}{n}} \int_{\frac{m-1}{n}}^{\frac{m}{n}} f(\alpha, \beta) \, d\alpha d\beta,$$

whenever $(x, y) \in \left(\frac{m-1}{n}, \frac{m}{n} \right] \times \left(\frac{p-1}{n}, \frac{p}{n} \right]$, for $m, p \in \{1, 2, ..., n\}$. Observe that $T^n(f)$ is constant over each square $\left(\frac{m-1}{n}, \frac{m}{n} \right] \times \left(\frac{p-1}{n}, \frac{p}{n} \right]$. Let us define the set $\mathcal{D}^n$ of simple distributions obtained as above, that is, $\mathcal{D}^n \equiv T^n(\mathcal{D})$.

Thus, a density function $f \in \mathcal{D}^n$ can be described by a matrix $A = (a_{ij})_{n \times n}$, as follows:

$$f(x, y) = a_{mp} \text{ if } (x, y) \in \left(\frac{m-1}{n}, \frac{m}{n} \right] \times \left(\frac{p-1}{n}, \frac{p}{n} \right], \tag{1}$$

for $m, p \in \{1, 2, ..., n\}$. The definition of f at the zero measure set of points $\{(x, y) = \left(\frac{m}{n}, \frac{p}{n} \right) : m = 0 \text{ or } p = 0\}$ is arbitrary.

We will assume full support and symmetry, that is, we will consider the set $\mathcal{S} \subset \mathcal{D}$ of density functions which are bounded away from zero and such that $f(x, y) = f(y, x)$. (Symmetry is not important for the results of this section , but it is important for sections 3 and **??**). The following lemma is immediate:

Lemma 1 *If $f \in \mathcal{S}$, then $T^n(f)$ can be represented by a matrix $A = (a_{ij})_{n \times n}$, as in (1), such that, for all $m, p \in \{1, 2, ..., n\}$, $a_{mp} > 0$ and $a_{mp} = a_{pm}$.*

Our method is to study the properties of the infinite dimensional set $\mathcal{S}$ by the properties of the finite dimensional sets $\mathcal{S}^n \equiv T^n(\mathcal{S})$, for each $n \in \mathbb{N}$. The following result provides the justification of the method:

Proposition 2 *$f \in \mathcal{S}$ satisfies Property VII if and only if $f^n = T^n(f)$ satisfies Property VII, for all n.*

For each $f \in \mathcal{S}$ that is affiliated, there is a $f^n = T^n(f) \in \mathcal{S}^n$ that is also affiliated. The converse does not hold: there are functions $f \in \mathcal{S}$ that are not affiliated, but such that $f^n = T^n(f) \in \mathcal{S}^n$ is affiliated for some n. Thus, the "proportion" of functions in $\mathcal{S}$ that are affiliated are overestimated by the proportion of the functions in $\mathcal{S}^n$ that are affiliated. (Here, we used quotation marks because "proportion" cannot be properly defined in $\mathcal{S}$, as we said before.)

Since $\mathcal{S}^n$ is finite dimensional, we can use the Lebesgue measure to evaluate the proportion of densities that are affiliated. This is done through a numerical experiment.

Numerical experiment
For $n = 3$, 4, 5 and 6, we repeated 10^8 trials of the following procedure:[9]

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[9] The results are already stable for 10^7 trials and even less.

1. Generate $N = \frac{n(n+1)}{2}$ random numbers $X_1, X_2, ..., X_N$ from a uniformly distribution on $[0, 1]$.

2. Define a symmetric matrix $(x_{ij})_{n \times n}$ with these numbers (see illustration below for the case $n = 3, N = 6$)

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} X_1 & X_4 & X_6 \\ X_4 & X_2 & X_5 \\ X_6 & X_5 & X_3 \end{bmatrix}$$

3. Define the symmetric matrix $A = (a_{ij})_{n \times n}$ by

$$a_{ij} = n^2 \frac{x_{ij}}{\sum_j \sum_i x_{ij}},$$

so that $\sum_j \sum_i a_{ij} = n^2$.

It is easy to see that each matrix $A = (a_{ij})_{n \times n}$ corresponds to a distribution $T^n(f) \in \mathcal{S}^n$ as in (1). For instance, if all X_k are equal, A is the matrix formed only by ones, which corresponds to the uniform distribution in $[0, 1]^2$.

Then, we calculate the correlation implied by $T^n(f)$ and verify whether it satisfies Property VII (affiliation). In fact, we checked whether it satisfies Properties IV, V, VI and VII.[10] For each n, we counted the number of cases in each interval of correlations and obtained the proportion of trials that satisfy such properties. The results are reported in the graphics below, for $n = 3$, 4 and 5, and are available upon request for all n up to 10.

For each n, we generated at least three times 10^8 different matrices (distributions). The results become stable (they reproduce with great precision), for $n \leq 7$ and correlation lower than 0.7 with just 10^7 trials.[11]

For the case $n = 3$, Figure 2 shows the proportion of distributions that satisfy properties IV, V, VI and VII (affiliation), for a range of positive correlation ρ, from 0 to 0.8 (correlations above 0.8 are very rare in our simulations). For instance, about 12% of the cases with correlation 0.4 satisfied affiliation.

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[10] Property I occurs in 50% of the cases. We did not find a good characterization of Properties II and III to include them in the tests.

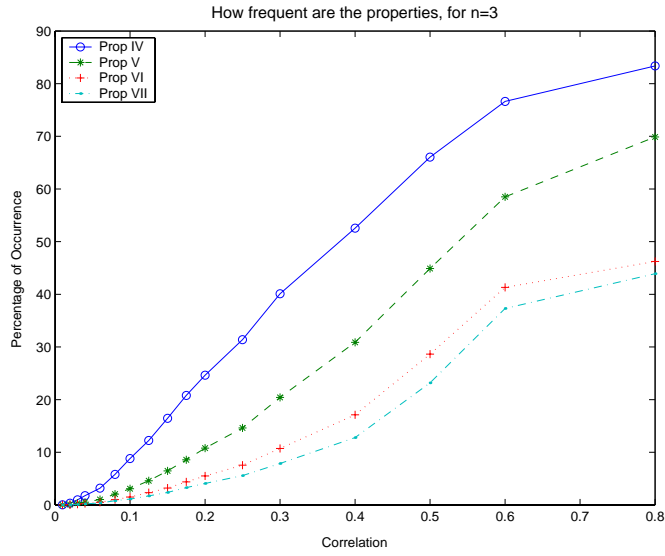[11] Kotz, Wang and Hung (1988) made a similar study, for $n = 3$.

Figure 2: Percentage of occurrence of properties IV, V, VI and VII (affiliation), when n=3.

If we increase n to 4, the proportion of distributions that satisfy each property falls sharply, especially for properties V, VI and VII, as Figure 3 shows. One sees that Property VII is satisfied for less than 2% of the cases, even for a correlation as high as 0.8.
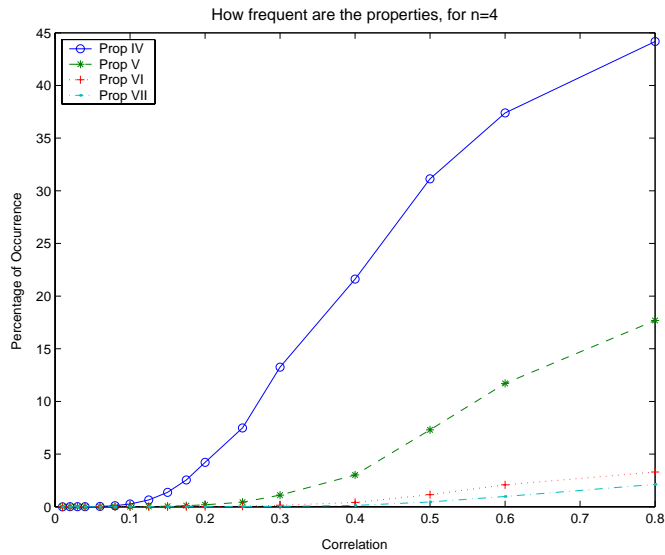

Figure 3: Percentage of occurrence of properties IV, V, VI and VII (affiliation), when n=4.

In the case $n = 5$, we did not obtain a significant proportion of cases of distributions satisfying properties VI and VII in 10^8 trials. See Figure 4.
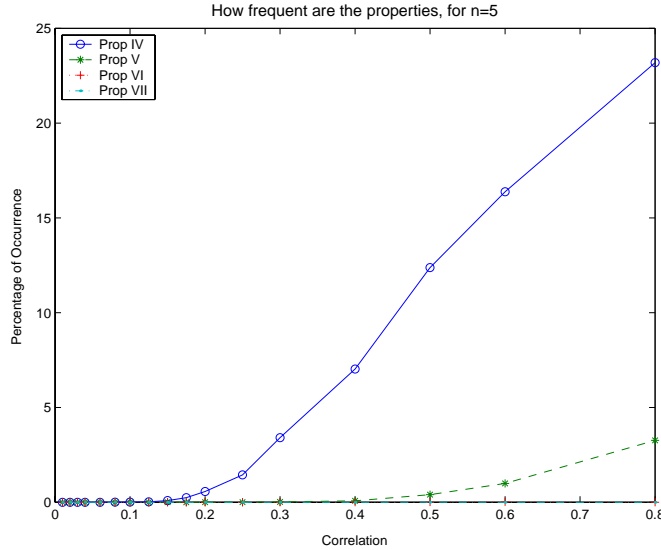

Figure 4: Percentage of occurrence of properties IV, V, VI and VII (affiliation) when n=5.

From the graphics above, we see that distributions rarely satisfy affiliation (property VII), even for a large correlation coefficient. The reason for that comes from the fact that affiliation requires an inequality to be satisfied for each quadruple of coordinates. If we consider the general definition of affiliation given in the appendix, we see that affiliation requires an inequality to be satisfied conditioned on *every* sublattice. This is a rather strong property to require and it is the source of its scarcity.

These findings seem to contradict the common understanding that affiliation is reasonable. For instance, models with conditional independence are considered very natural and it is usually supposed that they satisfy affiliation. We analyze them in the next subsection.

2.3 Conditional independence

Conditional independence models assume that the signals of the bidders are independent conditional to the value of the object (see Wilson 1969, 1977). Assume that $f(t_1, ..., t_N | v)$ is the p.d.f. of the signals conditional to the value and that it is C^2. It can be proven that the signals are affiliated if and only if

$$\frac{\partial^2 \log f(t_1, ..., t_N | v)}{\partial t_i \partial t_j} \geqslant 0,$$

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and

$$\frac{\partial^2 \log f\left(t_1, ..., t_N | v\right)}{\partial t_i \partial v} \geqslant 0, \tag{2}$$

for all i, j.[12] Conditional independence implies only that

$$\frac{\partial^2 \log f\left(t_1, ..., t_N | v\right)}{\partial t_i \partial t_j} = 0.$$

Thus, conditional independence is not sufficient for affiliation. To obtain the latter, one needs to assume (2) or that t_i and v are affiliated. In other words, to obtain affiliation from conditional independence, one has to assume affiliation itself. So, the justification of affiliation through conditional independence is meaningless.

This may bring to mind an usual method of obtaining affiliated signals: to assume that the signals t_i are a common value plus an individual error, that is, $t_i = z + \varepsilon_i$, where the ε_i are independent and identically distributed. This is not yet sufficient for the affiliation of t_1, ..., t_N. Indeed, let g be the p.d.f. of the ε_i, $i = 1$, ..., N. Then, t_1, ..., t_N are affiliated if and only if g is a strongly unimodal function.[13,14]

2.4   Affiliation in other sciences

The above discussion suggests that affiliation is, indeed, a narrow condition and probably not a good description of the world. Nevertheless, we know that affiliation is widely used in Statistics, reliability theory and in many areas of social sciences and economics (possibly under other names). Why is this so, if affiliation is restrictive?

In Statistics, affiliation is known as Positive Likelihood Ratio Dependence (PLRD), the name given by Lehmann (1966) when he introduced the concept. PLRD is widely known by statisticians to be a strong property and many papers in the field do use weaker concepts (such as given by properties V, IV or III).

In Reliability Theory, affiliation is generally referred to as Total Positivity of order two (TP$_2$), after Karlin (1968). Historical notes in Barlow and Proschan (1965) suggest why TP$_2$ is convenient for the theory. It is generally assumed that the failure rates of components or systems follow specific probabilistic distributions and such special distributions usually have the TP$_2$ property, so, it is natural to study its consequences. In contrast, in auction theory, the signals represent information gathered by the bidders and there is no reason for assuming that they have a specific distribution. Indeed, this is rarely assumed. Thus,

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[12] See Topkis (1978), p. 310.

[13] The term is borrowed from Lehmann (1959). A function is strongly unimodal if $\log g$ is concave. A proof of the affirmation can be found in Lehmann (1959), p. 509, or obtained directly from the previous discussion.

[14] Even if g is strongly unimodal, so that $t_1, ..., t_N$ are affiliated, it is not true in general that t_1, ..., t_N, ε_1, ..., ε_N, z are affiliated.

the reason for the use of TP_2 (or affiliation) in reliability theory does not apply to auction theory.

Finally, we stress that the kind of results of previous subsections are insufficient to regard a hypothesis as not useful or inadequate. This judgement has to be made in the context of the other assumptions of the theory. For instance, it is possible that the hypothesis is not so restrictive *given* the setting where it is assumed. Moreover, the judgment must take into account the most important of all criteria: whether the resulting theory "yields sufficiently accurate predictions"(Friedman (1953), p. 14). This paper addresses only the use of affiliation in auction theory, as we emphasize in the subsequent sections. It is a task for the specialists in other fields to analyze whether this assumption is appropriate for their applications.

3   Equilibrium existence

Before we present our main equilibrium existence result, we call the attention to the fact that the same proof from Milgrom and Weber (1982a) can be used to prove equilibrium existence for Property VI. Indeed, the following property is sufficient:

Property VI$'$ - The joint distribution of X and Y satisfy property VI$'$ if for all x, x' and y in the support of the distribution, $x \geq y \geq x'$ imply

$$\frac{F_{Y|X}(y|x')}{f_{Y|X}(y|x')} \geq \frac{F_{Y|X}(y|y)}{f_{Y|X}(y|y)} \geq \frac{F_{Y|X}(y|x)}{f_{Y|X}(y|x)}$$

and analogously, if we exchange the roles of X and Y.

It is easy to see that Property VI implies Property VI$'$. Unfortunately, however, it is impossible to generalize further the existence of equilibrium for the properties defined in subsection 2.2. Indeed, in the appendix, we give a example of a distribution in $\mathcal{S}^3$ which satisfies Property V, but does not have equilibrium. These facts are summarized in the following:

Theorem 3 *Consider a symmetric first price, private value auction between 2 bidders. Suppose also that bidders are risk-neutrals and there is a joint symmetric p.d.f., $f : \left[\underline{t}, \overline{t}\right]^2 \to \mathbb{R}_+$. If f satisfies property VI$'$, there is a symmetric pure strategy monotonic equilibrium. Moreover, property V is not sufficient for equilibrium existence.*

The most important message of Theorem 3 is the negative one: that it is impossible to generalize the equilibrium existence for the other still restrictive definitions of positive dependence. This mainly negative result leads us to consider another route to prove equilibrium existence.

We are interested in the density functions in the finite dimensional sets $\mathcal{S}^n = T^n(\mathcal{S})$ (see subsection 2.2 for the definition of T^n and $\mathcal{S}$). These cases

are interesting because: (1) they are natural approximations for all density functions; (2) they embrace all real world auctions and applications, where values and bids are never continuous (they are discretized at some level — units of dollars or cents, for instance); (3) they allow for the use of the convenient formulas and expressions of the continuous variables, as we show below; and, more important, (4) the sets involved are finite dimensional, which has a well defined "natural" measure on them (Lebesgue measure), as we explained in subsection 2.2.

In some sense, we are also using a discretization to obtain finite dimensional sets (this is done via T^n) but our method has the advantage of making the density functions simple functions, while maintaining the underlying variables continuous. This is what allows the use of the theory of differentiable symmetric pure strategy equilibrium. Indeed, consider the following well known lemma:

Lemma 4 *If $f \in \mathcal{S}$ has an increasing and differentiable symmetric pure strategy equilibrium, it is given by*

$$b(x) = x - \int_0^x \exp\left[-\int_u^x \frac{f(s|s)}{F(s|s)} ds\right] du. \tag{3}$$

Observe that even a discontinuous density function f can have a differentiable equilibrium. Thus, we can consider differentiable equilibria for functions $f \in \mathcal{S}^n$. This is the basic point of our method. The following result, which is proven in the supplement to this paper, shows that our method leads to the equilibrium in the limit (continuous case):

Proposition 5 *Let $f \in \mathcal{S}$ be continuous. If $T^n(f)$ has a differentiable symmetric pure strategy equilibrium for all $n \geq n_0$, then f also has, and it is the limit of the equilibria of $T^n(f)$ as n goes to infinity.*

With these preliminaries out of the way, we can consider the cases $f \in \mathcal{S}^n$. Let $b(\cdot)$ be given by (3). In the supplement, we prove that $b(\cdot)$ is always increasing. Let $\Pi(y, b(x)) = (y - b(x)) F(x|y)$ be the interim payoff of a player with type y who bids as type x, when the opponent follows $b(\cdot)$. Let $\Delta(x, y)$ represent $\Pi(y, b(x)) - \Pi(y, b(y))$. It is easy to see that $b(\cdot)$ is equilibrium if and only if $\Delta(x, y) \leq 0$ for all x and $y \in [0, 1]^2$. Thus, the content of the next theorem is that it is possible to prove equilibrium existence by checking this condition only for a finite set of points:

Theorem 6 *Let $f \in \mathcal{S}^n$. Then, f has a symmetric differentiable pure strategy equilibrium if and only if $\Delta(x, y) \leq 0$ for all $(x, y) \in P$, where $P \subset [0, 1]^2$ is a given finite set.*

In the supplement to this paper, we completely characterize the set P. It is useful to say that the theorem is not trivial, since $\Delta(x, y)$ is not monotonic in the squares $\left(\frac{m-1}{n}, \frac{m}{n}\right] \times \left(\frac{p-1}{n}, \frac{p}{n}\right]$. Indeed, the main part of the proof is the analysis of the non-monotonic function $\Delta(x, y)$ in the sets $\left(\frac{m-1}{n}, \frac{m}{n}\right] \times \left(\frac{p-1}{n}, \frac{p}{n}\right]$

and the determination of its maxima for each of these sets. Indeed, for each square we need to check at least one and at most 5 points. Thus, the set P has less than $5n^2$ points.[15]

Using Theorem 6, we can easily classify whether there is equilibrium or not, and, through numerical simulations, obtain the proportion of cases with pure strategy equilibrium.

Numerical experiment
In addition to the tests described in subsection 2.2, for each trial $f \in \mathcal{S}^n$, we test whether the auction with players with such distribution has a pure strategy equilibrium. This is shown in the Table 1 below.

For each n, 100%=distributions with equilibrium.

Distribution satisfying	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
Prop. IV	44.9%	21.2%	$5,5\%$	$0,93\%$	$0,0\%$
Prop. V	30.5%	9.7%	$1,0\%$	$0,026\%$	$0,0\%$
Prop. VI	21.3%	2.4%	$0,1\%$	$0,0\%$	$0,0\%$
Affiliation	16.8%	0.9%	$0,011\%$	$0,0\%$	$0,0\%$

Table 1 - Proportion of $f \in \mathcal{S}^n$ with pure strategy equilibrium

that satisfy properties IV-VII.

Table 1 shows that the available equilibrium results are too restrictive (this includes Theorem 3 above).[16]

4   The Revenue Ranking of Auctions

In this section we derive an expression for the difference in revenue from second and first price symmetric auctions. Indeed, we are interested in answering the question of whether the result on the rank of the auctions also holds for a concept weaker than affiliation.

We have the following:

Theorem 7 *Consider the auction of an indivisible object with 2 risk neutral bidders and with private values. Let $f(x)$ be a symmetric probability density function. If f satisfies Property VI′, then the second price auction gives greater revenue than the first price auction. Specifically, the revenue difference is given by*

$$\int_0^1 \int_0^x b'(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx$$

---

[15]Thus, the number of restrictions increase with n. As a consequence — not obvious, but true in the simulations —, we see that the proportion of distributions with pure strategy equilibrium falls when n increases.

[16]For $n \geq 7$, all the properties represent nearly 0% of the cases where there exists pure strategy equilibrium.

where $b\left(\cdot\right)$ is the first price equilibrium bidding function, or by

$$\int_0^1 \int_0^x \left[\int_0^y L\left(\alpha|y\right) d\alpha\right] \cdot \left[1 - \frac{F\left(y|x\right)}{f\left(y|x\right)} \cdot \frac{f\left(y|y\right)}{F\left(y|y\right)}\right] \cdot f\left(y|x\right) dy \cdot f\left(x\right) dx,$$

where $L\left(\alpha|t\right) = \exp\left[-\int_\alpha^t \frac{f(s|s)}{F(s|s)} ds\right]$. Moreover, Property V is not sufficient for this revenue rank.
Proof. *See the appendix.* ∎

From Theorem 7, we learn that the revenue superiority of the English (second price) auction over the first price auction seems to be strongly dependent on the condition required for Property VI′. More than that, the revenue rank is not valid even for such a strong positive dependence concept as Property V.

The counterexample mentioned in the proof of Theorem 3 is obtained for $f \in \mathcal{S}^3$, which is a relatively simple set. (It is not possible to provide the counterexample in $\mathcal{S}^2$ because the properties are equivalent in this set — see Esary et. al. 1967, condition 4.8 and its discussion).

In the supplement of this paper, we develop the expression of the revenue differences from the second price auction to the first price auction for $f \in \mathcal{S}^n$. We can make simulations, generating distributions as we have done in subsection 2.2 and section 3, to evaluate the revenue difference percentage, given by:

$$r_d = \frac{R_2 - R_1}{R_2} \cdot 100\%,$$

where R_2 is the expected revenue of the second price auction and R_1 is the expected revenue of the first price auction. That is, we carried out the following:

Numerical experiments
In each of the previously described trials (see subsection 2.2), we tested not only whether the first price auction has pure strategy equilibrium (see section 3), but we also obtained the revenue differences, for the cases where the symmetric pure strategy equilibrium exists.

We collected 10 sets of data, each with 10^7 trials. The average of r_d for the cases with positive correlation (and PSE) is given in Table 2 below:

Sets	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
Set 1	$8,25\%$	$11,04\%$	$10,38\%$	$8,89\%$	$-0,32\%$
Set 2	$8,24\%$	$11,06\%$	$10,42\%$	$9,15\%$	$24,29\%$
Set 3	$8,24\%$	$11,01\%$	$10,38\%$	$9,47\%$	$14,00\%$
Set 4	$8,25\%$	$11,03\%$	$10,45\%$	$9,29\%$	$6,62\%$
Set 5	$8,25\%$	$11,01\%$	$10,47\%$	$8,83\%$	$17,93\%$
Set 6	$8,24\%$	$11,00\%$	$10,48\%$	$9,71\%$	$20,53\%$
Set 7	$8,24\%$	$11,03\%$	$10,43\%$	$9,47\%$	$21,43\%$
Set 8	$8,24\%$	$11,04\%$	$10,42\%$	$9,36\%$	$19,78\%$
Set 9	$8,25\%$	$11,02\%$	$10,53\%$	$9,43\%$	$4,13\%$
Set 10	$8,23\%$	$10,98\%$	$10,65\%$	$9,81\%$	$16,83\%$
Average	$8,24\%$	$11,02\%$	$10,46\%$	$9,34\%$	$14,525\%$

Table 2 - Average of the relative revenue differences
for positive correlated distributions with PSE.

If we consider all distributions with PSE (and not only those with positive correlation), we obtain the following:

Sets	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
Set 1	$7,58\%$	$10,20\%$	$10,05\%$	$8,66\%$	$-0,32\%$
Set 2	$7,58\%$	$10,20\%$	$10,02\%$	$8,92\%$	$24,29\%$
Set 3	$7,58\%$	$10,15\%$	$10,04\%$	$9,40\%$	$14,00\%$
Set 4	$7,58\%$	$10,17\%$	$10,10\%$	$9,17\%$	$6,62\%$
Set 5	$7,59\%$	$10,18\%$	$10,10\%$	$8,81\%$	$17,93\%$
Set 6	$7,58\%$	$10,14\%$	$10,14\%$	$9,71\%$	$20,53\%$
Set 7	$7,58\%$	$10,15\%$	$10,11\%$	$9,47\%$	$21,43\%$
Set 8	$7,58\%$	$10,17\%$	$10,05\%$	$9,24\%$	$19,78\%$
Set 9	$7,58\%$	$10,17\%$	$10,16\%$	$9,19\%$	$4,13\%$
Set 10	$7,57\%$	$10,12\%$	$10,16\%$	$9,19\%$	$16,83\%$
Average	$7,58\%$	$10,17\%$	$10,09\%$	$9,17\%$	$14,52\%$

Table 3 - Average of the relative revenue differences
for all distributions with PSE.

We report below the variance of the distribution of r_d for all distributions with PSE.

Sets	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
Set 1	$6,52\%$	$10,03\%$	$11,47\%$	$11,75\%$	$9,70\%$
Set 2	$6,52\%$	$10,04\%$	$11,40\%$	$12,14\%$	$7,64\%$
Set 3	$6,52\%$	$10,02\%$	$11,52\%$	$11,49\%$	$8,79\%$
Set 4	$6,52\%$	$10,02\%$	$11,50\%$	$12,07\%$	$8,91\%$
Set 5	$6,51\%$	$10,02\%$	$11,47\%$	$11,51\%$	$8,90\%$
Set 6	$6,52\%$	$10,08\%$	$11,39\%$	$11,95\%$	$5,83\%$
Set 7	$6,52\%$	$10,07\%$	$11,35\%$	$12,15\%$	$6,93\%$
Set 8	$6,52\%$	$10,07\%$	$11,40\%$	$11,76\%$	$7,66\%$
Set 9	$6,52\%$	$10,02\%$	$11,43\%$	$11,45\%$	$12,58\%$
Set 10	$6,51\%$	$10,05\%$	$11,43\%$	$11,77\%$	$9,81\%$
Average	$6,52\%$	$10,04\%$	$11,44\%$	$11,80\%$	$8,67\%$

Table 4 - Variance of the relative revenue differences

for all distributions with PSE.

In what follows, we will treat the numerical simulations as giving an "experimental distribution" of r_d. No confusion should arise between the "experimental distribution" of r_d and the distributions generated by each $f \in \mathcal{S}^n$.

We first analyze the cases with positive correlation. The results for $n = 3$, 4 and 5and all distributions are shown in the following figures.
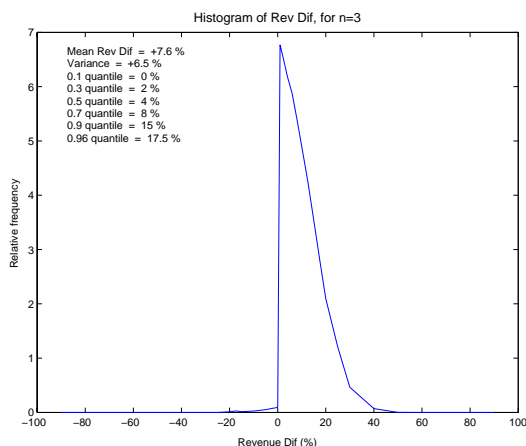

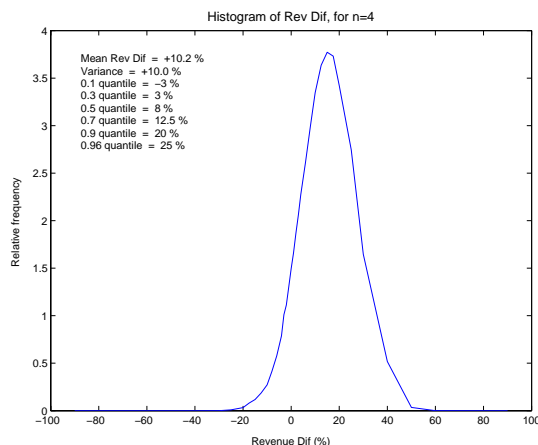Figure 5: histogram for r_d, $n = 3$ - all distributions with PSE.

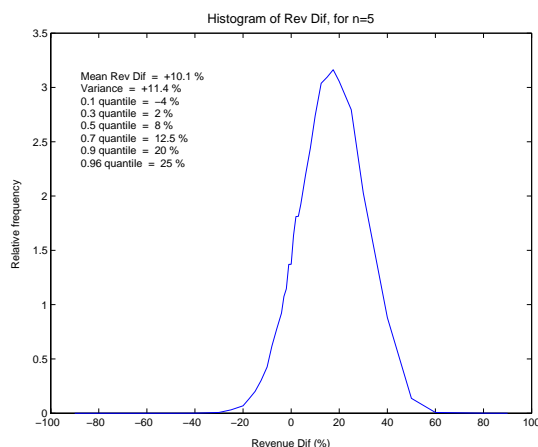Figure 6: histogram for r_d, $n = 4$ - all distributions with PSE.

Figure 7: histogram for r_d, $n = 5$ - all distributions with PSE.

5   Related literature and conclusion

A few papers pointed out restrictions or limitations to the implications of affiliation. Perry and Reny (1999) presented an example of a multi-unit auction where the linkage principle fails and the revenue ranking is reversed, even under affiliation. Thus, their criticism seems to be restricted to the generalization of the consequences of affiliation to multi-unit auctions. In contrast, we considered single-unit auctions and relaxed the requirement of affiliation of the distribution.

Klemperer (2003) argues that, in real auctions, affiliation is not as important as asymmetry and collusion. He illustrates his arguments with examples of the 3G auctions conducted in Europe in 2000-2001.

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Nevertheless, much more was written in accordance with the conclusions of affiliation. McMillan (1994, p.152) says that the auction theorists working as consultants to the FCC in spectrum auctions, advocated the use of an open auction using the linkage principle as an argument: "Theory says, then, that the government can increase its revenue by publicizing any available information that affects the licence's assessed value". The disadvantages of the open format in the presence of risk aversion and collusion were judged "to be outweighed by the bidders' ability to learn from other bids in the auction" (p. 152). Milgrom (1989, p. 13) emphasizes affiliation as the explanation of the predominance of English auction over the first price auction.

This paper presents evidence that affiliation is a restrictive condition. After developing an approach to test the existence of symmetric pure strategy equilibrium (PSE) for simple density functions, we are able to verify that many cases with PSE do not satisfy affiliation.[17] Also, the superiority of the English auction is not maintained even for distributions satisfying strong requirements of positive dependence.

Nevertheless, we show that the original insights of Milgrom and Weber (1982a) are surprisingly true, in a weak sense, for a much bigger set of cases. Indeed, we find evidence that positive dependence helps to ensure equilibrium existence and the revenue superiority of the English auction is true, on average, for our set of simple density functions.

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