Balanced allocation methods for claims problems with indivisibilities.*

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Abstract

In this work we deal with rationing problems. In particular with claims problems with indivisible goods, that is, problems in which a certain amount of indivisible units (of an homogeneous good), has to be distributed among a group of agents, when this amount is not enough to satisfy agents' demands. We use a subfamily of standards of comparisons (monotonic standards) to construct scheduling methods to solve this type of problems. The rules constructed in this way can be interpreted as discrete versions of the constrained equal awards and constrained equal losses rules when the good is perfectly divisible. They not only enjoy similar properties, but have stronger relations with the cea and cel rules in terms of expected values and the size of indivisibility.

Keywords: claims problems, indivisibilities, standard of comparisons, monotone standard, balancedness.

JEL Classification: D63

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1 Introduction.

A claims problem represents a situation in which a quantity of a certain commodity has to be distributed among some agents and the available resources fall short of total demand. The canonical example of this kind of problems is that in which a firm goes to bankruptcy, that is, the amount it owes to its creditors is greater than the firm's worth. In this problem, in general, a judge has to liquidate the firm and decide how to distribute the amount gotten in the liquidation among the creditors. In this example, as it happens in the classical claims literature, the good to be distributed is perfectly divisible, and so are the awards alloted to the agents. The reader is referred to the surveys by Moulin (2002) and Thomson (2003). Nonetheless, there are many claims situations involving the distribution of a commodity coming in indivisible units.

Consider the following examples: In order to carry out the administrative tasks at the university departments, a University contracts a certain number of secretaries. On one hand, this number depends on the financial capabilities of the University. On the other hand, each department has the right to receive, depending on its volume, a certain number of secretaries. It happens that the total number of secretaries the departments demand is larger than the available amount. How many secretaries correspond to each department? Another example is the case of renting cars firms. Some months, the demands of new cars from these firms to the cars manufacturer is so high that the production in that month is not enough to satisfy the whole demand. The manufacturer must decide how to distribute the available cars among the renting firms. Consider also the distribution of radio frequencies among the different broadcasting corporations, whenever there is no an auction mechanism. If the amount of frequencies requested by the firms is too large, the Government should decide how many frequencies are allotted to each corporation.

Previous examples belong to the so called *integer claims problems*. There, the departments, the renting firms, or media-corporations are referred to as the *agents*, the amount of indivisible units is called the *estate* and the demands or rights are called the *claims*. A *rule* is a way of distributing the available estate among the agents according to their claims. Rules can be either deterministic or probabilistic. We are interested here in deterministic single-valued rules.

In the axiomatic method rules are defended on the basis of the properties they fulfil. Among those properties, the general equity requirement is *equal treatment of equals* that prescribes equal awards to agents with equal demands. In general, this property cannot be met in the integer claims problem, and thus, the equal treatment of equals principle takes on a different form: whenever two claimants have identical claims, their awards differ by at most one unit. This property was introduced by Balinski and Young (1977) under the name of *balancedness* in the context of apportionment, and used also by Young (1994) in claims problems with indivisibilities.

The use of priority methods has been common in solving claims problems with indivisibilities. In Moulin (2000) the claimants arrive one at a time and they are fulfilled until the good is run out. The family of rules resulting from this procedure are the unique solutions fulfilling three interesting procedural properties: *composition up*, *composition down*, and *consistency*. *Composition up* states that the procedure works by allocating one unit at a time, provided that each time the claim of the recipient agent is decreased by one unit. *Composition down* states that the procedure works by allocating the units of deficit one at a time, provided the demand of the recipient is reduced accordingly. *Consistency* says that when one agent leaves with her allocation, the solution of the reduced problem is such that all remaining agents receive identical awards as in the original problem. Of course, the pure priority methods characterized this way fail to be balanced.

A different type of priority methods was introduced by Young (1994), and used also by Kaminski (2000) and Moulin (2002). They all use the idea of *Standard of Comparison*. A Standard of Comparison is simply a priority order defined over pairs of agents and claims so that it is increasing in claims, i.e., for all agent i, and for all claim x, we have that the pair $(i, x+1)$ has priority over the pair (i, x). That is, instead of simply consider a linear order on the set of agents, now a linear order on the cartesian product of the set of agents and the set of non-negative integer numbers is considered. If i, j are two agents, and x, y are their respective claims, and in the standard of comparison $\succ$, it happens that $(i, x) \succ (j, y)$, then a claim of x units of the good by agent i has priority over a claim of y units by agent j. Given a particular standard of comparison, a natural allocation method fulfilling by construction the property of Composition up is defined as follows: We allocate the first unit of the good to the agent i such that (i, c_i) has highest priority for the standard, and reduce agent $i's$ claim to $c_i - 1$. The second unit goes similarly to agent j (possibly the same) such that (j, c_j) is highest for the standard (given agent $i's$ reduced claim), and then we reduce the claim of agent j by one unit, and so on, until the moment in which the good is exhausted. We call these methods *UP-methods.* Similarly, a second rationing method can be defined associated to any standard of comparison, in a dual way. First, we allocate all agents their demands. Since the available amount of the good is not enough, we have to remove some units up to reaching that available amount. We now use the standard of comparison to allocate the successive units of deficit by using the same algorithm described before. Those mechanisms are referred to as *DOWN-methods*, and all of them satisfy by construction the property of Composition down. Previous methods were introduced in Moulin and Stong (2002) under the name of *standard of gains* and *standard of losses,* respectively.

Moulin and Stong (2002) characterize the family of *UP-methods* associated to a standard of comparison by means of three properties: *consistency*, *composition up*, and *claims* *monotonicity*, that requires no harm on any agent awards when her claim increases, leaving unchanged all others' claims and the estate. Similarly, the *DOWN-methods* associated to a standard of comparison are characterized by *consistency*, *composition down* and *loss monotonicity* that requires the loss of any agent not to decrease when her claim increases, leaving unchanged all others' claims and the total deficit.

Both the family of *UP-methods* and *DOWN-methods* associated to standards of comparisons contain allocation procedures that violate balancedness. In order to ensure this property, we should consider a subfamily of the standard of comparisons. We call them *monotonic standards*, and are those standards of comparison that give priority to a larger claim, i. e., $(i, x+1) \succ (j, x)$, for all i, j, x.

In this paper we concentrate on the *UP-methods* and *DOWN-methods* associated to *monotonic standards*. We call them *UP* and *DOWN monotonic methods*. It happens that these methods, apart from balancedness, also satisfy some properties that previously appeared in the literature in the study of two well-known solutions for claims problems when the good is perfectly divisible, the constrained equal awards and the constrained equal losses rules (see Moulin (1985), Chun (1988), Young (1988), Dagan (1996), Herrero and Villar (2001), Herrero and Villar (2002), and Yeh (2004)) . Furthermore, the family of UP monotonic methods is characterized by using properties also fulfilled by the constrained equal losses rule, whereas the family of DOWN monotonic methods is characterized by using properties fulfilled by the constrained equal awards rule. Somehow, any member of the family of UP monotonic methods can be looked at as a discrete version of the constrained equal losses rule, and, similarly, any member of the family of DOWN monotonic methods could be interpreted as a discrete version of the constrained equal awards rule. The relationship among the discrete and the continuous rules is stronger: the allocations prescribed by the constrained equal losses rule can be interpreted as the ex-ante expectations of the agents under the application of UP monotonic methods, if all plausible allocations prescribed by such methods are equally likely, and similarly the allocations prescribed by the constrained equal awards rule can be interpreted as the ex-ante expectations of the agents under the application of DOWN monotonic methods, if all plausible allocations prescribed by such methods are equally likely. Also, the allocations prescribed by any UP (DOWN) monotonic method converges to the allocation recommended by the constrained equal losses (awards) rule when the size of the indivisibilities goes to zero.

The rest of the paper is structured as follows: In Section 2 we set up the claims problems with indivisibilities and the different notions of allocations and rules. In Section 3 we introduce standards of comparison and the Up and Down methods. Section 4 is devoted to the properties our rules fulfil. In Section 5 we present our characterizations. Section 6 presents the relationship between the monotonic up and down methods and the constrained equal losses and constrained equal awards rules for the perfectly divisible case. Section 7, with final comments and remarks, concludes. Proofs are relegated to an Appendix.

2 Statement of the problem.

We devote this section to provide formal statements of claims problem, allocations and rules.

Let $\mathbb{N}$ be the set of all potential **agents**. Let $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$. A claims problem is a triple (N, E, c), where $N \in \mathcal{N}$ is the set of agents ($n = |N|$), $c \equiv (c_i)_{i \in N} \in \mathbb{Z}_+^N$ is the vector of claims, $E \in \mathbb{Z}_{++}$ is the amount to allocate. And it happens that $\sum_{i \in N} \geq E$. Let $\mathbb{C}^N$ denote the class of all claims problems with agent set N, and $\mathbb{C}$ the class of all claims problems, that is,

$$\mathbb{C}^N = \left\{ (N, E, c) \in \{N\} \times \mathbb{Z}_{++} \times \mathbb{Z}_+^N : \sum_{i \in N} c_i \geq E \right\}$$

and

$$\mathbb{C} = \bigcup_{N \in \mathcal{N}} \mathbb{C}^N.$$

Let $C, L : \mathbb{C}_{\mathbb{Z}} \longrightarrow \mathbb{Z}_+$ be the aggregate claim and aggregate loss functions respectively:

$$C(N, E, c) = \textstyle\sum_{i \in N} c_i, \qquad L(N, E, c) = C(N, E, c) - E.$$

For each problem, we face the question of finding a division of the estate among the agents.

An **allocation** for $(N, E, c) \in \mathbb{C}$ is a list $\boldsymbol{x} \in \mathbb{Z}_+^N$ satisfying two conditions: (a) Each agent receives a nonnegative amount that is not higher than his claim ($0 \leq x_i \leq c_i$ for each $i \in N$); and (b) the estate is distributed exactly ($\sum_{i \in N} x_i = E$). Let $\boldsymbol{X(N, E, c)}$ be the set of all allocations for (N, E, c). A **rule** is a function, $\boldsymbol{F}$, that selects, for each problem $(N, E, c) \in \mathbb{C}$, a unique allocation $F(N, E, c) \in X(N, E, c)$.[1]

[1] Notice that the notion of rule refers simply to the selection of a feasible allocation for any problem. This choice can be made via a direct formulation, as it is the case for the majority of rules used when the estate is perfectly divisible, or else, it can be made by using an algorithm, as we do in this work.

3 Standards of Comparison and Up and Down Methods.

A standard of comparison is a linear order (complete, antisymmetric and transitive) over the cartesian product of the set of potential agents and claims such that for any given agent, she has more priority the larger her claim is.

Standard of Comparison, $\boldsymbol{\sigma}$ (Young, 1994): Is a function $\sigma : \mathbb{N} \times \mathbb{Z}_{++} \longrightarrow \mathbb{Z}_{++}$ such that for each $i \in \mathbb{N}$, and each $a \in \mathbb{Z}_{++}$, $\sigma(i, a+1) < \sigma(i, a)$. Let Σ denote the class of all standards of comparison.

Associated to any standard of comparison, two natural methods for solving claims problems can be constructed. The first option is allocating all units of the estate one by one. The second one is subtracting all units of deficit one by one, after giving (temporarily) all agents their claims. We shall call those *up methods* and *down methods* respectively. For each problem $(N, E, c) \in \mathbb{C}$, the **agent with strongest claim** for (N, E, c) according to the standard of comparison σ is the agent $i \in N$ such that the pair (i, c_i) has the highest priority among all the pairs (j, c_j) involved in (N, E, c), according to σ. That is, i is the agent with the strongest claim for (N, E, c) according to σ if for each $j \in N \setminus \{i\}$, then $\sigma(i, c_i) < \sigma(j, c_j)$.

Up method associated to $\boldsymbol{\sigma}$, $\boldsymbol{U^{\sigma}}$ (Moulin & Stong, 2002): Let $(N, E, c) \in \mathbb{C}$. Give one unit of the estate to the agent with the strongest claim for (N, E, c) according to σ. Reduce the claim of this agent by one unit. Identify the agent with the strongest claim in the resulting problem according to σ, and proceed in the same way. Repeat this process until the estate runs out.

Down method associated to $\boldsymbol{\sigma}$, $\boldsymbol{D^{\sigma}}$ (Moulin & Stong, 2002): Let $(N, E, c) \in \mathbb{C}$. Fully compensate all agents. Subtract one unit from the agent with the strongest claim for (N, E, c) according to σ. Reduce the claim of this agent by one unit. Identify the agent with the strongest claim in the resulting problem according to σ, and proceed in the same way. Repeat this process until reaching the estate.

Example 3.1. *Let $N = \{1, 2, 3\}$, and assume that the standard of comparison is such that, restricted to agents in N, it happens that $\sigma(2, x) < \sigma(1, y) < \sigma(3, z)$, for all $x, y, z \in \mathbb{Z}_{++}$. Now, consider the problem (N, E, c), where $E = 6$, and $c = (1, 3, 5)$. For the pairs involved in the aforementioned problem, we have*

$$\sigma(2,3) < \sigma(2,2) < \sigma(2,1) < \sigma(1,1) < \sigma(3,5) < \sigma(3,4) < \sigma(3,3) < \sigma(3,2) < \sigma(3,1).$$

The next table shows the functioning of the up-method for this problem. The first column gives the kth unit of the estate. The second column gives the allocation up to that unit, $x^{(k)}$. The third column gives the updated vector of claims, $c^{(k)}$.

E	$x^{(k)}$	$c^{(k)}$
	(0,0,0)	*(1,3,5)*
1	*(0,1,0)*	*(1,2,5)*
2	*(0,2,0)*	*(1,1,5)*
3	*(0,3,0)*	*(1,0,5)*
4	*(1,3,0)*	*(0,0,5)*
5	*(1,3,1)*	*(0,0,4)*
6	*(1,3,2)*	*(0,0,3)*

Example 3.2. *Similarly, in the next table we show the functioning of the down-method for this same problem. In this case we start by fully compensating all agents. This implies allocating 9 units, but we only have 6 available. Thus we need to remove 3 units. The table shows the process of removing. The first column gives the kth unit of the estate. We start from 9 and we remove unit by unit up to reach 6. The second column gives the allocation up to that unit,* $x^{(k)}$*. The third column gives the updated vector of claims,* $c^{(k)}$*.*

E	$x^{(k)}$	$c^{(k)}$
9	*(1,3,5)*	
8	*(1,2,5)*	*(1,3,5)*
7	*(1,1,5)*	*(1,2,5)*
6	*(1,0,5)*	*(1,1,5)*

Previous examples illustrate how the up and down methods work. Additionally, they show that these methods could result in *pure priority rules*, depending upon the standard of comparison used. Given the standard of comparison in previous examples, the allocation obtained by means of the up-method is the allocation prescribed by a pure priority rule in which agent 2 is fully satisfied first, then agent 1 comes to the line and he is fulfilled, and, finally, any remaining units go to agent 3. As for the allocation obtained by the application of the down-method, it is simply the allocation suggested by the pure priority rule with the reverse order: Now, agent 3 is first to be served, up to the moment in which he is satisfied, next, agent 1 comes, and finally, agent 1 is served, provided there are some units available.

The next example considers a different type of standard of comparison

Example 3.3. *Let* $N = \{1, 2, 3\}$*, and assume that the standard of comparison is such that, restricted to agents in* N*, it happens that for all* $i, j \in N$*, and all* $x, y \in \mathbb{Z}_{++}$*, if* $x > y$*, then* $\sigma(i, x) < \sigma(j, y)$*. Furthermore,* $\sigma(1, x) < \sigma(2, x) < \sigma(3, x)$ *if* x *is odd, and* $\sigma(2, x) < \sigma(1, x) < \sigma(3, x)$ *if* x *is even. Now, let again* $E = 6$*, and* $c = (1, 3, 5)$*. The next table shows the functioning of the up-method associated to this standard of comparison*

E	$x^{(k)}$	$c^{(k)}$
	(0,0,0)	*(1,3,5)*
1	*(0,0,1)*	*(1,3,4)*
2	*(0,0,2)*	*(1,3,3)*
3	*(0,1,2)*	*(1,2,3)*
4	*(0,1,3)*	*(1,2,2)*
5	*(0,2,3)*	*(1,1,2)*
6	*(0,2,4)*	*(1,1,1)*

Example 3.4. *In a similar way, the table below illustrates the down method applied to the same problem and standard of comparisons as in the previous example.*

E	$x^{(k)}$	$c^{(k)}$
9	*(1,3,5)*	
8	*(1,3,4)*	*(1,3,5)*
7	*(1,3,3)*	*(1,3,4)*
6	*(1,2,3)*	*(1,3,3)*

As we see in this example, the allocations are far from resulting in pure priority rules, and the different units are allocated in an alternating way. As a consequence, the type of final allocation obtained heavily depends on the type of standard of comparison used.

4 Properties.

Here we look for properties our rules may fulfil. The most common and appealing requirement is a property of **impartiality**. In one if its forms, *equal treatment of equals,* it says that, in any problem, if two claimants have equal claims, then they should receive equal amounts. Obviously, in our context, no rule can fulfill this property. It is enough to consider a problem with two agents with equal claims, and an estate of one unit. Balinski and Young (1977) and Young (1994) consider a weaker version of this condition, that they call **balancedness**: If in a problem two agents have equal claims, then their allocations differ, at most, by one unit. It is interesting to observe that, when the allocation procedure is an Up or Down method, balancedness requires that agents with identical claims should be served alternately.

Balancedness: For each $(N, E, c) \in \mathbb{C}$ and each $\{i, j\} \subseteq N$, if $c_i = c_j$, then $|F_i(N, E, c) - F_j(N, E, c)| \leq 1$.

Note that, as it has shown in Examples 3.1 and 3.2, the application of the up and down methods to some standards of comparisons results in allocations violating balancedness.

The next group of properties refers to changes in the estate, when the set of agents and their claims remain fixed.

The first property is straightforward. It says that, if the estate increases, then no agent should be punished.

Estate monotonicity: For each (N, E, c), $(N, E', c) \in \mathbb{C}$, if $C(N, E, c) > E' \geq E$, then $F(N, E', c) \geq F(N, E, c)$.

The following two properties are *procedural* properties, and have to do with possible mistakes in the estimation of the estate, either from below or from above. They guarantee a sort of invariance in the final allocation, after correcting the mistakes.

Imagine that, when estimating the value of the estate, we were too pessimistic, and the actual value is larger than expected. Then two alternatives are open. Either we solve the new problem. Or we consider the problem with the underestimated estate. And then allocate the remaining estate, after reducing the claims by the amounts of the first step. The property of *composition up* asks for the final allocation to be independent of the chosen alternative.

Composition up (Young, 1988): For each $(N, E, c) \in \mathbb{C}$ and each each $E' \in \mathbb{Z}_{++}$ such that $E > E'$, then $F(N, E, c) = F(N, E', c) + F(N, E - E', c - F(N, E', c))$.

An alternative formulation of *composition up* states that the rule is completely determined by the allocation of the first unit, and it reads as follows: For each $(N, E, c) \in \mathbb{C}$, $F(N, E, c) = F(N, 1, c) + F(N, E - 1, c - F(N, 1, c))$.

Note that the *up-methods* satisfy this property, whereas the *down-methods* do not.

Dually, imagine that when estimating the value of the estate, we were too optimistic, and the actual value is smaller than expected. Again, two alternatives are open. Either we solve the new problem. Or we consider a the problem in which the estate is the reduced one, and the claims are the allocation obtained with the overestimated estate. The property of *composition down* asks for the final allocation to be independent of the chosen alternative.

Composition down (Moulin, 1987): For each $(N, E, c) \in \mathbb{C}$ and each $E' \in \mathbb{Z}_{+}$ such that $C(N, E, c) > E' > E$, then $F(N, E, c) = F(N, E, F(N, E', c))$.

In a similar way as before, an alternative formulation of *composition down* states that the rule is completely determined by the removing of the first unit of deficit, and it reads as follows: For each $(N, E, c) \in \mathbb{C}$, $F(N, E - 1, c) = F(N, E - 1, F(N, E, c))$.

Note that the *down-methods* satisfy this property, whereas the *up-methods* do not.

Trivially, both *composition up* and *composition down* imply *estate monotonicity*.

The next two properties refer to cases in which the claim of an agent rises, leaving unchanged all other agents demands. The first requirement, *claims monotonicity*, establishes that if, moreover, the estate remain unchanged, then this agent share should not decrease. The up methods satisfy this property, and the down methods do not.

Claims monotonicity: For each $(N, E, c) \in \mathbb{C}$ and for each $i \in N$ if $c'_i > c_i$ and for each $j \neq i$, we have $c'_j = c_j$, then $F_i(N, E, c') \geq F_i(N, E, c)$.

Alternatively, *losses monotonicity* states that if, moreover, the total deficit remains unchanged, then this agent share of deficit should not decrease. Now, the down methods satisfy this property, and the up methods do not.

Losses monotonicity: For each $(N, E, c) \in \mathbb{C}$ and for all $i \in N$, if $c'_i > c_i$ and for all $j \neq i$, we have $c'_j = c_j$, then $c_i - F_i(N, E, c) \leq c'_i - F_i(N, E', c')$, where $E' = E + (c'_i - c_i)$.

The two next properties exploit the idea that only claimants responsible for the problem should be rationed. In other words these properties are "protective" in favor of small claimants. They refer to how small a claim should be for its owner to receive his claim in full. One way to decide that threshold in a problem is the following. Substitute it for the claim of any other agents whose claim is higher, and check whether there would then be enough to compensate everyone.

Conditional full compensation (Herrero & Villar, 2002): For each $(N, E, c) \in \mathbb{C}$, if $\sum_{j=1}^{n} \min\{c_i, c_j\} \leq E$, then $F_i(N, E, c) = c_i$.

The following property proposes an alternative threshold. When an individual's claim is smaller than the equal division of the estate, the individual should be fully compensated.

Exemption (Herrero & Villar, 2001): For each $(N, E, c) \in \mathbb{C}$, if $c_i \leq \lfloor E/n \rfloor$, then $F_i(N, E, c) = c_i$

Note that *exemption* implies *conditional full compensation.* Moreover, in the two-agent case, both properties coincide.

The next group of properties are "protective" properties for agents with sufficiently large claims. They refer to cases where agents with claims small enough (below a certain threshold) should receive nothing, thereby favoring agents with larger claims. Different thresholds give rise to different properties.

Conditional null compensation (Herrero & Villar, 2002): For each $(N, E, c) \in \mathbb{C}$, if $\sum_{j=1}^{n} \min\{c_i, c_j\} \leq L$, then $F_i(N, E, c) = 0$.

Exclusion (Herrero & Villar, 2001): For each $(N, E, c) \in \mathbb{C}$, if $c_i \leq \lfloor L/n \rfloor$, then $F_i(N, E, c) = 0$

Note that *exclusion* implies *conditional null compensation.* Moreover, in the two-agent case, both properties coincide.

Finally, we consider properties that refer to changes in the set of agents. Suppose that, after solving a problem, $(N, E, c) \in \mathbb{C}$, a proper subset of the set of agents, $S \subset N$, decide to reallocate the total amount they have received. That is, they face the problem $(S, \sum_{i \in S} a_i, c_S)$, where $c_S \equiv (c_i)_{i \in S}$ and a is the allocation selected by the rule for (N, E, c). The requirement is such that the reallocation is the restriction, to the subset S, of the initial allocation.

Consistency (Aumann & Maschler, 1985): For each $(N, E, c) \in \mathbb{C}$, each $S \subset N$, and for each $i \in S$, $F_i(N, E, c) = F_i(S, \sum_{j \in S} F_j(N, E, c), c_S)$.

When the requirement of consistency affects only to groups of agents of size two, the property is called **bilateral consistency.**

The next property refers to situations in which, apart from the allocations in the two-agent case, we can recover the allocation for the general case. Let us consider a allocation for a problem, $(N, E, c) \in \mathbb{C}$, with the following feature: For each two-agent subset, the rule chooses the restriction of that allocation for the associated reduced problem to this agent subset. The requirement is such that the allocation is selected by the rule for the original problem (N, E, c).

Let $c.con(E, c; F) \equiv \{x \in \mathbb{Z}_+^N : \sum_{i \in N} x_i = E$ and for all $S \subset N$ such that $|S| = 2, x_S = F(S, \sum_{i \in S} x_i, c_s)\}$

Converse consistency (Chun, 1999): For each $(N, E, c) \in \mathbb{C}$, $c.con(E, c; F) \neq \phi$, and if $x \in c.con(E, c; F)$, then $x = F(N, E, c)$.

As in the continuous case, duality of properties can also be established for discrete solutions.

A property $\mathcal{P}^d$ is the **dual** of $\mathcal{P}$ if it is true that F satisfies $\mathcal{P}$ if and only if its dual rule, F^d, satisfies $\mathcal{P}^d$. A property is **self-dual** when it coincides with its dual.

The following proposition establishes the duality relations among the properties considered above.

Proposition 4.1. *The following pairs of properties are dual:*

- *Composition up and composition down.*
- *Claims monotonicity and losses monotonicity.*
- *Exemption and exclusion.*

- *Conditional full compensation and conditional null compensation.*

Moreover, balancedness, estate monotonicity, consistency, bilateral consistency, and converse consistency are self-dual.

The next four results (that were proved for claims problems with perfectly divisible goods) also valid (without modifications) in the presence of indivisibilities.

Theorem 4.1 (Herrero & Villar, 2001). *Let F be a rule characterized by a set of independent properties $\Pi = \{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$. Let $\mathcal{P}_i^d$ be the dual property of $\mathcal{P}_i$. Then the dual rule, F^d, is characterized by the corresponding set of dual properties $\Pi^d = \{\mathcal{P}_1^d, \ldots, \mathcal{P}_k^d\}$. Moreover, the properties in Π^d are also independent.*

Lemma 4.1 (Elevator lemma, Thomson 2000). *If a rule F is bilaterally consistent and coincides with a conversely consistent rule F' in the two agent case, then it coincides with F' in general.*

Proposition 4.2 (Chun, 1999). *Estate monotonicity and consistency together imply converse consistency.*

Proposition 4.3 (Chun, 1999). *Converse consistency implies consistency.*

The next result is similar to the one presented in Herrero and Villar (2001), but, unlike them, in this case we show the relation among the properties without the requirement of continuity.

Proposition 4.4. *In the two-agent case, composition down and conditional full compensation together imply balancedness.*

The proof is given in Appendix B.

5 Characterizations.

Up and down methods associated to any standard of comparison have been characterized by Moulin and Stong (2002).

Theorem 5.1 (Moulin & Stong, 2002). *A rule F satisfies claims monotonicity, composition up, and consistency if and only if there exists a standard of comparison $\sigma \in \Sigma$ such that $F = U^\sigma$. A rule F satisfies losses monotonicity, composition down, and consistency if and only if there exists a standard of comparison $\sigma \in \Sigma$ such that $F = D^\sigma$.*

The up and down methods associated to a standard of comparison in general violate *balancedness*. In order to guarantee this property, we should concentrate on a particular subfamily of standards of comparison, that we call *monotonic standards*.

Monotonic standard of comparison: For each $\{i, j\} \subseteq \mathbb{N}$, and each $x, y \in \mathbb{Z}_+$, if $x > y$, then $\sigma(i, x) < \sigma(j, y)$. Let Σ^M denote the subfamily of all monotonic standards of comparison.

In other words, monotonic standards of comparison always give priority to larger demands.

The following result is straightforward:

Proposition 5.1. *Let $\sigma \in \Sigma$ be an standard of comparison. Then, the associated up and down methods, U^σ and D^σ, satisfy balancedness if and only if σ is monotonic.*

This proposition together with Theorem 5.1 trivially imply the next result:

Theorem 5.2. *A rule F satisfies balancedness, claims monotonicity, composition up, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = U^\sigma$. A rule F satisfies balancedness, losses monotonicity, composition down, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = D^\sigma$.*

We shall call **up (down) monotonic methods** to the up (down) methods associated to monotonic standards of comparison.

The notion of *duality* refers to how a pair of rules allocates gains and losses. Two rules are *dual* if one of them allocates awards in the same way the other one allocates losses. Two rules, F and F^d, are **dual** if for each $(N, E, c) \in \mathbb{C}$, $F^d(N, E, c) = c - F(N, L, c)$. The next result is straightforward from the definition of the up and down monotonic methods.

Proposition 5.2. *Let $\sigma \in \Sigma^M$. The rules U^σ and D^σ are dual.*[2]

The following alternative characterizations of the up and down monotonic methods are also obtained. All the proofs in this section are in Appendix B.

Theorem 5.3. *A rule F satisfies conditional null compensation, composition up, and consistency if and only if it there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = U^\sigma$. A rule F satisfies conditional full compensation, composition down, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = D^\sigma$.*

[2]This result is more general. In fact, it is true not only for monotonic standards but also for any standard of comparison.

The next result also characterizes the up and down monotonic methods by changing *consistency* by *converse consistency.*

Theorem 5.4. *A rule F satisfies conditional null compensation, composition up, and converse consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = U^\sigma$. A rule F satisfies conditional full compensation, composition down, and converse consistency if and only if it there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = D^\sigma$.*

The last pair of results in this section exploit the relationship between *conditional null compensation* and *conditional full compensation* with *exclusion* and *exemption* respectively.

Theorem 5.5. *A rule F satisfies exclusion, composition up, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = U^\sigma$. A rule F satisfies exemption, composition down, and consistency if and only if it there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = D^\sigma$.*

Theorem 5.6. *A rule F satisfies exclusion, composition up, and converse consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = U^\sigma$. A rule F satisfies exemption, composition down, and converse consistency if and only if it there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = D^\sigma$.*

The results in this section can be summarized in Table 1. All characterizations are tight. The independence of the properties is proved in the Appendix A.

6 Up and Down Monotonic Methods and the constrained equal awards and losses rules.

In Theorems 5.3-5.6 above we obtained characterization results for the family of up and down monotonic methods. Some of those characterizations have analogous counterparts in characterization results of the continuous constrained equal awards and constrained equal losses rules.[3] Actually, the relationship between those monotonic methods and the con-

[3]Under the assumption that the estate were completely divisible, two of the most widely studied rules are the so called constrained equal awards and constrained equal losses rules. The idea underlying the first one is equality in gains, adjusting, if it is necessary, to ensure that no agent receives more than his claim.

Constrained equal awards rule, ***cea***: For each $(N, E, c) \in \mathbb{C}$, selects the unique vector $cea(N, E, c) = \min\{c, \lambda\}$ for some $\lambda \in \mathbb{R}$.

The idea of the second rule is also equality, but now focusing on losses. If necessary, adjustments are made to ensure that no agent receives a negative amount.

Constrained equal losses rule, ***cel***: For each $(N, E, c) \in \mathbb{C}$, selects the unique vector $cel(N, E, c) = \max\{0, c - \lambda\}$ for some $\lambda \in \mathbb{R}$.

Property	D^σ	U^σ
Balancedness	Y^5	Y^5
Claims monotonicity	N	Y^5
Losses monotonicity	Y^5	N
Conditional full compensation	$Y^{1,2}$	N
Exemption	$Y^{3,4}$	N
Conditional null compensation	N	$Y^{1,2}$
Exclusion	N	$Y^{3,4}$
Estate monotonicity	Y	Y
Composition down	$Y^{1,2,3,4,5}$	N
Composition up	N	$Y^{1,2,3,4,5}$
Consistency	$Y^{1,3,5}$	$Y^{1,3,5}$
Converse consistency	$Y^{2,4}$	$Y^{2,4}$

Table 1: This table summarizes the result in former sections; "Y" means that the rule satisfies that property for each $\sigma \in \Sigma^M$, while "N" that it does not. On the other hand Y^k means that this property, together with the others marked with $(^k)$ in the same column, characterize the rule.

strained equal awards and constrained equal losses rules is strongest. On the one hand, any monotonic up method can be interpreted as a discrete version of the constrained equal losses rule, and, similarly, any monotonic down method could be interpreted as a discrete version of the constrained equal awards rule. In this section we further explore the relationship between the family of monotonic up methods and the *cel* rule, and the relationship between the monotonic down methods and the *cea* rule. Two types of results are obtained. On the one hand, we see that, for any problem, the allocations prescribed by the constrained equal losses rule can be interpreted as the ex-ante expectations of the agents under the application of up monotonic methods, if all plausible allocations prescribed by such methods are equally likely, and similarly the allocations prescribed by the constrained equal awards rule can be interpreted as the ex-ante expectations of the agents under the application of down monotonic methods, if all plausible allocations prescribed by such methods are equally likely. On the other hand, the allocations prescribed by any up (down) monotonic method converges to the allocation recommended by the constrained equal losses (awards) rule when the size of the indivisibilities goes to zero. These results are presented below.

Proposition 6.1. *Let $(N, E, c) \in \mathbb{C}$. Let $\Sigma^M_{(N,c)}$ denote the subset of Σ^M of the different partial orders involved in the problem (N, E, c).*[4] *Then*

[4]In Σ^M we consider all possible orders over $\mathbb{N} \times \mathbb{Z}_{++}$. Notice that, for a given claims problem (N, E, c), no all of them rank the pairs (i, c_i) involved in that particular problem in different ways. $\Sigma^M_{(N,c)}$ denotes precisely the subset of those different orders.

(a) $cel(N,E,c) = \frac{1}{\left|\Sigma^M_{(N,c)}\right|} \sum_{\sigma \in \Sigma^M_{(N,c)}} U^\sigma(N,E,c)$

(b) $cea(N,E,c) = \frac{1}{\left|\Sigma^M_{(N,c)}\right|} \sum_{\sigma \in \Sigma^M_{(N,c)}} D^\sigma(N,E,c)$

Proof. Let us prove the result for the up monotone method. On one hand, we know that the constrained equal losses rule is *conversely consistent* (see Yeh (2004)). On the other hand, it is easy to check that the up monotone methods are *consistent*. Then, the average given by the right hand side in the formula is also consistent (see Thomson 2004). By the Elevator Lemma it is enough to consider the two-agent case. But it is straightforward that in this case, both the constrained equal losses rule and the average coincide. As a result, they are equal in general. We use an analogous argument for Statement b. □

We face know the question of making the size of the indivisibilities smaller and smaller. As a way of example, assume that we are distributing secretaries to the different Departments of the University, and consider the possibility of the hours of work of the secretary to be distributed between two different Departments. That is, it is possible to have a fraction of a secretary's time. The amount of secretaries to distribute is the same but the size of the indivisibility is reduced to half its previous size. Let $\sigma \in \Sigma^M$, and consider an up method associated to σ, U^σ. The next result says that if the the units the estate is expressed in are getting smaller and smaller. In the limit the allocation prescribed by U^σ coincides with the allocation recommended by the constrained equal losses rule. A similar statement can be obtained for the down methods associated to σ, and the constrained equal awards rule.

Proposition 6.2. *Let $(N,E,c) \in \mathbb{C}$ and let $k \in \mathbb{Z}_{++}$. Then, for each $\sigma \in \Sigma^M$,*

(a) $\lim_{k\to\infty} \frac{1}{k} U^\sigma(N,kE,kc) = cel(N,E,c)$.

(b) $\lim_{k\to\infty} \frac{1}{k} D^\sigma(N,kE,kc) = cea(N,E,c)$.

Proof. Let $\sigma \in \Sigma^M$. Let $(N,E,c) \in \mathbb{C}$. Let $k \in \mathbb{Z}_{++}$, then $(N,kE,kc) \in \mathbb{C}$. Let d_k be the distance between $\frac{1}{k}U^\sigma(N,kE,kc)$ and $cel(N,E,c)$:

$$d_k = \left\| \frac{1}{k} U^\sigma(N,kE,kc) - cea(N,E,c) \right\|_\infty .$$

We show that d_k goes to cero as k goes to infinity. We can manipulate d_k as follows:

$$\begin{aligned} d_k &= \max_{i\in N} \left| \tfrac{1}{k} U^\sigma(N,kE,kc) - cel(N,E,c) \right| \\ &= \max_{i\in N} \left| \tfrac{1}{k} \left(U^\sigma(N,kE,kc) - cel(N,kE,kc) \right) + \tfrac{1}{k} cel(N,kE,kc) - cel(N,E,c) \right| \\ &\le \max_{i\in N} \tfrac{1}{k} \left| U^\sigma(N,kE,kc) - cel(N,kE,kc) \right| + \left| \tfrac{1}{k} cel(N,kE,kc) - cel(N,E,c) \right| \\ &\le \max_{i\in N} \tfrac{1}{k} + \left| cel(N,E,c) - cel(N,E,c) \right| \\ &= \tfrac{1}{k} \end{aligned}$$

This proves Statement a. The second statement is proven in a similar way. □

7 Final Remarks

In this paper we have considered claims problems with indivisibilities, that is, problems in which the estate, the claims and the allocations are expressed in integer units. As was proved in Moulin (2000), if *composition up*, *composition down* and *consistency* together are imposed, only serial dictator solutions are left, leaving no room for any sort of compromise. Those pure priority methods, nonetheless, can be avoided if we dispense with one of the composition properties. Moulin and Stong (2002) consider two methods generating all rules satisfying *consistency, composition up*and alternatively *consistency* and *composition down* together with some additional mild restrictions, *claims monotonocity* and *losses monotonicity,*respectively. These two rich families of rules contain pure priority rules together with other rules satisfying some sort of egalitarian properties. In this paper we concentrate on the subfamily of those rules that, additionally, satisfy *balancedness*, a property that states that awards of agents with identical claims should differ, at most, in one unit, i.e., we isolate the family of rules that have an as egalitarian as possible behavior. Our family of rules can be constructed by using Moulin and Stong's up and down methods, provided that they are generated by *monotonic standards*, i.e., standards of comparison that give priority to higher claims, irrespective of the agents.

The rules shown here have been defined using two ingredientes: a standard of comparison and the way we use this standard. Hence the methods proposed by Moulin and Stong (2002) applied to the family of monotonic standards of comparison provide two subfamilies of rules. Nevertheless we can use monotonic standards of comparison together with different procedures. Suppose the claims problem we are dealing with is an apportionment problem, i.e., we have to allocate a given amount of seats in a Parliament among a group of parties or regions according to their votes or populations. Consider the following procedure. Give one unit of the estate to the agent with the strongest claim. Divide by two this agent's claim. Identify the agent with the strongest claim in the resulting problem. Give her the next unit. Now, if she is the agent whose claim was divided in the previous stage, then divide now the original claim by three; if she in not, divide her original claim by two. In general, once you have identified the agent with the strongest claim, divide her original claim by $k+1$ if it was divided by k in a previous stage. Repeat this process until the estate runs out. Allocations obtained by the aforementioned procedure coincide the ones coming from the D'Hont rule, used in most of the European elections.

Interestingly, the families of rules studied here have a close relationship with the familiar *constrained equal awards and constrained equal losses* rationing rules for the case in which the good is perfectly divisible. Actually, any rule generated by a monotonic up-method can be interpreted as a discrete version of the *constrained equal losses* rule, and similarly, any rule generated by a monotonic down-method can be interpreted as a discrete version of the *constrained equal awards* rule. They also not only fulfill balancedness and some of the procedural properties of the continuous rules, but also protective properties either to agents with large or small claims.

There is a close relationship between our balanced methods and the probabilistic *fair queuing* and *fair queuing** in Moulin and Stong (2002): Drawing a scheduling sequence from the monotonic up-methods at random and uniformly is a representation of fair queuing, and dually, drawing a scheduling sequence from the monotonic down-methods at random and uniformly is a representation of fair queuing*. In other words, any realization of the probabilistic *fair queuing* method coincides with the proposal of a particular monotonic up-method, and all the realizations via monotonic up-methods are equally likely, and similarly, any realization of the probabilisticnd *fair queuing** method coincides with the proposal of a particular monotonic down-method, and all the realizations via monotonic down-methods are equally likely. This explains the reason that for fair queuing and fair queuing* a similar result to Proposition 6.1 is obtained.

7.1 Appendix A

The characterizations in Theorems 5.2, 5.3, 5.4, 5.5 and 5.6 are tight. We here prove the independence of the properties.

Example 7.1. ***A rule satisfying balancedness, composition up, conditional full compensation, exemption, consistency, converse consistency and but neither claims monotonicity nor composition down***. *Such a rule, F, can be described as follows: Let $\succ: \mathbb{N} \longrightarrow \mathbb{Z}_{++}$ be an order defined over the set of potential agents. We start by dividing the estate among the agents with the lowest claims, attempting to give to each of them the same amount. If this is not possible, because of the indivisibility, then their allocations will differ by one unit. The agents with the highest priority according to $\succ$ are those who receive the extra unit. Then, if there is still some estate left, we divide it equally (again, respecting the weak equal treatment of equals principle) among agents with the second lowest claim. We continue the process until the estate runs out. Formally, let $(N, E, c) \in \mathbb{C}$, let $\mu^t(c) = t.th \min_{j \in N}\{c_j\}$, $M^t(c) = \{j \in N : \mu^t(c) = c_j\}$, $m^t(c) = |M^t(c)|$. Then for each $i \in M^k(c)$ let us define $\nu^k(c) = \sum_{s<k} m^s(c)\mu^s(c)$, then*

$$F_i(N,E,c) = \begin{cases} 0 & \textit{if } 0 \le E \le \nu^k(c) \\ \left\lfloor \frac{E-\nu^k(c)}{m^k(c)} \right\rfloor & \textit{if } \nu^k(c) \le E \le \nu^{k+1}(c) \textit{ and } i \notin Q^k_{E'}(c) \\ \left\lfloor \frac{E-\nu^k(c)}{m^k(c)} \right\rfloor + 1 & \textit{if } \nu^k(c) \le E \le \nu^{k+1}(c) \textit{ and } i \in Q^k_{E'}(c) \\ c_i & \textit{otherwise} \end{cases}$$

where $Q^k_{E'}(c)$ is the subset of $M^k(c)$ involving the E' agents in $M^k(c)$ with the highest priority according to $\succ$, defining $E' = E - \nu^k(c) - \sum_{j \in M^k(c)} \left\lfloor \frac{E-\nu^k(c)}{m^k(c)} \right\rfloor$.

Example 7.2. ***A rule satisfying balancedness, claims monotonicity, conditional full compensation, composition down, but neither consistency nor converse consistency***. *This rule, F, can be defined as follows. Let $\sigma_1, \sigma_2 \in \Sigma^M$ be two different monotone standards such that $\sigma_1(i,x) < \sigma_1(i+1,x)$ and $\sigma_2(i,x) > \sigma_2(i+1,x)$. Then, we define the solution $F^{(\sigma_1,\sigma_2)}$ as*

$$F^{(\sigma_1,\sigma_2)}(N,E,c) = \begin{cases} D^{\sigma_1}(N,E,c) & \textit{if } |N| = 2 \\ D^{\sigma_2}(N,E,c) & \textit{otherwise} \end{cases}$$

Consider now the problem $(\{1,2,3\}, 4, (3,5,6))$, $F^{(\sigma_1,\sigma_2)}(\{1,2,3\}, 4, (3,5,6)) = (2,2,1)$; but if $S = \{2,3\}$ then $F^{(\sigma_1,\sigma_2)}(S, 3, c_S) = (1,2)$.

Example 7.3. ***A rule satisfying balancedness, composition down, consistency and converse consistency, but not conditional full compensation***. *Let $\succ: \mathbb{N} \longrightarrow \mathbb{Z}_{++}$ be an order over the set of potential agents. The rule, P, is defined as follows: First, let us assume for a while that the estate were completely divisible. Let p denote the rule that, for each problem distributes the estate proportionally to the claims. Now,* *we distinguish two types of agents: those who have already received an integer amount according to p, i.e.,* $p_i(N, E, c) \in \mathbb{Z}_+$ *(and then* $\lfloor p_i(N, E, c) \rfloor = p_i(N, E, c)$*); and those agents whose allocation is not an integer number. Let us denote by* $Q(p; N, E, c)$ *this last group of agents:* $Q(p; N, E, c) = \{j \in N : p_j(N, E, c) \notin \mathbb{Z}_+\}$*. Let* $q = |Q(p; N, E, c)|$*. In the second stage we distribute the* E' *remaining units among some agents in* $Q(p; N, E, c)$ *according to the order* $\succ$*. We give one and only one unit to each of the* E' *agents with the highest priority in* $Q(p; N, E, c)$*. Let* $Q^{\succ}(p; N, E, c)$ *be the ordered set* $Q(p; N, E, c)$ *with the restriction of* $\succ$*. Let* $Q^{\succ}_a(p; N, E, c)$ *be the set of the a first agents in* $Q^{\succ}(p; N, E, c)$*. Therefore*

$$P^{\succ}_j(N, E, c) = \begin{cases} \lfloor p_j(N, E, c) \rfloor + 1 & \text{if } j \in Q^{\succ}_{E'}(p; N, E, c) \\ \lfloor p_j(N, E, c) \rfloor & \text{otherwise} \end{cases}$$

where $E' = E - \sum_{i \in N} \lfloor p_i(N, E, c) \rfloor > 0$

Example 7.4. ***A rule satisfying claims monotonicity, composition up, consistency, but not balancedness****. Let us consider the up method,* U^σ*, where* $\sigma(1, .) < \sigma(2, .) < \sigma(3, .) < \ldots$ *This rule is called arrival rule under the under first agent 1, if she is present, second agent 2, if she is present, and so on.*

Example 7.5. ***A rule satisfying balancedness, claims monotonicity, consistency, but not composition up****: The dual rule of the rule introduced in Example 7.1.*

Appendix B. Proof of Theorems

Proof of Proposition 4.4

Let $(N, E, c) \in \mathbb{C}$ such that $N = \{1, 2\}$ and $c = (c_1, c_2)$, where $c_1 = c_2$. Let us suppose that the result is not true and for some estate E it happens that $x_1 = F_1(N, E, c) < F_2(n, E, c) - 1 = x_2 - 1$ ($x = (x_1, x_2)$). Note that in this case $x_2 \neq 0$ (otherwise $x_1 < 0$). Let $E' = 2x_1$, then, by *conditional full compensation*, $F(E', x) = (x_1, x_1)$. By *composition down*,

$$(x_1, x_1) = F(E', x) = F(E', F(E, c)) = F(E', c).$$

Let $\overline{E} \in [E', E]$, by *estate monotonicity* (implied by *composition down*), $F(E', c) \leq F(\overline{E}, c) \leq F(E, c)$, hence $F(\overline{E}, c) = (x_1, \overline{E} - x_1)$. Let $E_1 \geq E$ such that $F(E_1, c) = x_1$ and for all $\widetilde{E} > E_1$, $F(\widetilde{E}, c) > x_1$. Let us take, in particular, $\widetilde{E} = E_1 + 1$. Let $(\widetilde{x}_1, \widetilde{x}_2) = F(E_1 + 1, c)$ and $E_2 = 2\widetilde{x}_1$. Then

- By *conditional full compensation*, $F(E_2, \widetilde{x}) = (\widetilde{x}_1, \widetilde{x}_1)$. By *composition down*, $F(E_2, c) = (\widetilde{x}_1, \widetilde{x}_1)$.
- *Estate monotonicity* together with the fact that $\widetilde{x}_1 > x_1$, we obtain that $\widetilde{x}_1 = x_1 + 1$. Then, $E_2 \in [E', E]$, since $E_2 = 2\widetilde{x}_1 > 2x_1 = E'$ and $E_2 = 2\widetilde{x}_1 = 2(x_1 + 1) = (x_1 + 1) + (x_1 + 1) < x_1 + x_2 + 1 = E + 1$, i.e., $E_2 \leq E$. By the reasoning above, $F(E_2, c) = (x_1, E_2 - x_1)$.

Taking into account both facts, $(x_1 + 1, x_1 + 1) = (\widetilde{x}_1, \widetilde{x}_1) = F(E_2, c) = (x_1, E_2 - x_1)$, which is a contradiction. Therefore $|x_1 - x_2| \leq 1$, hence F satisfies balancedness.

Proof of Theorem 5.3

It is easy to check that all up monotonic methods satisfy the three properties. Conversely, let us suppose that F is a rule satisfying *conditional null compensation*, *composition up*, and *consistency*. First we define an order $\sigma \in \Sigma^M$. Afterwards we show that F coincides with U^σ.

Step 1. *Definition of the monotonic standard of comparison.* Let $\sigma \in \Sigma^M$ be defined as follows

$$x > y \Rightarrow \sigma(i, x) < \sigma(j, y)$$
$$x = y \Rightarrow [\sigma(i, x) < \sigma(j, y) \Leftrightarrow F(\{i, j\}, 1, (x, y)) = e_i].$$

It is straightforward to see that σ is complete and antisymmetric. Let us show that σ is transitive. Suppose that there exist $\{i, j, k\} \subseteq N$ such that $\sigma(i, x) < \sigma(j, y)$, $\sigma(j, y) < \sigma(k, z)$, but $\sigma(i, x) > \sigma(k, z)$. By construction, this can only happen when $x = y = z$. By definition of σ, in such a case, $F(\{i, j\}, 1, (x, y)) = e_i$, $F(\{j, k\}, 1, (x, z)) = e_j$ and $F(\{k, i\}, 1, (z, x)) = e_k$. Consider the problem $(\{i, j, k\}, 2, (x, y, z))$. It only admits three possible $\mathbb{Z}$-allocations: $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$. Suppose that $F(\{i, j, k\}, 2, (x, y, z)) = (1, 1, 0)$, by *consistency*, $F(\{i, k\}, 1, (x, z)) = e_i$, achieving in this way a contradiction with $F(\{i, k\}, 1, (x, z)) = e_k$. An analogous argument is applied if $F(\{i, j, k\}, 2, (x, y, z)) = (1, 0, 1)$, or if $F(\{i, j, k\}, 2, (x, y, z)) = (0, 1, 1)$. Therefore $\sigma(i, x) < \sigma(k, z)$, and then σ is transitive.

Step 2. *Let us prove now that $F = U^\sigma$.* By Proposition 4.2 and Lemma 4.1 it is sufficient to consider the two-claimant case. By Propositions 4.4, 4.1, and 5.2, F satisfies balancedness. We make the proof in two steps. Then, let us consider the problem $(S, E, c) \in \mathbb{C}_\mathbb{Z}$ where $S = \{i, j\} \subseteq N$. Without loss of generality we can assume that $c_i \leq c_j$. We analyze the following cases:

Case 1. If $c_i = c_j = x$ and E is even. Then, by *balancedness*, $F(S, E, c) = \left(\frac{E}{2}, \frac{E}{2}\right) = U^\sigma(S, E, c)$.

Case 2. If $c_i = c_j = x$ and E is odd. Then, by *composition up*, $F(S, E, (x, x)) = F(S, 2\lambda, (x, x)) + F(S, 1, (x, x) - F(S, 2\lambda, (x, x)))$ where $\lambda \in \mathbb{Z}$ is such that $E = 2\lambda + 1$. By definition of σ, and applying *balancedness* in $(S, 2\lambda, (x, x))$, we conclude that

$$\begin{aligned} F(S, E, c) &= (\lambda, \lambda) + U^\sigma(S, 1, (x - \lambda, x - \lambda)) \\ &= U^\sigma(S, 2\lambda, (x, x)) + U^\sigma(S, 1, (x - \lambda, x - \lambda)) \\ &= U^\sigma(S, E, c) \end{aligned}$$

Case 3. If $E \leq c_j - c_i$. By *conditional null compensation*, $F(S, E, c) = (0, E) = U^\sigma(S, E, c)$.

Case 4. If $E > c_j - c_i$. By *composition up*, $F(S, E, c) = F(S, c_j - c_i, c) + F(S, E - c_j + c_i, (c_i, c_i))$. Applying the arguments of Cases 1, 2 and 3, we have that $F(S, E, c) = U^\sigma(S, c_j - c_i, c) + U^\sigma(S, E - c_j + c_i, (c_i, c_i)) = U^\sigma(S, E, c)$.

Then, F coincides with U^σ in the two agents case, and therefore they also coincide in general.

The second part of the theorem follows from the first part shown above, Theorem 4.1, and Propositions 4.1 and 5.2.

Proof of Theorem 5.4

The first part is straightforward from Theorem 5.3 and Proposition 4.3. The second part follows from the first part, Theorem 4.1, and Propositions 4.1 and 5.2.

Proof of Theorem 5.5

Since exclusion implies conditional null compensation, the first part of the result is trivial from Theorem 5.3. Regarding to the second part, it follows from the first part, Theorem 4.1, and Propositions 4.1 and 5.2.

Proof of Theorem 5.6

The first part comes from Theorem 5.5 and Proposition 4.3. The second part follows from the first part, Theorem 4.1, and Propositions 4.1 and 5.2.

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