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A characterization of a new value and an existing value for cooperative games with levels structure of cooperation

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Abstract: We present parallel characterizations of two different values in the framework of restricted cooperation games. The restrictions are introduced as a finite sequence of partitions defined on the player set, each of them being coarser than the previous one, hence forming a structure of different levels of a priori unions. On the one hand, we consider a value first introduced by [18], which extends the Shapley value to games with different levels of a priori unions. On the other hand, we introduce another solution for the same type of games, which extends the Banzhaf value in the same manner. We characterize these two values in terms of easily comparable sets of properties and we illustrate them by means of an example.

Keywords: Shapley value, Banzhaf value, levels structure of cooperation

JEL Classification: C71

Resum: En aquest treball presentem dues caracteritzacions de dos valors diferents en el marc dels jocs coalicionals amb cooperació restringida. Les restriccions són introduïdes com una seqüència finita de particions del conjunt de jugadors, de manera que cada una d'elles és més grollera que l'anterior, formant així una estructura amb diferents nivells d'unions a priori. Per una banda, considerem el valor introduït per [18], que extèn el valor de Shapley a jocs amb diferents nivells d'unions a priori. D'altra banda, introduïm una altra solució, que extèn el valor de Banzhaf de manera similar. Caracterizem els dos valors anteriors en termes de conjunts de propietats fàcilment comparables lògicament i els ilustrem a partir d'un exemple.

1 Introduction

Transferable utility cooperative games (just games from now on) are used to describe situations in which agents cooperate to obtain some gains, e.g. building a road to connect a number of towns or reaching an agreement to pass a bill. These gains are assumed to be divisible and transferable among players without any loss. The problem of allocating the gains that the cooperation generates among the players is one of the main topics tackled in the literature. Therefore, assessing the strategic position of each player in a given game is crucial in order to find a share-out that respects to some extent the power of each player. The Shapley value [15] is the best known concept in this respect, together with the Banzhaf value [6, 12].

In the original model where both the Shapley value and the Banzhaf value are typically used there is no restriction to the cooperation, and the game is defined by the worth that any coalition can obtain by its own. However, there are many real situations in which there is a priori information about the behavior of the players or there are environmental restrictions and only partial cooperation occurs. Different approaches have been used to address this type of situations and different models of games with restricted cooperation have been studied. In particular, players may form coalitions and these coalitions may bargain for the division of the worth of the grand coalition. [5] suppose that the restrictions to the cooperation are given by a partition of the set of players. The model with both a game and a partition of the set of players is called a game with a priori unions. For these games, [13] proposes and characterizes the Owen value, an extension of the Shapley value [15] to allocate the gains generated by the grand coalition. Following a similar procedure, in a subsequent paper [14] defines an extension of the Banzhaf value [12] known as the Banzhaf-Owen value. The first characterization of this solution concept is presented in [4]. [2] give parallel characterizations of the two aforementioned values which eases the comparison between them.

[18] takes one step beyond by introducing games with levels structure of cooperation, which extends the model of games with a priori unions. He proposes and characterizes an extension of the Owen value for this kind of situations, which we will call the Shapley levels value. As before, players are assumed to be organized in unions as pressure groups for the division of the worth available (first level of cooperation). Nevertheless, this time the formed unions may again organize themselves in larger groups (second level of cooperation) while they maintain their internal obligations of the first level, and so on and so forth. Hence, this time the restrictions to the cooperation are described by a sequence of partitions of the player set, each of them

being coarser than the previous ones. [7] give an alternative characterization of the Shapley levels value using a balanced contributions property and [17] implement the Shapley levels value in a subgame perfect equilibrium of a particular bidding mechanism.

In the present paper, we first propose an extension of the Banzhaf-Owen value for games with levels structure of cooperation, which we call the Banzhaf levels value. Next, we provide two parallel characterizations of both the Shapley levels value and the Banzhaf levels value which reveals the differences between both solution concepts. On the one hand, the Shapley levels value is characterized using the level game property and the level balanced contributions property. The level game property states that in each level, the sum of the payoffs of the players of any union equals the payoff of the same union when considering it as a player in the corresponding level game. The level balanced contributions is a reciprocity property that asserts that the change on the payoff of a player caused by the isolation of another player of her same union of the first level remains invariant if we permute both players. A similar property is used in [7], but in this case the player set is assumed to change. On the other hand, the characterization of the Banzhaf levels value is based on the singleton level game property and the level neutrality under individual desertion property. The singleton level game property is the restriction of the level game property to singleton unions of a given level, whereas the level neutrality under individual desertion property states that the payoff of a player is not affected by the isolation of another player in her same union of the first level. Hence, level neutrality under individual desertion implies level balanced contributions and level game property implies singleton level game property and thus the main properties used to characterize each of the two values are logically related, which eases the comparison between the two values.

There is a variety of reasons to seek for parallel axiomatizations of two different values using a minimal set of logically related properties. In the first place, from a mathematically point of view, characterizing a value using a few independent properties may be more appealing than just giving an explicit formula or procedure to calculate it. In the second place, deciding on whether to use a value or not can be made more easily using a set of properties instead of a formula. Lastly, depending on the framework, one set of properties or another shall fit better, and hence one value or the other shall be used.

The rest of the paper is organized as follows. Section 2 is mainly devoted to present the model of games with levels structure of cooperation, and in particular the Shapley levels value introduced by [18]. In Section 3 we define the Banzhaf levels value. In Section 4 we introduce and explain some properties that a value for games with levels structure of cooperation might satisfy, and we provide a characterization for each of the two aforementioned values. Section 5 concludes with an example.

2 Preliminaries

An *n-person cooperative game with transferable utility (a game)* is a pair (N, v), where $N = \{1, \ldots, n\}$ is the finite set of players and v, the *characteristic function*, is a real valued function on $2^N = \{S | S \subseteq N\}$ with $v(\emptyset) = 0$. We denote by G^N the set of games with player set N. For each $S \subseteq N$ and $i \in N$ we will write $S \cup i$ instead $S \cup \{i\}$ and $S \setminus i$ instead $S \setminus \{i\}$.

Given $(N, v) \in G^N$, a player $i \in N$ is a *dummy* if $v(S \cup i) = v(S) + v(i)$ for all $S \subseteq N \setminus i$, that is, if all her marginal contributions are equal to $v(i)$. A player $i \in N$ is called a *null player* if she is a dummy and $v(i) = 0$. Two players $i, j \in N$ are *symmetric* if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, that is, if their marginal contributions to each coalition coincide.

A *value on G^N* is a map f that assigns to every game $(N, v) \in G^N$ a vector $f(N, v) \in \mathbb{R}^n$. The following definitions provide the explicit expressions of two well-known values in the literature.

Definition 2.1. [15] Given a game (N, v), the *Shapley value*, ϕ, is a vector in $\mathbb{R}^n$ where each coordinate is defined as follows:

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left[v(S \cup i) - v(S) \right], \text{ for every } i \in N,$$

where $s = |S|$ and $n = |N|$.[1]

Definition 2.2. [6, 12] Given a game (N, v), the *Banzhaf value*, ψ, is a vector in $\mathbb{R}^n$ where each coordinate is defined as follows:

$$\psi_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{1}{2^{n-1}} \left[v(S \cup i) - v(S) \right], \text{ for every } i \in N.$$

We denote by $\mathcal{P}(N)$ the set of all partitions of a finite set of players N, and for each $P \in \mathcal{P}(N)$ and each $S \subseteq N$, $P_{|S} \in \mathcal{P}(S)$ is the partition of S induced by P, i.e., $P_{|S} = \{U \cap S : U \in P\}$. A *levels structure of cooperation* is a pair $(N, \underline{B})$, where N is the set of players and $\underline{B} = \{B_0, \ldots, B_k\}$ is a

[1] We use the $|\cdot|$ operator to denote the cardinality of a finite set.

sequence of partitions of N such that $B_0 = \{\{i\} : i \in N\}$ and, for each $r \in \{0, \ldots, k-1\}$, B_{r+1} is coarser than B_r. That is to say, for each $r \in \{1, \ldots, k\}$ and each $S \in B_r$, there is $B \subseteq B_{r-1}$ such that $S = \cup_{U \in B} U$. Each $S \in B_r$ is called a *union* and B_r is called the *r-th level of* $\underline{\mathrm{B}}$. We denote by $\mathcal{L}(N)$ the set of all levels structures of cooperation over the set N. The following example illustrates the above definitions.

Example 2.3. Let $N = \{1, 2, 3, 4, 5, 6\}$ and $\underline{B} = \{B_0, B_1, B_2\}$ be a levels structure of cooperation over N with two levels, where

$$\begin{aligned} B_2 &= \{\{1,2,3\},\{4,5,6\}\}, \\ B_1 &= \{\{1,2\},\{3\},\{4\},\{5,6\}\}, \text{ and} \\ B_0 &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}. \end{aligned}$$

A *cooperative game with levels structure of cooperation* is a triple $(N, v, \underline{B})$, where $(N, v) \in G^N$ and $(N, \underline{B}) \in \mathcal{L}(N)$. We denote by GL^N the set of all cooperative games with levels structure of cooperation. Given $(N, \underline{B}) \in \mathcal{L}(N)$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $i \in N$, $(N, \underline{B^{-i}}) \in \mathcal{L}(N)$ is the levels structure of cooperation obtained from $(N, \underline{B})$ by isolating player i from the union she belongs to at each level, i.e., $\underline{B^{-i}} = \{B_0^{-i}, \ldots, B_k^{-i}\}$, where, for all $r \in \{0, \ldots, k\}$, $B_r^{-i} = \{U \in B_r : i \notin U\} \cup \{S \setminus i, \{i\}\}$ given that $i \in S \in B_r$. Note that $B_0^{-i} = B_0$. For each $r \in \{1, \ldots, k\}$ and each $U \in B_r$, $[U]$ denotes U considered as a player at level r, whereas $[B_r]$ denotes the set of players at level r, i.e, $[B_r] = \{[U] : U \in B_r\}$ and $([B_r], \underline{B_r}) \in GL^{[B_r]}$, where $\underline{B_r} = \{B_r, \ldots, B_k\}$. Given $(N, v, \underline{B}) \in GL^N$ with $\underline{B} = \{B_0, \ldots, B_k\}$, for each $r \in \{0, \ldots, k\}$ we define the *r-th level game* $([B_r], v^r, \underline{B_r}) \in GL^{[B_r]}$ as the game induced from (N, v) by considering the coalitions of B_r as players, i.e., $v^r([U]) = v(U)$ for each $r \in \{0, \ldots, k\}$ and each $U \in B_r$.

In the framework of games with levels structure of cooperation we assume that players are initially organized into the coalition structure B_k as pressure groups for the division of $v(N)$. Then, each union of the last level is divided again according to the coalition structure B_{k-1} as pressure group for the division of the amount that the unions of the last level have obtained, and so on and so forth until the last level, B_0, is reached.

A *value on* GL^N is a map f that assigns to every game with levels structure of cooperation $(N, v, \underline{B}) \in GL^N$ a vector $f(N, v, \underline{B}) \in \mathbb{R}^n$. We denote by $\Pi(N)$ the set of permutations of N, and given $\pi \in \Pi(N)$, $\pi\underline{B} = \{\pi B_0, \ldots, \pi B_k\}$, where $\pi B_k = \{\pi S_1, \ldots, \pi S_p\}$ given that $B_k = \{S_1, \ldots, S_p\}$, and $\pi v(S) = v(\pi^{-1} S)$. Consider the following properties that a value on GL^N may satisfy:

- A value f on GL^N satisfies *efficiency* if for all $(N, v, \underline{B}) \in GL^N$,

$$\sum_{i \in N} f_i(N, v, \underline{B}) = v(N).$$

- A value f on GL^N satisfies *additivity* if for all $(N, v, \underline{B}), (N, w, \underline{B}) \in GL^N$,

$$f(N, v + w, \underline{B}) = f(N, v, \underline{B}) + f(N, w, \underline{B}).$$

- A value f on GL^N satisfies *individual symmetry* if for all $(N, v, \underline{B}) \in GL^N$ and each $\pi \in \Pi(N)$,

$$\pi f(N, v, \underline{B}) = f(\pi N, \pi v, \pi \underline{B}).$$

- A value f on GL^N satisfies *coalitional symmetry* if for all $(N, v, \underline{B}) \in GL^N$ and each level $r \in \{1, \cdots, k\}$, if $[S], [U] \in [B_r]$ are symmetric players in the game $([B_r], v^r)$ and S, U are subsets of the same union in B_l for each $l > r$, then

$$\sum_{i \in S} f_i(N, v, \underline{B}) = \sum_{i \in U} f_i(N, v, \underline{B}).$$

- A value f on GL^N satisfies the *null player property* if for all $(N, v, \underline{B}) \in GL^N$ such that $i \in N$ is a null player for the game (N, v),

$$f_i(N, v, \underline{B}) = 0.$$

The above five properties are natural extensions of the properties used in [13] within the framework of GL^N.

Next, let the sets $\Omega(\underline{B}) = \Omega_1(\underline{B}) \subseteq \Omega_2(\underline{B}) \subseteq \cdots \subseteq \Omega_k(\underline{B}) \subseteq \Pi(N)$ defined as follows. First of all,

$$\Omega_k(\underline{B}) = \{\sigma \in \Pi(N) : \forall S \in B_k,\ \forall i, j \in S \in B_k \text{ and } l \in N, \\ \text{if } \sigma(i) < \sigma(l) < \sigma(j) \text{ then } l \in S\}.$$

Then, for $r \in \{k-1, \ldots, 1\}$ we recursively define

$$\Omega_r(\underline{B}) = \{\sigma \in \Omega_{r+1}(\underline{B}) : \forall i, j \in S \in B_r \text{ and } l \in N, \\ \text{if } \sigma(i) < \sigma(l) < \sigma(j) \text{ then } l \in S\}.$$

Hence, $\Omega_r(\underline{B})$ denotes the permutations of $\Omega_{r+1}(\underline{B})$ such that the elements of each union of B_r are consecutive. Let us see an example to illustrate the above definitions.

Example 2.4. For the levels structure of cooperation of Example 2.3, $|\Omega_2(\underline{B})| = 72$, $|\Omega_1(\underline{B})| = 36$, $(1,2,4,3,5,6) \notin \Omega_2(\underline{B})$, $(1,3,2,4,5,6) \in \Omega_2(\underline{B}) \setminus \Omega_1(\underline{B})$ and $(3,2,1,5,6,4) \in \Omega_1(\underline{B})$.

Next, we recall the definition of the already known solution concept for games with levels structure of cooperation.

Definition 2.5. Given a game with levels structure of cooperation $(N, v, \underline{B}) \in GL^N$, the *Shapley levels value* [18], Φ, is a vector in $\mathbb{R}^n$ where each coordinate is defined as follows:

$$\Phi_i(N, v, \underline{B}) = \frac{1}{|\Omega(\underline{B})|} \sum_{\sigma \in \Omega(\underline{B})} (v(P_i^\sigma \cup i) - v(P_i^\sigma)),$$

where $P_i^\sigma = \{j \in N : \sigma(j) < \sigma(i)\}$ is the set of predecessors of i at σ.

[18] characterizes the Shapley levels value using the above five properties.

Theorem 2.6. [18] *The Shapley levels value is the unique value on GL^N satisfying efficiency, additivity, individual symmetry, coalitional symmetry, and the null player property.*

3 A new value on GL^N

In this section we introduce a new value on GL^N that coincides with the Banzhaf-Owen value [14] when the levels structure of cooperation has just one level, i.e., when $\underline{B} = \{B_0, B_1\}$. The idea for defining this new value is to induce, for each player, a partition of the set of players that respects the restrictions of the levels structure of cooperation. In other words, instead of looking at which permutations are feasible for the given levels structure, as in [18], for each player we look at which coalitions are feasible for the given levels structure of cooperation.

Given a levels structure of cooperation $(N, \underline{B}) \in \mathcal{L}(N)$, for each player $i \in N$, let $i \in U_0 = \{i\} \subseteq U_1 \subseteq \cdots \subseteq U_k$ such that $U_r \in B_r$ for all $r \in \{0, \ldots, k\}$. Then, the *partition induced by $\underline{B}$ on i* is defined as follows,

$$P(i, \underline{B}) = \bigcup_{r=0}^{k} (B_r)_{|U_{r+1} \setminus U_r},$$

where $U_{k+1} = N$ by convenience. Then, $P(i, \underline{B}) \in \mathcal{P}(N \setminus i)$. We denote $|P(i, \underline{B})|$ by m_i, and the unions of the partition induced by $\underline{B}$, by $P(i, \underline{B}) = \{T_1, \ldots, T_{m_i}\}$. Finally the set of indices of the partition induced by $\underline{B}$ is denoted by $M_i = \{1, \ldots, m_i\}$ which can be seen as the set of representatives of the unions of $P(i, \underline{B})$.

Example 3.1. For the levels structure of cooperation of Example 2.3 we have, for instance, $P(1,\underline{B}) = \{\{2\},\{3\},\{4,5,6\}\}$ and $P(3,\underline{B}) = \{\{1,2\},\{4,5,6\}\}$.

Using this partition induced by the levels structure for each player, we define a new value on GL^N, namely the Banzhaf levels value, which is built based on the Banzhaf-Owen value for games with a priori unions.

Definition 3.2. Given a cooperative game with levels structure of cooperation $(N,v,\underline{B}) \in GL^N$, the *Banzhaf levels value*, Ψ, is a value on GL^N defined, for every $i \in N$, as follows:

$$\Psi_i(N,v,\underline{B}) = \sum_{R\subseteq M_i} \frac{1}{2^{m_i}} \left[v(T_R \cup i) - v(T_R)\right],$$

where $T_R = \cup_{r\in R} T_r$.

One can easily check that the coalitions considered in each marginal contribution, T_R, are the coalitions for which there exists a $\sigma \in \Omega(\underline{B})$ such that $T_R = P_i^\sigma$. Therefore, exploiting the link between coalitions of elements of $P(i,\underline{B})$, for each $i \in N$, and the permutations of $\Omega(\underline{B})$ we propose an alternative expression of the Shapley levels value, Φ.

Remark 3.3. Given a cooperative game with levels structure of cooperation $(N,v,\underline{B}) \in GL^N$,

$$\Phi_i(N,v,\underline{B}) = \sum_{R\subseteq M_i} \frac{c_R^i}{|\Omega(\underline{B})|} \left[v(T_R \cup i) - v(T_R)\right],$$

where $c_R^i = |\{\sigma \in \Omega(\underline{B}) : P_i^\sigma = T_R\}|$.

Note that the expressions of Φ and Ψ above lead to the Owen and Banzhaf-Owen values respectively for levels structure of cooperation with a single and trivial level.

4 Two parallel axiomatic characterizations

In this section we characterize both Φ and Ψ based on two different groups of properties. The first group applies only to games with the trivial levels structure $\underline{B} = \{B_0\} = \{\{i\} : i \in N\}$ and points out which value on G^N does the value on GL^N generalize, either the Shapley value or the Banzhaf value. The second group of properties describes the performance of the values on GL^N with respect to the levels structure.

We consider a number of properties that a value on GL^N, f, might be asked to satisfy. We start with a first set of properties.

EFF A value f on GL^N satisfies *efficiency* if for every $(N,v) \in G^N$,

$$\sum_{i \in N} f_i(N,v,\{B_0\}) = v(N).$$

2-EFF A value f on GL^N satisfies *2-efficiency* if for every $(N,v) \in G^N$ and any $i,j \in N$,

$$f_i(N,v,\{B_0\}) + f_j(N,v,\{B_0\}) = f_p(N^{ij}, v^{ij}, \{B_0\}^{ij}),$$

where $(N^{ij}, v^{ij}, \{B_0\}^{ij})$ is the game such that player i and j have merged into the new player p, i.e., $N^{ij} = (N \setminus \{i,j\}) \cup p$, $\{B_0\}^{ij} = \{\{i\} : i \in N^{ij}\}$, and

$$v^{ij}(S) = \begin{cases} v(S) & \text{if } p \notin S \\ v((S \setminus p) \cup i \cup j) & \text{if } p \in S \end{cases} \quad \text{for every } S \subseteq N^{ij}.$$

DPP A value f on GL^N satisfies *the dummy player property* if for every $(N,v) \in G^N$, if $i \in N$ is a dummy player on (N,v),

$$f_i(N,v,\{B_0\}) = v(i).$$

SYM A value f on GL^N satisfies *symmetry* if for every $(N,v) \in G^N$, if $i,j \in N$ are symmetric players on (N,v),

$$f_i(N,v,\{B_0\}) = f_j(N,v,\{B_0\}).$$

EMC A value f on GL^N satisfies *equal marginal contributions* if for every $(N,v),(N,w) \in G^N$ and every $i \in N$ such that $v(S \cup i) - v(S) = w(S \cup i) - w(S)$ for all $S \subseteq N \setminus i$,

$$f_i(N,v,\{B_0\}) = f_i(N,w,\{B_0\}).$$

The above properties are standard in the literature for games without restricted cooperation. The EFF property states that the whole worth available is shared among the players. The 2-EFF property is a collusion neutrality property which states that the payoff of two players does not change if they decide to artificially merge in a new player. Properties of this kind are used in many characterizations of the Banzhaf value, see for instance [10], [8] or [11]. The SYM and DPP properties are clear by themselves. The property of EMC states that if a player's marginal contributions to any coalition in two games coincide, then her payoffs also coincide. Stronger versions of EMC have been used in characterizations of both Shapley and Banzhaf values and are called *monotonicity* [19]. Even so some of the stated properties are also satisfied by the two values considered in this paper for more general levels structures of cooperation than the trivial one, in order to obtain our results there is no need to consider stronger properties. Let us now consider another set of properties.

LGP A value f on GL^N satisfies *level game property* if for every $(N, v, \underline{B}) \in GL^N$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $U \in B_r$ for some $r \in \{1, \cdots, k\}$,

$$\sum_{i \in U} f_i(N, v, \underline{B}) = f_{[U]}([B_r], v^r, \underline{B_r}).$$

SLGP A value f on GL^N satisfies the *singleton level game property* if for every $(N, v, \underline{B}) \in GL^N$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $U \in B_r$ for some $r \in \{1, \cdots, k\}$, such that $U = \{i\}$ for some $i \in N$,

$$f_i(N, v, \underline{B}) = f_{[U]}([B_r], v^r, \underline{B_r}).$$

LBC A value f on GL^N satisfies *level balanced contributions* if for every $(N, v, \underline{B}) \in GL^N$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $i, j \in U \in B_1$,

$$f_i(N, v, \underline{B}) - f_i(N, v, \underline{B^{-j}}) = f_j(N, v, \underline{B}) - f_j(N, v, \underline{B^{-i}}).$$

LNID A value f on GL^N satisfies *level neutrality under individual desertion* if for every $(N, v, \underline{B}) \in GL^N$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $i, j \in U \in B_1$,

$$f_i(N, v, \underline{B}) = f_i(N, v, \underline{B^{-j}}).$$

The LGP is based on a property used in [13] to characterize the Owen value. It states that the total payoff obtained by the members of a union in a given level equals the payoff obtained by the union when considering it as a player in the corresponding level game. The SLGP is a weaker version of LGP, which states that any union which is composed of a single player gets the same payoff in the original game and in the corresponding level game when considering the union as a player. The idea behind SLGP was also used in [3] and more recently in [1].

The LBC property is a reciprocity property that states that the isolation of a player from the levels structure affects the players in her same union of the first level in the same amount as if it happens the other way around. This property has been used in the context of games with a priori unions, e.g. [16] and [3]. The LNID property is a stronger version of LBC and states that the isolation of a player from the levels structure does not affect the payoffs of the players which are in her same union in all the levels. LNID was introduced in [3] and also used in [1] to characterize extensions of the Banzhaf value to different classes of games.

Next we state and prove the two characterization results, one for the Shapley levels value (Theorem 4.1) and one for the Banzhaf levels value (Theorem 4.2). We start characterizing the Shapley levels value.

Theorem 4.1. *The Shapley levels value, Φ, is the unique value on GL^N satisfying* EFF*,* SYM*,* EMC*,* LGP*, and* LBC*.*

Proof. First we show that Φ satisfies the properties and then we prove that it is the only value on GL^N satisfying them.

(1) *Existence.* Note that, by definition, for every $(N, v) \in G^N$, $\Phi(N, v, \{B_0\}) = \phi(N, v)$. Hence, from [19] we have that Φ satisfies EFF, SYM, and EMC.

In the case of LGP, let $(N, v, \underline{B}) \in GL^N$ with $\underline{B} = \{B_0, \dots, B_k\}$, and consider some $U \subseteq N$ such that $U \in B_r$ for some $r \in \{1, \cdots, k\}$. We prove that LGP holds by induction over r. If $r = 1$, from the definition of the induced partition, $P(i, \underline{B}) \setminus \{\{j\} : j \in U \setminus i\}$ is the same partition for each $i \in U$. Hence, take $i \in U$ and let us define $P(U, \underline{B}) = P(i, \underline{B}) \setminus \{\{j\} : j \in U \setminus i\}$, $m_U = |P(U, \underline{B})|$, and $M_U = \{1, \dots, m_U\}$. Then,

(1)
$$\sum_{i\in U} \Phi_i(N, v, \underline{B}) = \frac{1}{|\Omega(\underline{B})|} \sum_{i\in U} \sum_{R\subseteq M_U} \sum_{S\subseteq U\setminus i} c^i_{R+S} \cdot (v(T_R \cup S \cup i) - v(T_R \cup S)),$$

where $c^i_{R+S} = |\{\sigma \in \Omega(\underline{B}) : P^\sigma_i = T_R \cup S\}|$ for each $i \in U$, $R \subseteq M_U$, and $S \subseteq U \setminus i$. Since $U \in B_1$ and from the way $\Omega(\underline{B})$ is constructed, for a given $R \subseteq M_U$ and $S \subseteq U \setminus i$, there is an integer c^U_R such that $\frac{c^U_R}{c^i_{R+S}} = \binom{|U|-1}{|S|}$ for all $i \in U$ and all $S \subseteq U \setminus i$, and thus eq. (1) reduces to

$$\begin{aligned}
&\sum_{i\in U} \Phi_i(N,v,\underline{B}) \\
= \quad & \frac{1}{|\Omega(\underline{B})|} \sum_{i\in U} \sum_{R\subseteq M_U} \sum_{S\subseteq U\setminus i} \frac{c_R^U}{\binom{|U|-1}{|S|}} \cdot (v(T_R \cup S \cup i) - v(T_R \cup S)) \\
= \quad & \frac{1}{|\Omega(\underline{B})|} \sum_{R\subseteq M_U} c_R^U \cdot \sum_{i\in U} \sum_{S\subseteq U\setminus i} \frac{1}{\binom{|U|-1}{|S|}} \cdot (v(T_R \cup S \cup i) - v(T_R \cup S)) \\
= \quad & \frac{1}{|\Omega(\underline{B})|} \sum_{R\subseteq M_U} c_R^U \cdot \left(\sum_{\varnothing \subsetneq S \subsetneq U} \left(\frac{|S|}{\binom{|U|-1}{|S|-1}} - \frac{|U\setminus S|}{\binom{|U|-1}{|S|}} \right) v(T_R\cup S) + |U| v(T_R \cup U) - |U| v(T_R) \right) \\
= \quad & \frac{1}{|\Omega(\underline{B})|} \sum_{R\subseteq M_U} |U| \cdot c_R^U \cdot [(v(T_R\cup U) - v(T_R))] \\
= \quad & \frac{1}{|\Omega(\underline{B_1})|} \sum_{R\subseteq M_{[U]}} c_R^{[U]} \cdot \left[\left(v^1(T_R \cup [U]) - v^1(T_R) \right) \right] \\
= \quad & \Phi_{[U]}([B_{r-1}], v^{r-1}, \underline{B_{r-1}}),
\end{aligned}$$

where the fourth equality holds since $\frac{|S|}{\binom{|U|-1}{|S|-1}} - \frac{|U\setminus S|}{\binom{|U|-1}{|S|}} = 0$ for each $\varnothing \subsetneq S \subsetneq U$ and the fifth equality is explained as follows. From the definition of the induced partition, it is straightforward to check that $P(U,\underline{B}) = P([U],\underline{B_1})$. Moreover, let $c_R^{[U]} = |\{\sigma \in \Omega(\underline{B_1}) : P_{[U]}^{\sigma} = T_R\}|$. Then, it can be easily checked that $\frac{c_R^{[U]}}{c_R^U} = |U| \cdot \frac{|\Omega(B_1)|}{|\Omega(\underline{B})|}$, which completes the first step of the induction.

Now suppose that for any $S \in B_{r-1}$, $\sum_{i\in S} \Phi_i(N,v,\underline{B}) = \Phi_{[S]}([B_{r-1}], v^{r-1}, \underline{B_{r-1}})$, and let $U \in B_r$. Then

$$\sum_{i\in U} \Phi_i(N,v,\underline{B}) = \sum_{\substack{S\in B_{r-1}\\ S\subseteq U}} \sum_{i\in S} \Phi_i(N,v,\underline{B}) = \sum_{\substack{S\in B_{r-1}\\ S\subseteq U}} \Phi_{[S]}([B_{r-1}], v^{r-1}, \underline{B_{r-1}})$$

by the induction hypothesis. Observe that $([B_{r-1}], v^{r-1}, \underline{B_{r-1}})$ is a levels structure of cooperation of $k-r+1$ levels. Hence, we can follow the argument from eq. (1) with $[B_{r-1}]$ instead N and $[S]$ instead i to obtain

$$\sum_{\substack{S\in B_{r-1}\\ S\subseteq U}} \Phi_{[S]}([B_{r-1}], v^{r-1}, \underline{B_{r-1}}) = \Phi_{[U]}([B_r], v^r, \underline{B_r}),$$

which completes the induction.

In the case of LBC, let $(N, v, \underline{B}) \in GL^N$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $i, j \in U \in B_1$. Then, it is easy to check that $P(i, \underline{B}) \cup i = P(j, \underline{B}) \cup j$. Hence, let us define $P(ij, \underline{B}) = P(i, \underline{B}) \setminus j = P(j, \underline{B}) \setminus i$, $m_{ij} = |P(ij, \underline{B})|$, and $M_{ij} = \{1, \ldots, m_{ij}\}$. Then,

$$\begin{aligned}
&\Phi_i(N, v, \underline{B}) - \Phi_i(N, v, \underline{B^{-j}}) \\
&= \sum_{R \subseteq M_{ij}} \frac{c^i_{R+j}}{|\Omega(\underline{B})|} \left(v(T_R \cup j \cup i) - v(T_R \cup j)\right) + \frac{c^i_R}{|\Omega(\underline{B})|} \left(v(T_R \cup i) - v(T_R)\right) \\
&- \sum_{R \subseteq M_{ij}} \frac{c^{i,-j}_{R+j}}{|\Omega(\underline{B^{-j}})|} \left(v(T_R \cup j \cup i) - v(T_R \cup j)\right) + \frac{c^{i,-j}_R}{|\Omega(\underline{B^{-j}})|} \left(v(T_R \cup i) - v(T_R)\right) \\
&= \sum_{R \subseteq M_{ij}} \left[\left(\frac{c^i_{R+j}}{|\Omega(\underline{B})|} - \frac{c^{i,-j}_{R+j}}{|\Omega(\underline{B^{-j}})|} \right) \left(v(T_R \cup j \cup i) - v(T_R \cup j)\right) \right. \\
&\qquad \left. + \left(\frac{c^i_R}{|\Omega(\underline{B})|} - \frac{c^{i,-j}_R}{|\Omega(\underline{B^{-j}})|} \right) \left(v(T_R \cup i) - v(T_R)\right) \right],
\end{aligned}$$

where for each $R \subseteq M_{ij}$, $c^{i,-j}_R = |\{\sigma \in \Omega(\underline{B^{-j}}) : P^\sigma_i = T_R\}|$ and $c^{i,-j}_{R+j} = |\{\sigma \in \Omega(\underline{B^{-j}}) : P^\sigma_i = T_R \cup j\}|$. Note that by definition, $c^i_R = c^j_R$, $c^i_{R+j} = c^j_{R+i}$, $c^{i,-j}_R = c^{j,-i}_R$, and $c^{i,-j}_{R+j} = c^{j,-i}_{R+i}$. We additionally claim (see a proof in the Appendix) that

$$\frac{c^i_R + c^i_{R+j}}{|\Omega(\underline{B})|} = \frac{c^{i,-j}_R + c^{i,-j}_{R+j}}{|\Omega(\underline{B^{-j}})|}. \tag{2}$$

Then $\Phi_i(N, v, \underline{B}) - \Phi_i(N, v, \underline{B^{-j}})$ depends on i in the same way it depends on j, which concludes the proof.

(2) *Uniqueness.* In [19] it is proved that any value on GL^N that satisfies EFF, SYM, and EMC is unique for games with the trivial levels structure of cooperation. In other words, let f^1 and f^2 be two values on GL^N satisfying EFF, SYM, and EMC, then

$$f^1(N, v, \{B_0\}) = f^2(N, v, \{B_0\}) = \phi(N, v) \quad \text{for any } (N, v) \in G^N.$$

Hence, let f^1 and f^2 be two values on GL^N satisfying LGP and LBC and such that $f^1(N, v, \{B_0\}) = f^2(N, v, \{B_0\})$ for all $(N, v) \in G^N$. We prove that for any $(N, v, \underline{B}) \in GL^N$, with $\underline{B} = \{B_0, \ldots, B_k\}$, $f^1(N, v, \underline{B}) = f^2(N, v, \underline{B})$ by induction on the number k of levels of $\underline{B}$. The case $k =$

1 is proved in [16]. Now suppose that $f^1(N,v,\underline{B}) = f^2(N,v,\underline{B})$ for any $(N,v,\underline{B}) \in GL^N$ such that $|\underline{B}| \leq k$ and let $(N,v,\underline{B}) \in GL^N$ with $|\underline{B}| = k+1$ and $i \in N$. We prove that $f_i^1(N,v,\underline{B}) = f_i^2(N,v,\underline{B})$ by a second induction on $u = |U|$, where $i \in U \in B_1 \in \underline{B}$. If $u = 1$, i.e. $U = \{i\}$, since f^1 and f^2 satisfy LGP, we have

$$f_i^1(N,v,\underline{B}) = f_{[U]}^1([B_1],v^r,\underline{B_1}) = f_{[U]}^2([B_1],v^r,\underline{B_1}) = f_i^2(N,v,\underline{B}),$$

where the second equality holds by the first induction hypothesis. Hence, suppose that $f_i^1(N,v,\underline{B}) = f_i^2(N,v,\underline{B})$ for any $(N,v,\underline{B}) \in GL^N$, with $|\underline{B}| = k+1$ and any $i \in U \in B_1$ that satisfies $|U| \leq u$. Now suppose that $|U| = u+1$ and let $j \in U \setminus i$. Since f^1 and f^2 satisfy LBC, we have

$$\begin{aligned}(3) \quad f_i^1(N,v,\underline{B}) - f_j^1(N,v,\underline{B}) &= f_i^1(N,v,\underline{B^{-j}}) - f_j^1(N,v,\underline{B^{-i}}) \\ &= f_i^2(N,v,\underline{B^{-j}}) - f_j^2(N,v,\underline{B^{-i}}) = f_i^2(N,v,\underline{B}) - f_j^2(N,v,\underline{B}),\end{aligned}$$

where the second equality follows from the second induction hypothesis, since $i \in U \setminus \{j\} \in B_{1,-j}$ and $j \in U \setminus \{i\} \in B_{1,-i}$ with $|U \setminus \{j\}| = |U \setminus \{i\}| = u$, where $|B^{-j}| = |B^{-i}| = k+1$. Now, adding up eq. (3) for each $j \in U \setminus i$, we have

(4)

$$(t+1)f_i^1(N,v,\underline{B}) - \sum_{j\in U} f_j^1(N,v,\underline{B}) = (t+1)f_i^2(N,v,\underline{B}) - \sum_{j\in U} f_j^2(N,v,\underline{B}).$$

Finally, by LGP we have that

(5)

$$\sum_{j\in U} f_j^1(N,v,\underline{B}) = f_{[U]}^1([B_1],v^r,\underline{B_1}) = f_{[U]}^2([B_1],v^r,\underline{B_1}) = \sum_{j\in U} f_j^2(N,v,\underline{B}),$$

where the second equality holds by the first induction hypothesis since $|\underline{B_1}| = k$. Combining eq. (4) and (5) we obtain $f_i^1(N,v,\underline{B}) = f_i^2(N,v,\underline{B})$, which completes the proof. □

In the next theorem we characterize the Banzhaf levels value with a set of six properties which are easily comparable to the properties used to characterize the Shapley levels value.

Theorem 4.2. *The Banzhaf levels value, Ψ, is the unique value on GL^N satisfying* 2-EFF, DPP, SYM, EMC, SLGP, *and* LNID.

Proof. As before, first we show that Ψ satisfies the properties and then we prove that it is the only value satisfying them.

(1) *Existence.* Note that, by definition for every $(N, v) \in G^N$, $\Psi(N, v, \{B_0\}) = \psi(N, v)$. Hence, from [11] we have that Ψ satisfies 2-EFF, DPP, SYM, and EMC.

In the case of SLGP, the proof follows immediately taking into account the fact that for any $(N, v, \underline{B}) \in GL^N$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $U = \{i\} \in B_r$ for some $r \in \{1, \cdots, k\}$, $P(i, \underline{B}) = P([U], \underline{B_r})$.

In the case of LNID, we only need to check that for any $(N, v, \underline{B}) \in GL^N$ with $\underline{B} = \{B_0, \ldots, B_k\}$, and any $i, j \in U \in B_1$, $P(i, \underline{B}) = P(i, \underline{B^{-j}})$, which follows from the definition of the partition induced by $\underline{B}$.

(2) *Uniqueness.* From the characterization in [11], we have that any value on GL^N that satisfies 2-EFF, DPP, SYM, and EMC is unique for games with the trivial levels structure of cooperation. In other words, let f^1 and f^2 be two values on GL^N satisfying 2-EFF, DPP, SYM, and EMC, then

$$f^1(N, v, \{B_0\}) = f^2(N, v, \{B_0\}) = \psi(N, v) \quad \text{for any } (N, v) \in G^N.$$

Now let f^1 and f^2 be two values on GL^N satisfying SLGP and LNID such that $f^1(N, v, \{B_0\}) = f^2(N, v, \{B_0\})$ for all $(N, v) \in G^N$. We prove that for any $(N, v, \underline{B}) \in GL^N$, with $\underline{B} = \{B_0, \ldots, B_k\}$, $f^1(N, v, \underline{B}) = f^2(N, v, \underline{B})$ by induction on the number k of levels of $\underline{B}$. The case $k = 1$ is proved in [2]. Hence suppose that $f^1(N, v, \underline{B}) = f^2(N, v, \underline{B})$ for any $(N, v, \underline{B}) \in GL^N$ such that $|\underline{B}| \leq k$ and let $(N, v, \underline{B}) \in GL^N$ such that $|\underline{B}| = k + 1$. Let $i \in U \in B_1$ be an arbitrary player that belongs to an arbitrary union of the first level. We prove that $f_i^1(N, v, \underline{B}) = f_i^2(N, v, \underline{B})$ by a second induction on $u = |U|$. If $u = 1$, i.e. $U = \{i\}$, since f^1 and f^2 satisfy SLGP, we have

$$f_i^1(N, v, \underline{B}) = f_{[U]}^1([B_1], v^1, \underline{B_1}) = f_{[U]}^2([B_1], v^1, \underline{B_1}) = f_i^2(N, v, \underline{B}),$$

where the second equality holds by the first induction hypothesis since $\underline{B_1} \in \mathcal{L}(N)$ is a levels structure with k levels. Now suppose that $f_i^1(N, v, \underline{B}) = f_i^2(N, v, \underline{B})$ for any $(N, v, \underline{B})$ such that $|\underline{B}| = k + 1$ and any $i \in U \in B_1$ where $|U| \leq u$. Finally, suppose that $|U| = u + 1$ and let $j \in U \setminus i$. Since f^1 and f^2 satisfy LNID we have

$$f_i^1(N, v, \underline{B}) = f_i^1(N, v, \underline{B^{-j}}) = f_i^2(N, v, \underline{B^{-j}}) = f_i^2(N, v, \underline{B}),$$

where the second equality holds by the second induction hypothesis since $i \in U \setminus j \in \underline{B_1^{-j}}$, $\underline{B_1^{-j}}$ has $k + 1$ levels of cooperation and $|U \setminus j| = u$, which concludes the proof. □

Finally, we check that the proposed properties are independent axioms, and hence we cannot drop any of them from the characterizations. We start examining the properties used for the characterization of the Shapley levels value, Φ.

Remark 4.3. Independence of properties for Theorem 4.1

(i) The value on GL^N, g, given by $g(N,v,\underline{B})=0$ for all $(N,v,\underline{B})\in GL^N$ satisfies SYM, EMC, LGP, LBC but not EFF.

(ii) Let g be the value on GL^N defined as follows:

- If $N=\{i,j\}$ and $\underline{B}=\{\{i\},\{j\}\}$,

$$\begin{aligned} g_i(N,v,\underline{B}) &= \frac{3}{4}\left(v(N)-v(j)\right)+\frac{1}{4}v(i) \quad \text{and} \\ g_j(N,v,\underline{B}) &= \frac{1}{4}\left(v(N)-v(i)\right)+\frac{3}{4}v(j). \end{aligned}$$

- Otherwise, $g(N,v,\underline{B})=\Phi(N,v,\underline{B})$.

  Thus, g satisfies EFF, EMC, LGP, LBC, but not SYM.

(iii) Consider the value on GL^N, g, given by

$$g(N,v,\underline{B})=\begin{cases}\Phi(N,v,\underline{B}) & \text{if } (N,v,\underline{B})\notin\mathcal{C}\\ a_{i(N,v)}\mathbf{1}_{i(N,v)} & \text{if } (N,v,\underline{B})\in\mathcal{C}\end{cases}$$

where

$$\mathcal{C}=\left\{\begin{array}{l}(N,v,\underline{B})\in GL^N: v=b_i\tau_i+(a_i-b_i)\delta_N,\\ \text{for some } i=i(N,v)\in N \text{ and } 0\le b_i<a_i\end{array}\right\}$$

such that for every $S\subseteq N$,

$$\tau_i(S)=\begin{cases}1 & \text{if } i\in S\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad \delta_N(S)=\begin{cases}1 & \text{if } S=N\\ 0 & \text{otherwise}\end{cases},$$

and $\mathbf{1}_k\in\mathbb{R}^n$ is such that $\mathbf{1}_k(l)=1$ if $k=l$ and $\mathbf{1}_k(l)=0$ if $k\neq l$. Then g satisfies EFF, SYM, LGP, LBC, but not EMC.

(iv) The value on GL^N, g, given by $g(N,v,\underline{B})=\phi(N,v)$ for all $(N,v,\underline{B})\in GL^N$ satisfies EFF, SYM, EMC, LBC, but not LBC.

(v) Let g be the value on GL^N defined as follows:

- If $N=\{i,j\}$ and $\underline{B}=\{\{\{i\},\{j\}\},N\}$, $g(N,v,\underline{B})=(\frac{v(N)}{2},\frac{v(N)}{2})$.
- Otherwise, $g(N,v,\underline{B})=\Phi(N,v,\underline{B})$.

  Thus, g satisfies EFF, SYM, EMC, LGP but not LBC.

Lastly, we examine the properties used for the characterization of the Banzhaf levels Ψ.

Remark 4.4. Independence of axioms for Theorem 4.2

(i) The value on GL^N, g, given by

$$g_i(N,v,\underline{B}) = \sum_{R\subseteq M_i} \frac{|R|!(m_i-|R|-1)!}{m_i!}\left(v(T_R\cup i)-v(T_R)\right),$$

satisfies DPP, SYM, EMC, SLGP, LNID but not 2-EFF.

(ii) The value on GL^N, g, given by $g(N,v,\underline{B})=0$ for all $(N,v,\underline{B})\in GL^N$ satisfies 2-EFF, SYM, EMC, SLGP, LNID, but not DPP.

(iii) Let g be the value on GL^N defined as follows:

- If $N=\{i,j\}$ and $\underline{B}=\{\{i\},\{j\}\}$,

$$\begin{aligned} g_i(N,v,\underline{B}) &= \frac{3}{4}\left(v(N)-v(j)\right)+\frac{1}{4}v(i) \quad \text{and} \\ g_j(N,v,\underline{B}) &= \frac{1}{4}\left(v(N)-v(i)\right)+\frac{3}{4}v(j). \end{aligned}$$

- Otherwise, $g(N,v,\underline{B})=\Psi(N,v,\underline{B})$.

  Thus, g satisfies 2-EFF, DPP, EMC, SLGP, LNID, but not SYM.

(iv) The value on GL^N, g, given by

$$g(N,v,\underline{B}) = \begin{cases} \Psi(N,v,\underline{B}) & \text{if } (N,v,\underline{B})\notin\mathcal{C} \\ 0 & \text{if } (N,v,\underline{B})\in\mathcal{C} \end{cases}$$

where $\mathcal{C}=\{(N,v,\underline{B})\in GL^N : v=a_S\delta_S, \text{ for some } S\subseteq N\}$, satisfies 2-EFF, DPP, SYM, SLGP, LNID, but not EMC.

(v) The value on GL^N, g, given by $g(N,v,\underline{B})=\psi(N,v)$ for all $(N,v,\underline{B})\in GL^N$ satisfies 2-EFF, DPP, SYM, EMC, LNID, but not SLGP.

(vi) The value on GL^N, g, given by

$$g_i(N,v,\underline{B}) = \sum_{R\subseteq M_i} \frac{1}{2^{m_i-|T_R\cap U_k|}}\cdot\frac{|T_R\cap U_k|!(|U_k\setminus T_R|-1)!}{|U_k|!}\left(v(T_R\cup i)-v(T_R)\right),$$

satisfies 2-EFF, DPP, SYM, EMC, SLGP, but not LNID, where recall that U_k is the union of the k-th level to which player i belongs.

It should be pointed out that, from the proofs above it follows that in both Theorems, the group of properties that apply only for the trivial levels structure can be replaced by any other group of properties that characterize either the Shapley or the Banzhaf value.

5 Conclusions and an example

In the present paper we have proposed a new value for games with levels structure of cooperation and we have provided parallel characterizations of this new value, the Banzhaf levels value, and the Shapley levels value. Since the main properties used in both characterizations are logically comparable, our paper serves in the purpose of deciding which value to use in any framework of restricted cooperation given by a sequence of union levels.

We conclude the paper by examining an example to help us illustrate the use of the two different values in a decision problem. Before doing so, we make a comment on the validity of the application of the Banzhaf levels value. [9] claim that, in the context of voting games with a single level structure of cooperation and the Banzhaf levels value, only comparisons between players that belong to the same union of the first level are meaningful. The reason why they state so is that the number that Φ assigns to player i can be interpreted as the mathematical expectation of the decisiveness of player i when considering the probability distribution defined on the set of permutations of players conditional to the partition induced by the levels structure on player i. Since players that belong to different unions give rise to different induced partitions, their corresponding probability distributions are different and hence [9] conclude that they cannot be compared. Nevertheless, when the levels structure of cooperation is pin down and the players cannot behave strategically and change their position in the structure, as it is the case in the example below, we can do compare the values of players belonging to different unions, even in the case of simple games. We argue that even so the probability distribution of each agent is different, all of them are obtained from the same fixed structure following the same rules, which can be seen as public knowledge. Therefore, we may interpret the Banzhaf levels value as the subjective expectation of any player about the outcome of the game, provided the following condition holds: all agents believe that, for any arbitrary given agent, all possible coalitions that may form before she takes a decision -which may be different depending on the player considered- are equiprobable.

Example 5.1. Consider a grid computing network to which some departments of several universities contribute with resources, e.g., memory, databases or processing capacity. The whole network resources are used for purposes of calculations demanding massive levels of resources such as climate predictions. The departments involved are willing to use the grid computing network for their investigations and the problem arises when more than one department simultaneously request access to the common resources, which can only be accessed by one department at a time.

Moreover, consider a numerical example where the amount of resources that each department contributes with can be measured, e.g., either TB or Ghz. The total amount of resources add up to 41 units that are provided by 10 departments namely A, B, C, D, E, F, G, H, I and J, which respectively contribute 3, 1, 2, 10, 3, 5, 2, 3, 2, 10 units.

In order to measure the contribution of each department to the network we assume that a grid computing network needs a minimum of 21 units to operate. Hence, any group of departments whose resources add up to 21 units or more could form a smaller network. Even though all departments prefer to be part of a network as big as possible, we consider this possibility in order to measure the bargaining strength of each department.

The situation described so far can be modeled by a simple game (N, v), where N is the set of departments and the characteristic function $v(S)$ equals 1 if the aggregate amount of resources of coalition S is at least equal to 21 and 0 otherwise. Therefore, the priority rule needed to decide which department will use the grid first can be based on either the Shapley or the Banzhaf value, ϕ or ψ respectively. More precisely, we first normalize the Banzhaf value and the value of each department is interpreted as the probability - henceforth just *priority*- that the corresponding department can make use of the common resources when all departments simultaneously request access. These values (ϕ and $\overline{\psi}$) are depicted in Table 5.1.

However, each department involved is part of a university which, in turn, is in a given country. It may happen that when bargaining for the priority the departments are not autonomous anymore and need the permission of the university or country they belong to. If we take into account these restrictions, a levels structure of cooperation emerges naturally, and hence, the Shapley and Banzhaf levels values, Φ and Ψ, could be used as basis for a priority rule. Consider for instance, that the 10 departments are part of 6 different universities which, in turn, are in 4 countries. More precisely, suppose there is the following levels structure of cooperation, $\{\{A\}, \{B, C\}, \{D\}, \{E, F\}, \{G, H, I\}, \{J\}\}$ and $\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J\}\}$, i.e. for instance Dept. B and Dept. C belong to the same university, which at its turn it is located in the same region as the university which Dept. A belongs to.

Table 5.1 below comprises the different values considered in this paper[2].

From Table 5.1, it follows that when considering the restrictions given

[2]By $\bar{f}$ we denote the normalized f value. The different values have been calculated using a MATLAB© routine, which can be provided by the authors upon request.

Dep.	Resources	ϕ	Φ	ψ	Ψ	$\overline{\psi}$	$\overline{\Psi}$
A	3	0.0690	0.0833	0.1523	0.1250	0.0736	0.0800
B	1	0.0341	0.0417	0.0724	0.0625	0.0358	0.0400
C	2	0.0405	0.0417	0.0898	0.0625	0.0434	0.0400
D	10	0.2579	0.2500	0.4961	0.3750	0.2396	0.2400
E	3	0.0690	0.0417	0.1523	0.0625	0.0736	0.0400
F	5	0.1214	0.2083	0.2773	0.3125	0.1340	0.2000
G	2	0.0405	0.0278	0.0898	0.0625	0.0434	0.0400
H	3	0.0690	0.1110	0.1523	0.1875	0.0736	0.1200
I	2	0.0405	0.0278	0.0898	0.0625	0.0434	0.0400
J	10	0.2579	0.1667	0.4961	0.2500	0.2396	0.1600

Table 1: The different measures of priority.

by the levels structure of cooperation the priorities change significantly. For instance, a relevant such difference the change in the priority assigned to Dep. J. When the departments are considered autonomous it is given top priority together with Dep. D. However, when the universities and countries are taken into account it ranks third, having Dep. F priority over Dep. J. This is explained by the fact that even so Dep. J is one of the departments whose contribution is highest, the aggregate resources of its country are not so high compared to the aggregate resources of the remaining countries. Finally, the difference between Φ and Ψ reveals intensely on the values of Dept. E and Dept. I, since Ψ gives equal priority to both of them, whereas Φ doubles the value of Dept. E.

6 Appendix

Proof of the claim in the Proof of Theorem 4.1.

Let $(N, v, \underline{B}) \in GL^N$ with $\underline{B} = \{B_0, \ldots, B_k\}$, $i, j \in U_1 \subseteq \ldots \subseteq U_k$ with $U_r \in B_r$ for each $r \in \{1, \cdots, k\}$, and $R \subseteq M_{ij}$. Let us define, for $r \in \{1, \cdots, k\}$,

$$\begin{aligned} \lambda_R^r &= |\{\sigma \in \Omega_r(\underline{B}) : P_i^\sigma = T_R\}| + |\{\sigma \in \Omega_r(\underline{B}) : P_i^\sigma = T_R \cup j\}| \quad \text{, and} \\ \lambda_R^{-r} &= |\{\sigma \in \Omega_r(\underline{B^{-j}}) : P_i^\sigma = T_R\}| + |\{\sigma \in \Omega_r(\underline{B^{-j}}) : P_i^\sigma = T_R \cup j\}|. \end{aligned}$$

Observe that $\lambda_R^1 = c_R^i + c_{R+j}^i$ and $\lambda_R^{-1} = c_R^{i,-j} + c_{R+j}^{i,-j}$. We prove that $\frac{\lambda_R^r}{|\Omega_r(\underline{B})|} = \frac{\lambda_R^{-r}}{|\Omega_r(\underline{B^{-j}})|}$ for all $r \in \{1, \cdots, k\}$ by backward induction on r. For each $r \in \{1, \cdots, k\}$, let $b_r = |B_r|$, $u_r = |U_r|$, $A^r = |\{U \in B_r \setminus U_r : U \subseteq U_{r+1} \text{ and } U \cap T_R = \emptyset\}|$, and $B^r = |\{U \in B_r \setminus U_r : U \subseteq U_{r+1} \text{ and } U \subseteq T_R\}|$. Recall that by convenience $U_{k+1} = N$. Observe that $A^k + B^k + 1 = b_k$ and that, for each $r \in \{1, \cdots, k\}$, $|U_r \cap T_R| + |U_r \setminus T_R| = u_r$.

We start proving the case $r = k$. Recall that $U_k \in B_k$ is such that $i, j \in U_k$. In particular, $i, j \in U_k \setminus T_R$ and thus $|U_k \setminus T_R| \geq 2$. By definition of λ_R^r,

$$\begin{aligned}\lambda_R^k = &\prod_{S \in B_k \setminus \{U_k\}} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 1)! \\ &+ \prod_{S \in B_k \setminus \{U_k\}} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R| + 1)! \cdot (|U_k \setminus T_R| - 2)! \\ &= \prod_{S \in B_k \setminus \{U_k\}} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)! \cdot u_k.\end{aligned}$$

Similarly, by definition of λ_R^{-k},

$$\begin{aligned}\lambda_R^{-k} = &\prod_{S \in B_k \setminus \{U_k\}} |S|! \cdot (A^k + 1)! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)! \\ &+ \prod_{S \in B_k \setminus \{U_k\}} |S|! \cdot A^k! \cdot (B^k + 1)! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)! \\ &= \prod_{S \in B_k \setminus \{U_k\}} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)! \cdot (b_k + 1).\end{aligned}$$

Hence, for every $R \subseteq M_{ij}$, $\frac{\lambda_R^k}{\lambda_R^{-k}} = \frac{u_k}{b_k + 1}$. To conclude with the first step of the induction one can easily check that $\frac{\Omega_k(\underline{B})}{\Omega_k(\underline{B^{-j}})} = \frac{u_k}{b_k+1}$.

Now suppose that for every $R \subseteq M_{ij}$, $\frac{|\Omega_{r+1}(\underline{B})|}{|\Omega_{r+1}(\underline{B^{-j}})|} = \frac{\lambda_R^{r+1}}{\lambda_R^{-,r+1}}$, for some

$r \in \{2, \dots, k\}$. By definition of λ_R^k,

$$\frac{\lambda_R^r}{\lambda_R^{r+1}} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot A^r! \cdot B^r! \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq U_{r+1} \setminus U_r}} |S'|!$$
$$\cdot \frac{(|U_r \cap T_R|)! \cdot (|U_r \setminus T_R| - 1)! + (|U_r \cap T_R| + 1)! \cdot (|U_r \setminus T_R| - 1)!}{(|U_{r+1} \cap T_R|)! \cdot (|U_{r+1} \setminus T_R| - 1)! + (|U_{r+1} \cap T_R| + 1)! \cdot (|U_{r+1} \setminus T_R| - 2)!}$$
$$= \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot A^r! \cdot B^r! \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq U_{r+1} \setminus U_r}} |S'|!$$
$$\cdot \frac{(|U_r \cap T_R|)! \cdot (|U_r \setminus T_R| - 2)!}{(|U_{r+1} \cap T_R|)! \cdot (|U_{r+1} \setminus T_R| - 2)!} \cdot \frac{u_{r+1}}{u_r},$$

where $h(S) = |\{S' \in B_r : S' \subseteq S\}|$ for each $S \in B_{r+1}$. Similarly, by definition of λ_R^{-k},

$$\frac{\lambda_R^{-,r}}{\lambda_R^{-,r+1}} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot A^r! \cdot B^r! \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq U_{r+1} \setminus U_r}} |S'|!$$
$$\cdot \frac{(|U_r \cap T_R|)! \cdot (|U_r \setminus T_R| - 2)!}{(|U_{r+1} \cap T_R|)! \cdot (|U_{r+1} \setminus T_R| - 2)!}.$$

Combining the two above expressions we obtain

$$\frac{\lambda_R^r}{\lambda_R^{-,r}} = \frac{\lambda_R^{r+1}}{\lambda_R^{-,r+1}} \cdot \frac{u_r}{u_{r+1}}. \tag{6}$$

Furthermore,

$$\frac{|\Omega_r(\underline{B})|}{|\Omega_{r+1}(\underline{B})|} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot \frac{h(U_{r+1})!}{u_{r+1}!} \cdot \left(\prod_{\substack{S' \in B_r \setminus U_r \\ S' \subseteq U_{r+1}}} |S'|! \right) \cdot u_r!,$$

and

$$\frac{|\Omega_r(\underline{B^{-j}})|}{|\Omega_{r+1}(\underline{B^{-j}})|} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot \frac{h(U_{r+1})!}{(u_{r+1} - 1)!} \cdot \left(\prod_{\substack{S' \in B_r \setminus U_r \\ S' \subseteq U_{r+1}}} |S'|! \right) \cdot (u_r - 1)!.$$

Thus

$$\frac{|\Omega_r(\underline{B})|}{|\Omega_r(\underline{B^{-j}})|} = \frac{|\Omega_{r+1}(\underline{B})|}{|\Omega_{r+1}(\underline{B^{-j}})|} \cdot \frac{u_r}{u_{r+1}}. \tag{7}$$

Hence, from eq. (6) and (7), using the induction hypothesis we obtain

$$\frac{\lambda_R^r}{|\Omega_r(\underline{B})|} = \frac{\lambda_R^{-,r}}{|\Omega_r(\underline{B^{-j}})|},$$

which concludes the proof. □

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