Multi-Agent Initiatives with Type Dependent Externalities*

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Abstract

We model situations in which a principal provides incentives to a group of agents to participate in a project (such as a social event or a commercial activity). Agents' benefits from participation depend on the identity of other participating agents. We assume bilateral externalities and characterize the optimal incentive mechanism. Using a graph-theoretic approach we show that the optimal mechanism provides a ranking of incentives for the agents, which can be described as arising from a virtual popularity tournament among the agents. One implication of our analysis is that higher levels of asymmetry of externalities enable a reduction of the principal's payment. In addition, a slight change in the externality that an agent induces on others can result in a substantial change in the payment that he receives from the principal.

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1 Introduction

By *multi-agent initiative* we refer to a venture initiated by a certain party (henceforth a principal), the success of which depends on the participation of other agents. An agent willingness to participate depends, among other factors, on the participation of others. More specifically, when an agent decides whether or not to participate she takes into account not only how many other agents are expected to participate but, more importantly, *who* is expected to participate. Finally, the principal provides incentives to the agents and has to design these incentives optimally in view of the prevailing externalities. Obvious examples of multi-agent initiatives are various hedonic activities such as throwing a party or organizing a scientific conference where an agent simply enjoys the activity with some of his peers more than with others. But there are plentiful non-hedonic examples where the type-dependent externalities are pecuniary. Consider a firm (the principal) that makes acquisition offers to several owners of other firms (agents). A successful initiative would be a gathering of enough market power by the principal firm to maximize its profits, and participation would be willingness on the part of the agents to sell. An owner's willingness to sell is also affected by the question of which of his rivals are also expected to be purchased. An owner of a mall (or the organizer of some other marketplace) who needs a certain mixture of stores in order to maximize profits may offer incentives to a set of store owners to open up a store in his mall. Store's performance is affected by the other stores in the mall, since the externalities among stores is a crucial factor in the decision on whether to participate. Another relevant example is that of a firm that is trying to sell a network technology. A buyer's (agent's) willingness to pay for the technology is affected not only by the number of buyers that are expected to adopt the technology but also by the identity of these buyers and the nature of the agent's bilateral relationship with these buyers. A similar example is that of a standardization agency that tries to introduce a new standard for a certain technology. The producers' willingness to support the initiative is clearly exposed to type-dependent externalities, and a success is considered an implementation (participation) of the standard by the firms in the market. Finally, sports club trying to recruit a set of players faces the same participation problem with type-dependent externalities. The same holds for a start up company trying to convince a group of venture capital funds to invest in their project.

The planning and execution of a multi-agent initiative consists of two some what distinct stages: the selection stage, in which the principal selects the target audience for the new venture, and the participation stage, in which the principal introduces a set of incentives in order to induce the participation of his selected target audience. While we devote most of our attention in this paper to the second stage and investigate the optimal incentive mechanism for full participation of a pre-selected audience, our results will also hinge on the selection problem that the principal faces. The externalities among agents will be given in our model by a matrix whose typical entry $w_i(j)$ represents the extent to which agent i is attracted to the initiative when agent j participates. An optimal mechanism is a vector of rewards (offered by the principal to the agents) that sustains full participation at minimal total cost (or maximal total extraction) to the principal. In characterizing the optimal mechanisms we will focus on three main questions: 1. What is the right order of incentives across agents as a function of the externalities; i.e., who should be getting a higher-powered incentives and who should be rewarded less? 2. How does the structure of externalities affect the principal's cost of sustaining full participation? 3. How does a slight change in the externality that an agent induces on the others affect his reward and the principal's benefits?

We expose a surprising connection between the answer to the first question listed above and the way in which teams and players are ranked in sports tournaments. We show that under positive externalities the incentives are determined by an interesting virtual popularity tournament among the agents. In this tournament agent i beats agent j if j is attracted by i more than i is attracted by j. In other words, agent j values the participation of agent i more than agent i values the participation of agent j. This binary relation among the agents gives rise to a network described by a directed graph. We will use some graph theory methods to characterize the optimal mechanism and discuss the connection with sports tournaments.

The idea that under positive externalities players who induce stronger externalities on other receive a higher powered incentive is supported by an interesting empirical paper by Gould et al (2005). This paper demonstrates how externalities between stores in malls affect contracts offered by the malls owners. As in our model, stores are heterogeneous in the externalities they induce on each others. Anchor stores (such as department stores, stores with national brand name, etc) generate large positive externalities by attracting most of the customer traffic to the mall, and therefore increase the sales of non anchor stores. The most noticeable characteristic of anchor contracts is that most anchors either do not pay any rent or pay only a trivial amount. On average, anchor stores occupy over 58% of the total leasable space in the mall and yet pay only 10% of the total rent collected by the mall's owner.

In answering the second question we will point to a striking feature of the optimal mechanism. It turns out that the principal's cost of incentivizing the agents increases with the level of mutuality among the agents. Put differently, the principal gains whenever the attraction between any two agents is distributed asymmetrically. Such greater asymmetry, we will show, allows the principal more leverage in exploiting the externalities. This observation has an important implication on the principal's choice of group for the initiative in the selection stage.

With regard to the third question, we show that a strict increase in the (positive) externalities among agents does not necessarily entail that the principal will be strictly better off. The structure of the optimal mechanism has some interesting implications on the way in which agents choose to affect the externalities they induce on others. We argue that the slightest change in the externality that an agent induces on the others can result in a substantial change in the payment that this agent receives from the principal. This observation suggests that interaction of the sort that we describe may give rise to a preliminary game in which agents attempt to affect the externalities they generate, so as to improve their future rewards. We briefly discuss this issue at the end of the paper. Finally, while the payment for each agent in an optimal mechanism for a given set of externalities relies on a combinatorial problem (i.e., the directed graph) we provide a simple and intuitive formula for the total payment (i.e., the cost of the principal to incentivize the agents). This formula is of significant importance for the selection problem.

This work is part of an extensive literature on multi-agent incentive mechanisms, in which externalities arise between the agents. The structure of our game, in which the principal offers a set of incentives and the agents can either accept or reject the offer, is akin to various applications introduced in the literature. These include vertical contracting models (Katz and Shapiro 1986a; Kamien, Oren, and Tauman 1992) in which the principal supplies an intermediate good, which is a fixed input (a license to use the principal's patent) to N identical downstream firms (agents), who then produce substitute consumer goods; exclusive dealing models (Rasmusen, Ramseyer, and Wiley 1991; Segal and Whinston 2000) in which the principal is an incumbent monopolist who offer exclusive dealing contracts to N identical buyers (agents) in order to deter entrance of a rival; acquisition for monopoly models (Lewis 1983; Kamien and Zang 1990; Krishna 1993) in which the principal makes acquisition offers to N capacity owners (agents), and these capacities are used to produce homogeneous consumer good and network externalities models (Katz and Shapiro 1986b).

Our general approach is closely related to the seminal papers by Segal (1999, 2003) on contracting with externalities. These papers present a generalized model for the applications mentioned above as well as others. Our approach is also related to the incentive schemes investigated by Winter (2004) in the context of organizations. We follow Segal (2003) and Winter (2004) in that we do not assume that the principal can coordinate agents on his preferred equilibrium; that is we are looking for contracts that sustain participation as a unique Nash equilibrium. Indeed, recent experimental papers (see, for example, Brandt and Cooper 2005) indicate that in an environment of positive externalities players typically are trapped in the bad equilibrium of no-participation. Furthermore, as we show in the appendix, the optimal strategy for a principal that maximizes the net benefit under the worst-case scenario would be to offer a mechanism that sustains full participation as a unique Nash equilibrium. In addition, we assume that contracts are simple and descriptive in the sense that the principal cannot provide payoffs that are contingent on the participation behavior of other agents. Notably, all the examples discussed above seem to share this feature.

Our main departure from the above-mentioned literature lies in the fact that we focus on the case of heterogeneous agents with type-dependent externalities. The papers mentioned above, indeed most of the literature, assume that externalities depend on the volume of aggregate trade and not on the identity of the agents. Our emphasis on heterogeneous agents and type-dependent externalities allows us to capture what is not only a more realistic ingredient of multilateral contracts, but the key ingredient in the surprising relation between contracting and tournaments that we expose in this paper. Identity-type externalities were used in Jehiel and Moldovanu (1996) and Jehiel, Moldovanu, and Stachetti (1996), which consider the sale of a single indivisible object by the principal to multiple heterogeneous agents using auctions, when the utilities of the agents depend on which agent ultimately receives the good.

Finally, in contrast to the above literature which typically separates between analysis of positive externalities and analysis of negative externalities, we allow both positive and negative externalities to coexist.

The rest of the paper is organized as follows. We start in Section 2 with a simple two-agent example and demonstrate the role mutuality plays in the principal's cost of incentivizing the agents. Section 3 introduces the general model and Section 4 provides the solution for the participation problem under positive externalities between the agents. In this section we examine the influence of some attributes of the multilateral externalities on the cost of incentivizing the agents with an optimal mechanism, and provide a few conclusions for the selection problem. In Section 5 we consider the solution of participation problems under negative externalities and show that agents must be fully compensated in order to sustain full participation equilibrium. Section 6 provides a solution for the most general case in which positive and negative externalities coexist in the same participation problem. In Section 7 we demonstrate how this model can be used to solve optimally the selection problem. We suggest a preliminary game in Section 8 in which agents invest effort to increase the positive externalities that they impose on others in order to raise future rewards, so that externalities become endogenous. We conclude in Section 9.

2 A Simple Two-Agent Example

The key ideas behind some of this paper's results can be illustrated by using a simple two-agent example. For this purpose, suppose that the principal wishes to induce the full participation of agents 1 and 2 in his multi-agent initiative as a unique Nash equilibrium. The agents' outside option in case they decline the offer is $c > 0$. Assume that the additional utility of agent i from participating jointly with agent j in the initiative is $w_i(j)$. The principal offers agent i a payment v_i if agent i decides to participate. We assume that the two agents act simultaneously and none of them is informed about the participation decision of the other.

Let's consider the situation in which mutual externalities $w_1(2)$ and $w_2(1)$ are strictly positive and symmetric, both agents value each other equally such that $w_1(2) = w_2(1) > 0$. How much should the principal be paying in order to induce both agents to participate in a unique Nash equilibrium? To answer this question first note that in order to avoid an equilibrium in which no one is participating, the principal must offer at least one agent, say agent 1, a reward that induces him to participate even if the other agent is expected to decline. Hence offer him a reward of c. Now the principal can offer agent 2 a reward lower than c due to the externalities between the agents. Since agent 2's payoff from participation will be $w_2(1) + v_2$ and c in case she declines, we must have that $v_2 \geq c - w_2(1)$. Hence, the optimal principal's reward for agent 2 should be $c - w_2(1)$. The total reward is thus given by $2c - w_2(1)$. Note that if we reverse the order of the payments and pay agent 2 a reward of c and agent 1 a reward of $c - w_1(2)$ the total payment of rewards of the principal doesn't change. This result is generalized in our model and states that whenever two agents value each other equally, the determination of which agent receives a higher-powered incentive has no influence on the principal's optimal cost of inducing participation. The situation is different when externalities are not fully mutual.

Consider now a situation in which agents' externalities are still positive but asymmetric, i.e., $w'_1(2) = w_1(2) + \varepsilon$ and $w'_2(1) = w_2(1) - \varepsilon$ respectively, when $\varepsilon > 0$, so that $w'_1(2) > w'_2(1)$. Similar to the argument above, promising agent 1 a payoff of c and agent 2 a payoff of $c - w'_2(1)$ will result in both agents participating in a unique equilibrium. The cost of this mechanism for the principal would be $2c - w'_2(1) = 2c - w_2(1) + \varepsilon$. However, if we reverse the order of incentives by paying agent 2 a payoff of c and agent 1 a payoff of $c - w'_1(2)$ we again sustain full participation as a unique equilibrium but with a lower cost, i.e., $2c - w'_1(2) = 2c - w_1(2) - \varepsilon$. Thus, the principal should exploit the fact that agent 1 favors 2 more than agent 2 favors 1 by giving preferential treatment to 2 and providing him with a higher incentive. In the two-agent case there are only two choices for the order of incentives. We will later provide a general result, and demonstrate that the set of contracts that minimize the principal's cost of sustaining full participation is derived from a virtual popularity tournament. Our example here also demonstrates that the principal benefits from higher asymmetry in agents' externalities (i.e., lower mutuality). Indeed, the higher ε is, the lower the principal's payment in the optimal incentive mechanism. This observation will be extended later in the paper.

If externalities are negative, i.e., $w_i(j) < 0$ and $w_j(i) < 0$, the principal has to compensate each agent for the damage caused by the participation of the other. Therefore, the optimal mechanism that induces full participation is $v_1 = c + |w_1(2)|$ and $v_2 = c + |w_2(1)|$. This feature of full compensation in the case of negative externalities will also be generalized later.

3 The Model

A participation problem is given by a triple (N, w, c) where N is a set of n agents. The structure of externalities w is an $n \times n$ matrix specifying the bilateral externalities among the agents. A typical entry $w_i(j)$ represents the extent to which agent i is attracted to the initiative when agent j is participating. In addition, agents gain no additional benefit from their own participation, and so $w_i(i) = 0$. We assume that the agents' preferences are additively separable, i.e., agent i's utility from participating jointly with a group of agents C is $\sum_{j \in C} w_i(j)$ for every $C \subseteq N$. We assume that the externality structure w is fixed and exogenous. Finally, c is the vector of the outside options of the agents. For simplicity and in a slight abuse of notation, we assume that c is constant over all agents. Our results extend trivially to the case of differential costs.

In order to induce participation the principal sets up an incentive mechanism $v = (v_1, v_2, ..., v_n)$ by which agent i receives a payoff of v_i if he decides to participate and zero otherwise. v_i are not constrained in sign and the principal can either pay or charge his agents. Given a mechanism v agents face a normal form game $G(v)$. Each agent has two possible strategies in the game: participation or default. For a given set C of participating agents, each agent in C earns $\sum_{j \in C} w_i(j) + v_i$ and every agent outside C earns zero. We assume agents' participation decisions are taken simultaneously. We say that an incentive mechanism v is *incentive-inducing* (INI) if it induces all agents to participate in a unique equilibrium of $G(v)$. Mechanism v is said to be *optimal* if in addition the sum of rewards among all INI mechanisms is minimal. We regard an optimal mechanism as a solution for the participation problem.[1]

We view the participation problem as a reduced form of the global optimization problem faced by the principal which involves both the selection of the optimal group for the initiative and the design of incentives. Specifically, let U be a (finite) universe of potential participants. For each $N \subseteq U$ let $v^*(N)$ be the total payment made in an optimal mechanism that sustains the participation of the set of agents N in a unique equilibrium. If the principal wishes to avoid the strategic uncertainty involving multiplicity of equilibria,

[1] We allow the reward-minimizing mechanism not to be INI in itself. The formal definition should be the following: v is an optimal INI mechanism if (1) there exists no INI mechanism with less total reward and (2) for any $\varepsilon > 0, \{v_i + \varepsilon\}_{i \in N}$ is an INI mechanism. The caveat is needed because rewards take continuous values.

then the maximal level of net benefit she can guarantee herself is given by the following optimization problem: $\max_{N \subseteq U}[u(N) - v^*(N)]$, where $u(N)$ is the principal's gross benefit from the participation of the set N of agents. In the appendix we show that this optimization problem is identical to the one in which the principal maximizes the net benefit under the worst-case scenario arising from the fact that she cannot coordinate the agents to her preferred equilibrium. While most of our analysis will concern with the structure of incentives within the selected set N,our results will also shed light on the selection problem.

4 Positive Externalities

In this section we describe the optimal mechanism under positive externalities, i.e., $w_i(j) > 0$ for all $i, j \in N$, such that $i \neq j$. In this case, agents are more attracted to the initiative the more other agents participate. We demonstrate how an agent's payment is influenced by the externalities that he induces on others as well as by the externalities that others induce on him. We will also refer to how changes in the structure of externalities affect the principal's welfare.

In Proposition 1 we show that the optimal mechanism is part of a general set of mechanisms characterized by the *divide and conquer*[2] property. This set of mechanisms is constructed by ordering agents in an arbitrary fashion, and offering each agent a reward that would induce him to participate in the initiative under the belief that all the agents who are before him participate and all the agents who are after him default. Due to the positive externalities, "later" agents are induced (implicitly) by the participations of others and thus can be offered smaller (explicit) incentives. More formally, the *divide and conquer* (*DAC*) mechanisms have the following structure:

$$v = (c, c - w_{i_2}(i_1), c - w_{i_3}(i_1) - w_{i_3}(i_2), ..., c - \sum_k w_{i_n}(i_k))$$

where $\varphi = \{i_1, i_2, ..., i_n\}$ is an arbitrary order of agents. We refer to this order as the ranking of the agents and say that v is a DAC mechanism

[2]See Segal (2003) and Winter (2004) for a similarly structured optimal incentive mechanism.

with respect to the ranking φ. Note that the reward for agent i_k is v_k, i.e., the k'th coordinate of the vector v. The reward for agent i_k increases along with his position in the ranking. More specifically, the higher agent i_k is located in the ranking, the higher is the payment that he is offered. Note that given mechanism v, agent i_1 has a dominant strategy in the game $G(v)$,[3] which is to participate. Given the strategy of agent i_1, agent i_2 has a dominant strategy to participate as well. In general, agent i_j has a dominant strategy to participate provided that agents i_1 to i_{j-1} participate as well. Therefore, mechanism v sustains full participation through an iterative elimination of dominated strategies. The following proposition provides a necessary condition for the optimal mechanism.

Proposition 1 If v is an optimal mechanism then it is a *divide and conquer* mechanism.

Proof. Let's assume mechanism $v = \{v_{i_1}, v_{i_2}, ..., v_{i_n}\}$ is an optimal mechanism of the participation problem (N, w, c). Hence, under mechanism v full participation is a unique Nash equilibrium. Since no-participation by all agents is not an equilibrium, at least a single agent, henceforth i_1, receives a reward equal to his outside option c. Agent i_1 chooses to participate under any profile of other agents' decisions. Given that agent i_1 participates and an equilibrium of a single participation is not possible, at least one other agent, henceforth i_2, must receive a reward weakly higher than $c - w_{i_2}(i_1)$. Since v is the optimal mechanism, i_2's reward cannot exceed $c - w_{i_2}(i_1)$, and under any profile of decisions i_2 will participate. Applying this argument iteratively on the first $k-1$ agents, at least one other agent, henceforth i_k, must be incentivized with a reward equal to $c - \sum_{j=1}^{k-1} w_k(j)$. Otherwise an equilibrium with partial participation may arise. Hence, the optimal mechanism v must satisfy the *divide and conquer* property and therefore it is a DAC mechanism under a certain ranking φ. ■

4.1 Optimal Ranking

Our construction of the optimal mechanism for the participation problem (N, w, c) relies on Proposition 1. Since the optimal mechanism is a DAC, we have to search for the DAC mechanism with the lowest sum of rewards. The ranking that corresponds to this mechanism will be called the optimal

[3] Since rewards take continuous values we assume that if an agent is indifferent then he chooses to participate.

ranking. We show that under positive externalities the optimal ranking is determined by a virtual popularity tournament among the agents in which every agent is challenged by all the other agents. The results of these matches between all pairs of agents are described by a directed graph (digraph) $G(N, A)$, where the vertices set $N = \{1, 2, ..., n\}$ represents the agents, and $A \subset N \times N$ is a binary relation on N that defines the set of arcs. Our graphs will be *simple* and *complete* digraphs. A digraph $G(N, A)$ is *simple* if $(i, i) \notin A$ for every $i \in N$ and *complete* if for every $i, j \in N$ at least $(i, j) \in A$ or $(j, i) \in A$. We refer to digraphs with such properties as **tournaments**. Note that we allowed both $(i, j) \in A$ and $(j, i) \in A$ unless $i = j$. We define the tournament $G(N, A)$ with the set of arcs A as follows:

(1) $w_i(j) < w_j(i) \iff (i, j) \in A$

(2) $w_i(j) = w_j(i) \iff (i, j) \in A \text{ and } (j, i) \in A$

The interpretation of a directed arc (i, j) in the tournament G is that agent j values mutual participation with agent i more than agent i values mutual participation with agent j. We will also use the term agent i beats agent j whenever $w_i(j) < w_j(i)$. A two-sided arc represents a fully mutual situation, i.e., $w_i(j) = w_j(i)$. In this case we will say that i is even with j and the match ends in a tie.

In our solution analysis we distinguish between acyclic and cyclic graphs. We say that a tournament is cyclic if there exists at least a single vertex v for which there is a directed path starting and ending at v.[4]

4.2 Optimal Ranking for Acyclic Tournaments

A ranking φ is said to be **consistent** with tournament $G(N, A)$ if for every pair $i, j \in N$ if i appears before j in φ then i beats j in the tournament G. In other words, if agent i is ranked higher than agent j in a consistent ranking, then agent j values agent i more than agent i values j. We start with the following, probably known, useful lemma:

Lemma 1 If tournament $G(N, A)$ is acyclic, then there exists a unique ranking that is consistent with $G(N, A)$.

Proof. First we demonstrate that there is a single node that has $n - 1$ outgoing arcs. Since the tournament is complete, directed, and acyclic there cannot be two such nodes. If we assume that such a node doesn't exist, then all nodes in G have both incoming and outgoing arcs. Since the num-

[4]By definition, if $(i, j) \in A$ and $(j, i) \in A$, then the tournament is cyclic.

ber of nodes is finite, we get a contradiction for the assumption that G is acyclic. Let's denote this node as i_1 and place its corresponding agent first in the ranking (hence this agent beats all other agents). Now let's consider the subgraph $G(N^1, A^1)$ which results from the removal of node i_1 and its corresponding arcs. Graph $G(N^1, A^1)$ is directed, acyclic, and complete and, therefore, following the previous argument, has a single node that has exactly $n-2$ outgoing arcs. We denote this node as i_2, and place its corresponding agent at the second place in the ranking. Note that agent i_1 beats agent i_2 and therefore the ranking is consistent so far. After the removal of node i_2 and its arcs we get subgraph $G(N^2, A^2)$ and consequentially node i_3 is the single node that has $n-3$ outgoing arcs in subgraph $G(N^2, A^2)$. Following this construction, we can easily observe that the ranking $\varphi = \{i_1, i_2, ..., i_n\}$ satisfies consistency among all pairs of agents and due to its construction is unique. ■

We refer to the unique consistent ranking proposed in Lemma 1 as the tournament ranking.[5] From the consistency property, if agent i is ranked above agent j in the tournament ranking, then i beats j. Moreover, each agent's location in the tournament ranking is determined by the number of his winnings. Hence, the agent ranked first is the agent who won all matches and the agent ranked last lost all matches. As we will demonstrate later a tournament ranking is not well defined if the tournament $G(N, A)$ is cyclic.

In Proposition 2 a solution for the participation problem (N, w, c) with acyclic tournaments is provided. We show that the optimal ranking is the tournament ranking, and therefore the optimal mechanism is a DAC mechanism with respect to the tournament ranking. Moreover, since the tournament ranking is unique, the optimal mechanism for the participation problem is unique as well.

The intuition behind Proposition 2 is based on the notion that if agents $i, j \in N$ satisfy $w_i(j) < w_j(i)$ then the principal should exploit the fact that j favors i more than i favors j by giving preferential treatment to i (putting him higher in the ranking) and using agent i's participation to incentivize agent j. We used similar argument for the two-agent example earlier in this paper. Applying this notion upon all pairs of agents minimizes the principal total payment to the agents. One may think of the tournament $G(N, A)$ as a set of constraints that the optimal mechanism has to satisfy that eventually

[5] The tournament ranking is actually the unique Hamiltonian Path in tournament $G(N, A)$.

leads to a ranking.

Proposition 2 Let (N, w, c) be a participation problem for which the corresponding tournament G is acyclic. Let φ be the tournament ranking, such that the optimal mechanism is given by the DAC mechanism with respect to ranking φ.

Proof. According to Proposition 1 the optimal INI mechanism is a DAC mechanism. Hence the optimal mechanism is derived from constructing the optimal ranking and is equivalent to the following optimization problem:

$$\begin{aligned}
& \min_{\{j_1, j_2, \ldots, j_n\}} \left[c + [c - w_{j_2}(j_1)] + \ldots + [c - \sum_{k=1}^{n} w_{j_n}(j_k)] \right] \\
= & \min_{\{j_1, j_2, \ldots, j_n\}} \left[n \cdot c - \left\{ \sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + \ldots + \sum_{k=1}^{n} w_{j_n}(j_k) \right\} \right] \\
= & \max_{\{j_1, j_2, \ldots, j_n\}} \left[\sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + \ldots + \sum_{k=1}^{n} w_{j_n}(j_k) \right]
\end{aligned}$$

Since no externalities are imposed on nonparticipants, the outside options of the agents have no role in the determination of the optimal mechanism. We will next show that the ranking that solves the maximization problem of the principal is the tournament ranking. Let's assume, without loss of generality, that the tournament ranking φ is the identity permutation and $W_\varphi = \sum_{k=1}^{1} w_1(k) + \sum_{k=1}^{2} w_2(k) + \ldots + \sum_{k=1}^{n} w_n(k)$. W_φ is the principal's revenue extraction due to the fact that agents enjoy positive externalities. We need to show that this maximal revenue is attained when the order of agents corresponds to the tournament order.

By way of contradiction let's assume that there exists a different ranking denoted by σ such that $W_\varphi \leq W_\sigma$. First assume that σ is obtained from φ by having two adjacent agents i and j trade places such that i precedes j in φ. By Lemma 1 and its proof, agent i beats agent j and $w_i(j) < w_j(i)$. Having these two agents trade places means that $W_\sigma = W_\varphi - w_j(i) + w_i(j)$ and we have $W_\sigma < W_\varphi$. Consider now the case in which i and j are not adjacent. Using the same argument iteratively we get again that $W_\sigma < W_\varphi$ since any substitution is a result of a series of adjacent substitutions. Consider now any arbitrary ranking σ different from φ. Since we can move from σ to φ by a finite number of swaps of the sort described above we get again that $W_\sigma < W_\varphi$. Therefore the DAC mechanism with respect to the tournament ranking is unique and optimal. ■

Proposition 2 indicates that the optimal mechanism can be viewed as follows. First the principal pays the outside option c for each of his agents. Then the agents participate in a tournament that matches each agent against all other agents. The winner of each match is the agent who imposes a higher externality on his competitor. The loser of each match pays the principal an amount equal to the benefit that he acquires from the participation of his competitor. Note that if agent i is ranked higher than agent j in the tournament then it is not necessarily the case that j pays back more than i in total. The total amount paid depends on the size of bilateral externalities and not merely on the number of winning matches. However, the higher agent i is located in the tournament, the lower is the total amount paid to the principal.

It is interesting to note that the reward for the agents is not increasing continuously as a function of the externality that each imposes on the others. It turns out that the slightest change may increase rewards significantly. This is due to the fact that a minor change in externalities may change the optimal ranking and thus affect agents' payoffs substantially.

When discussing multi-agent initiatives one possible and intuitive solution might be to reward agents according to their measure of popularity such that the most popular agents would be rewarded the most. This follows from the argument that once a popular agent agrees to participate it is easier to convince the others to join. While the term "popularity" can be defined in many ways, most of them are vague, they all come down to the quality of being widely accepted by others. In our context agent i's popularity will be $\sum_{j=1}^{n} w_j(i)$, which is the sum of externalities imposed on others by his participation. However, as we have seen, agents' rankings in the optimal mechanism are determined by something more refined than the standard definition of popularity. Agent i's position in our ranking depends on the set of peers that value agent i's participation more than i values theirs. This two-way comparison may result in a different ranking than the one imposed by a standard definition of popularity.

Example 1 *Consider a group of 4 agents with identical outside option* $c = 20$*. The externalities structure of the agents is given by matrix* w *as shown in Figure 1. The tournament* G *is acyclic and the tournament ranking is* $\varphi = (3, 1, 2, 4)$*. Consequently, the optimal mechanism is* $v = (20, 17, 14, 10)$*, which is the divide and conquer mechanism with respect to the tournament* *ranking. Note that agent* 3 *who is ranked first does not have the maximal* $\sum_{j=1}^{n} w_j(i)$.

$$w = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 4 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 5 & 2 & 0 \end{pmatrix}$$

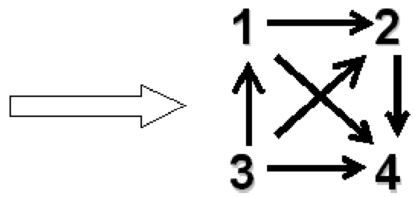

Figure 1

We note that while the principal's cost of incentivizing full participation is weakly decreasing with respect to the entries in the matrix of externalities it is not strictly decreasing. Consider a two-agent example in which $w_i(j) > w_j(i)$. If we increase $w_i(j)$ by a small ε the total payment will decrease by ε. However, if $w_j(i)$ is increased by ε, the total payment in the optimal mechanism will remain unchanged.[6] That is, the principal does not exploit the externality j induces on i since the reverse externality is greater. In general, let V be the optimal sum of payments of a participation problem (N, w, c). If $w_i(j) > w_j(i)$ then V is strictly decreasing with $w_i(j)$.

Solving for the optimal mechanism with an arbitrary matrix of externalities may prove to be quite hard as it requires going through the combinatorial problem of identifying the tournament ranking. However, we can express the total cost of the optimal mechanism with a simple and intuitive formula, one that is also a useful tool for the principal to solve the selection problem. Two crucial terms play a role in this formula: The first measures the aggregate level of externalities, i.e., $K_{agg} = \frac{1}{2} \sum\limits_{i\,j} w_i(j)$; the second measures the bilateral asymmetry among the agents and is given by $K_{asym} = \frac{1}{2} \sum\limits_{i<j} |w_i(j) - w_j(i)|$. K_{asym} stands for the extent to which agents induce mutual externalities on each other. The smaller the value of K_{asym} the higher the degree of mutuality of the agents. Proposition 3 shows that the cost of the optimal mechanism is additive and declining in these two measures.

Proposition 3 Let (N, w, c) be a participation problem and V be the principal's optimal cost of inducing participation. If the corresponding tournament $G(N, A)$ is acyclic then $V = n \cdot c - K_{agg} - K_{asym}$.

[6]As long as the inequality holds.

Proof. Without loss of generality, let's assume that the tournament ranking φ is the identity permutation. Hence, under the optimal mechanism, the principal's payment is $V = n \cdot c - \left[\sum_{j=1}^{1} w_1(j) + ... + \sum_{j=1}^{n} w_n(j)\right]$.

Let's denote $s_i(j) = s_j(i) = [w_i(j) + w_j(i)]/2$; $a_i(j) = [w_i(j) - w_j(i)]/2$ and $a_j(i) = [w_j(i) - w_i(j)]/2$. We can present K_{agg} and K_{asym} in the following manner: $K_{agg} = \frac{1}{2} \sum_{i\ j} w_i(j) = \sum_{i<\ j} \frac{w_i(j)+w_j(i)}{2} = \sum_{i<j} s_i(j)$ and $K_{asym} = \sum_{i<j} |a_i(j)|$. Since $w_i(j) = s_i(j) + a_i(j)$ we can rewrite the principal's payment as $V = n \cdot c - \left[\sum_{j=1}^{1} \{s_1(j) + a_1(j)\} + ... + \sum_{j=1}^{n} \{s_n(j) + a_n(j)\}\right] = n \cdot c - \sum_{i<j} s_i(j) - \sum_{i<j} a_i(j)$. Note that K_{asym} is always non-negative. Since we assumed that the tournament is acyclic, the tournament ranking satisfies $w_i(j) < w_j(i)$ when agent i is positioned above agent j in the ranking. Hence $a_i(j) > 0$ for all agents, K_{asym} is defined properly, and thus $V = n \cdot c - K_{agg} - K_{asym}$. ■

An interesting consequence of Proposition 3 is that the principal can benefit from a low degree of mutuality among the agents. Corollary 3.1 argues that the cost of the optimal mechanism for the principal is increasing with the level of mutuality (decreasing in the level of asymmetry in the externalities among agents). The intuition behind this result is rather simple if we consider again the virtual tournament discussed above. In each matching that takes place the principal extracts a "fine" from the losing agents. It is clear that these fines are increasing with the level of asymmetry (assuming $w_i(j)+w_j(i)$ is kept constant). Hence, a higher level of asymmetry allows the principal more leverage in exploiting the externalities. This observation may have important implications on the principal's selection stage.

Corollary 3.1 Let V be the principal's cost of the mechanism for the participation problem (N, w, c) in an acyclic tournament, then V is strictly decreasing with the asymmetry level.

Proof. Let's assume without loss of generality that the tournament ranking is the identity permutation φ. Then the minimal payment of the principal is $V = n \cdot c - \left[\sum_{j=1}^{1} w_1(j) + \sum_{j=1}^{2} w_2(j) + ... + \sum_{j=1}^{n} w_n(j)\right]$. Increasing the asymmetry level while the aggregate level of externalities remains constant requires that a given pair of agents $i, j \in N$ such that $w_i(j) < w_j(i)$ satisfy $\widehat{w_i(j)} = w_i(j) - \varepsilon$ and $\widehat{w_j(i)} = w_j(i) + \varepsilon$ when $\varepsilon > 0$. Consequently this implies that $\widehat{V} = V - \varepsilon$. This result can be immediately observed from Proposition 3. ■

If we relax the requirement of a unique Nash equilibrium and assume that the principal wishes to sustain full participation as a not necessarily unique equilibrium, then the cost of the optimal mechanism is substantially less. In the lowest cost mechanism that sustains full participation, it is easy to verify that in this partial implementation framework each agent i should be getting $v_i = c - \sum_j w_i(j)$. However, in this mechanism no-participation is an additional equilibrium. The total cost of this mechanism is $V_{multiple} = n \cdot c - \sum\limits_{i\ j} w_i(j)$. In other words, under partial implementation the principal can extract the full revenue generated by the externalities. Our emphasis on unique implementation is motivated by the fact that under most circumstances the principal cannot coordinate the agent to play his most-preferred equilibrium. Brandts and Cooper (2005) report experimental results that speak to this effect. Agents' skepticism about the prospects of the participation of others trap the group in the worst possible equilibrium even when the group is small. Nevertheless, one might be interested in evaluating the cost of moving from partial to unique implementation. Corollary 3.2 points out that this extra cost is decreasing with the level of asymmetry. More specifically, if the asymmetry level $K_{asym} = 0$ (or, equivalently, when $w_i(j) = w_j(i)$ for all pairs), it is going to be most expensive for the principal to move from partial to unique implementation. The other extreme case is when the externalities are always one-sided, i.e., for each pair of agents $i, j \in N$ satisfies that either $w_i(j) = 0$ or $w_j(i) = 0$. In this case we can show that the extra cost is zero and full implementation is as expensive as partial implementation. In general we have:

Corollary 3.2 Let V be the principal's cost of the optimal (unique implementation) mechanism for the problem (N, w, c) with acyclic tournament. Then $V - V_{multiple} = K_{agg} - K_{asym}$ and $V - V_{multiple}$ is strictly decreasing with the level of asymmetry.

Proof. The result follows immediately both from Proposition 3, where we show that $V = n \cdot c - \frac{1}{2} \sum\limits_{i\ j} w_i(j) - \frac{1}{2} \sum\limits_{i<j} |w_i(j) - w_j(i)|$, and from the fact that $V_{multiple} = n \cdot c - \sum\limits_{i\ j} w_i(j)$. Taken together, the two yield $V - V_{multiple} = \frac{1}{2} \sum\limits_{i\ j} w_i(j) - \frac{1}{2} \sum\limits_{i<j} |w_i(j) - w_j(i)| = K_{agg} - K_{asym}$. ■

4.3 Optimal Ranking of Cyclic Tournaments

In Section 4.2 we have shown that the optimal mechanism for the participation problem can be derived from a virtual tournament among the agents in which agent i beats agent j if $w_i(j) < w_j(i)$. The discussion was based however, on this tournament being acyclic. If the tournament is cyclic, then the choice of the optimal DAC mechanism (i.e., the optimal ranking) is more delicate since Lemma 1 does not hold anymore. Any ranking is prone to yield inconsistencies in the sense that there must be a pair i, j such that i is ranked above j although j beats i in the tournament.[7]

We will show that this problem is solved in very much the same way that it is solved in sports tournaments. Soccer, tennis, and football tournaments involve a sequence of bilateral matches that may turn out to yield a cyclic outcome. Various sports organizations (such as the National Collegiate Athletic Association - NCAA) nevertheless provide rankings of teams/players based on the cyclic tournament outcome. Extensive literature in operations research suggests solution procedures for determining the "minimum violation ranking" (MVR) (Kendall 1955, Ali et al. 1986, Cook and Kress 1990, Coleman 2005 are a few examples) that selects the ranking for which the number of inconsistencies is minimized. It can be shown that this ranking can be obtained as follows. Take the cyclic (directed) graph obtained by the tournament and find the smallest set of arcs such that reversing the direction of these arcs results in an acyclic graph. The desired ranking is taken to be the consistent ranking (per Lemma 1) with respect to the resulting acyclic graph.[8] In graph theory terminology this corresponds to the problem of "minimum feedback arc set". It turns out that the solution to our problem follows a very similar path. In our framework arcs are not homogeneous and so they will be assigned weights determined by the volume of externalities. Instead of looking at the smallest set of arcs for which the graph becomes acyclic, we will look for the set of arcs with minimal total weight for which the graph is acyclic. We will explain more formally how these weights are determined and how the optimal ranking is derived.

For a participation problem (N, w, c) and for each arc $(i, j) \in A$ we define by $t(i, j) = w_j(i) - w_i(j)$ the weight of the arc from i to j. Note that $t(i, j)$ is always non-negative because an arc (i, j) refers to a situation in which j favors i more than i favors j.[9] Hence $t(i, j)$ refers to the extent of the one-

[7]Consider, for example, the three-agent case where agent i beats j, agent j beats k, and agent k beats i. The tournament is cyclic and any ranking of these agents necessarily yields inconsistencies. The ranking $[i, j, k]$, for instance, yields an inconsistency involving the pair (i, k) since k beats i and i is ranked above agent k.

[8]Note that there may be multiple rankings resulting from this method.

[9]If the arc is two sided then $t(i, j) = 0$

sidedness of the externalities between the pairs of agents. If an inconsistency involves the arc (i, j), i.e., j precedes i although i beats j, the additional payment for the principal relative to the consistent ordering of the pair is $t(i, j)$.[10] For each subset of arcs $S = \{(i_1, j_1), (i_2, j_2), ..., (i_k, j_k)\}$ we define $t(S) = \sum_{(i,j)\in S} t(i, j)$, which is the total weight of the arcs in S. For each graph G and subset of arcs S we denote by G_{-S} the graph obtained from G by reversing the arcs in the subset S. Consider a cyclic graph G and let S^* be the subset of arcs that satisfies the following:

(1) G_{-S*} is acyclic

(2) $t(S^*) \leq t(S)$ for all S that satisfy the first condition.

Then, G_{-S*} is the acyclic graph obtained from G by reversing the set of arcs with the minimal total weight. Proposition 4 shows that the optimal ranking of G is the tournament ranking of G_{-S*} since the additional cost from inconsistencies, $t(S^*)$,[11] is the lowest.

Proposition 4 Let (N, w, c) be a participation problem with a cyclic tournament G. Let φ be the tournament ranking of G_{-S*}. Then, the optimal mechanism is the DAC mechanism with respect to φ.

Proof. Consider a subset of arcs S where G_{-S} is acyclic, and assume that the tournament ranking of G_{-S} is $\varphi = \{j_1, j_2, ..., j_n\}$. The payment of the principal V under the DAC mechanism with respect to φ is $V = n \cdot c - \left\{\sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k)\right\}$. Note that each $(i, j) \in S$ satisfies an inconsistency in tournament ranking φ. More specifically, if $(i, j) \in S$, then although i beats j, agent j is positioned above agent i (since arc (i, j) is reversed in G_{-S}) and the reward of agent i is reduced by $w_i(j)$. Note that $w_i(j) = w_j(i) - t(i, j)$. Let's provide the following substitution: If $(i, j) \in S$ then $w_i(j) = \widehat{w_j(i)} - t(i, j)$; otherwise $w_i(j) = \widehat{w_i(j)}$ and rewrite the principal's payment as $V = n \cdot c - \left\{\sum_{k=1}^{1} \widehat{w}_{j_1}(j_k) + ... + \sum_{k=1}^{n} \widehat{w}_{j_n}(j_k)\right\} + t(S)$. Note that $\widehat{w_i(j)} = \max(w_i(j), w_j(i))$; therefore all rankings differ only in the level of $t(S)$. Therefore, the subset S with the lowest $t(S)$ brings V to a minimum. Hence, the optimal mechanism is the DAC mechanism with

[10] Consider an inconsistency that arise from a pair of agents (i, k), when i beats k. Since agent k precedes i the payment for agent i is reduced by $w_i(k)$. However, in a consistent order of the agents (in which i precedes k) the payment for agent k is reduced by $w_k(i)$. Since $w_i(k) < w_k(i)$ the principal is forced to pay an additional cost of $w_i(k) - w_k(i)$ relative to the consistent ranking of the pair, which is equivalent to the weight $t(i, k)$.

[11] Note that each $(i, j) \in S^*$ satisfies an inconsistency in the tournament ranking of G_{-S*}.

respect to the tournament ranking of G_{-S*}. ■

Example 2 *Consider a group of 4 agents each having identical outside option $c = 20$. The externality structure and the equivalent cyclic tournament are demonstrated in Figure 2. The reversion of the arcs of both subsets $S_1^* = \{(2,4)\}$, $S_2^* = \{(1,2),(3,4)\}$ provide acyclic graphs $G_{-S_1^*}$ and $G_{-S_2^*}$ with minimal weights. The corresponding tournament rankings are $\varphi_1 = (4,3,1,2)$ and $\varphi_2 = (3,2,4,1)$. Hence, the optimal mechanisms are $v_1 = (20,13,13,12)$ and $v_2 = (20,16,10,12)$.*

$$\boldsymbol{w} = \begin{pmatrix} 0 & 1 & 2 & 5 \\ 2 & 0 & 4 & 2 \\ 1 & 2 & 0 & 7 \\ 3 & 4 & 6 & 0 \end{pmatrix}$$

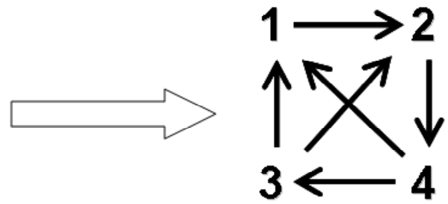

Figure 2

A participation problem is said to be *symmetric* if $w_i(j) = w_j(i)$ for all pairs $i, j \in N$. In the symmetric case, the principal cannot exploit the externalities among the agents, and the total payment made by the principal is identical for all rankings. This follows from Proposition 4 by noting that the tournament has two-way arcs connecting all pairs of agents, and so $t(i,j) = 0$ for all i, j and $t(S)$ is uniformly zero.

Corollary 5.1 When the externality structure w is symmetric then all DAC mechanisms are optimal.

It is also easy to verify that for any two agents with $w_i(j) = w_j(i)$, changing the position of i and j in the optimal ranking results in another optimal ranking.

We can now provide the analogue of Proposition 3 for the cyclic case. In this case, the optimal ranking has an additional term $K_{cyclic} = t(S^*)$ representing the cost of making the tournament acyclic.

Proposition 6 Let (N, w, c) be a participation problem. Let V be the principal's optimal cost of inducing participation. Then $V = n \cdot c - K_{agg} - K_{asym} + K_{cyclic}$.

Proof. Consider a participation problem (N, w, c) with corresponding cyclic tournament $G(N, A)$. Assume that φ is both the optimal ranking and, without the loss of generality, the identity permutation. In Proposition 3 we have demonstrated that the principal's payment in the optimal mechanism can be presented in the following way: $V = n \cdot c - \left[\sum_{j=1}^{1} s_1(j) + ... + \sum_{j=1}^{n} s_n(j)\right] - \left[\sum_{j=1}^{1} a_1(j) + ... + \sum_{j=1}^{n} a_n(j)\right]$, when $s_i(j) = s_j(i) = \frac{w_i(j)+w_j(i)}{2}$; $a_i(j) = \frac{w_i(j)-w_j(i)}{2}$ and $a_j(i) = \frac{w_j(i)-w_i(j)}{2}$. If the pair (i,j) displays an inconsistency with respect to φ, i.e., i beats j and j precedes i in φ, then $(i,j) \in S^*$ and $a_i(j) < 0$. Let's denote $\widehat{a_i(j)} = |a_i(j)|$ so that we can express the principal's payment, given the optimal ranking, in the following way: $V = n \cdot c - \sum_{i\ j} s_i(j) - \sum_{i\ j} \widehat{a_i(j)} + t(S^*)$. Therefore, we can conclude that under cyclic tournaments the principal's payment is $V = n \cdot c - K_{agg} - K_{asym} + K_{cyclic}$. ■

We note that Corollary 3.1 still holds for pairs of agents that are not in S^*. More specifically, if we decrease the level of mutuality over pairs of agents that are outside S^*, we reduce the total expenses that the principal incurs in the optimal mechanism.

5 Negative Externalities

So far we have limited our discussion to environments in which an agent's participation positively affects the willingness of other agents to participate; i.e., we assumed that externalities are positive. We now turn to the orthogonal case in which externalities are all negative. Later in Section 6 we discuss the general case of mixed externalities.

The most relevant environments of negative externalities are those of congestions. Traffic, market entry, and competition among applicants are all examples that share the property that the more agents that participate, the lower the utility of each participant is. The type-dependent property in our framework seems quite descriptive in some of these examples. In the context of competition it is clear that a more competitive candidate/firm induces a larger externality (in absolute value) than a less competitive one. It is also reasonable to assume, at least for some of these environments, that the principal desires a large number of participants in spite of the negative externalities that they induce on each other.

We show that in order to sustain full participation as a unique Nash equilibrium under negative externalities the principal has fully to compensate all agents for the participation of the others. This means that the optimal mechanism under negative externalities is not a DAC mechanism. As we have seen, positive externalities allow the principal to exploit the participation of some agents in order to incentivize others. With negative externalities this is not the case as agents' incentives to participate decline with the participation of the others. Hence it remains for the principal simply to reimburse the agents for the disutility arising from the participation of the others.

Proposition 6 Let (N, w, c) be a participation problem with negative externalities. Then the unique optimal mechanism v is given by $v_i = c + \sum_{i \neq j} |w_i(j)|$

Proof. Mechanism v induces a dominant strategy for all agents, which is to participate, since for all agents $u_i = \sum_{i=1}^{n} w_i(j) + v_i = c$, which means that it is optimal for a player to participate under the worst-case scenario in which all other players participate. Hence, participation by all players is a unique Nash equilibrium. To show that v is optimal, consider a mechanism m for which $m_i < v_i$ for some agents and $m_i = v_i$ for the rest. Let's assume by contradiction that full participation equilibrium holds under mechanism m. Consider an agent i for which $m_i < v_i$. If all other players are participating, then player i's best response is not to participate because $u_i = \sum_{i=1}^{n} w_i(j) + m_i < c$. Hence, v is a unique and optimal mechanism. ■

6 Mixed Externalities

While our solution for positive externalities builds on a virtual tournament among the agents, we demonstrated that when externalities are negative the optimal mechanism is simple and unique: agents are fully compensated for the disutility imposed by the participation of others. We are now going to deal with the general case of mixed externalities where the matrix of externalities consists both positive and negative entries. It turns out that the optimal mechanism for this case is a hybrid solution combining the structure of the optimal mechanisms in the two special cases (positive and negative externalities). Specifically, we shall show that the optimal mechanism for the mixed case can be derived by decomposing the problem into two separate problems, one with positive externalities and the other with negative externalities. The optimal mechanism for the original (mixed) problem will be obtained by adding agents' compensation payoffs to the solution of the positive participation problem. Formally:

Proposition 7 Let v be the optimal mechanism of a participation problem (N, w, c). Let (N, q, c) be an amended participation problem such that

$q_i(j) = w_i(j)$ if $w_i(j) > 0$ and $q_i(j) = 0$ if $w_i(j) < 0$, and let u be the optimal mechanism of (N, q, c). Then, $v_i = u_i + \sum_{j \in D_i} |w_i(j)|$ where $D_i = \{j \mid w_i(j) < 0 \ s.t. \ i, j \in N\}$.

Proof. Let's assume without loss of generality that the optimal ranking of the participation problem (N, q, c) is $\varphi = \{1, 2, ..., n\}$ and the optimal mechanism is u. In addition, let D_i be the set of agents who induce negative externalities on agent i, and hence $D_i = \{j \mid w_i(j) < 0 \ s.t. \ j \in N\}$ and $g_i : C \rightarrow R$ for $C \subseteq N$ the compensation function, which is defined by $g_i(D_i) = \sum_{j \in D_i} |w_i(j)|$. Finally, let v be an optimal mechanism of the mixed participation problem (N, w, c). To induce full participation as a unique equilibrium, we must rule out the "no participation" equilibrium. This is achieved by providing at least one agent with the incentive to participate under any profile of strategies. Consider the reward u_1 of the agent ranked first at φ. If $D_1 = \emptyset$ then agent 1 will participate under any set of profiles and $v_1 = u_1$. However, if $D_1 \neq \emptyset$, then agent 1 must be rewarded $v_1 = c + g_1(D_1)$ in order to enable his participation under the worst-case scenario in which only agents who negatively affect agent 1 participate. To deter a single participation equilibrium, agent 2's reward is $v_2 = u_2$ if $D_2 = \emptyset$. However, if $D_2 \neq \emptyset$, he also must be compensated for the worst-case scenario and therefore the minimal reward is $v_2 = u_2 + g_2(D_2)$. Following this construction, since agents $\{1, ..., k-1\}$ are participating the agent located at the k'th position in the optimal ranking φ will be offered $v_k = u_k$ if $D_k = \emptyset$, and $v_k = u_k + g_k(D_k)$ if $D_k \neq \emptyset$. Under this mechanism, the participation game is dominant solvable, and full participation is a unique equilibrium. In addition, the sum of rewards $V = \sum u_i + \sum |D_i|$ is optimal since mechanism u is optimal. ■

Proposition 7 implies that the virtual tournament we discussed in earlier sections plays a central role also in the mixed externalities case because it determines payoffs for the positive component of the problem. In this tournament i beats j whenever (1) $w_j(i) > 0$, and (2) $w_j(i) > w_i(j)$ (where $w_i(j)$ can be either positive or negative) We use the following example to demonstrate how the optimal mechanism is derived in the mixed externalities case.

Example 3 *Consider a group of 4 agents each having identical outside option $c = 20$. The externality structure of the agents is demonstrated by matrix w, as shown in Figure 3. The positive externality component (N, q, c) of the decomposition yields the optimal ranking $\varphi = (4, 3, 2, 1)$. The corresponding* *optimal mechanism of the positive component is* $u = (20, 16, 3, 15)$. *Adding compensations for negative externalities results in the optimal mechanism* $v = (20, 20, 4, 17)$. *Note that* $S^* = \{(1, 3)\}$.

$$w = \begin{pmatrix} 0 & 2 & -2 & 3 \\ -1 & 0 & 8 & 9 \\ -4 & 1 & 0 & 4 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

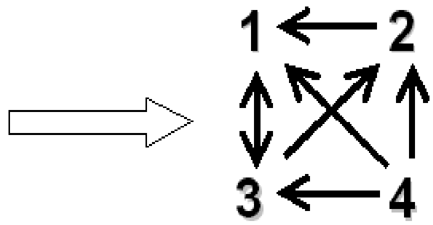

Figure 3

We conclude this section by deriving the analogous result to Propositions 3 and 6 in the case of mixed externalities. We show that the principal's cost of incentivizing his agents is decomposed in pretty much the same way as in the positive externalities case, only that now the principal has to add the compensation for the negative externalities. Specifically:

Proposition 8 Let (N, w, c) be a mixed participation problem and V be the payment of the optimal mechanism v. Let (N, q, c) be an amended participation problem such that $q_i(j) = w_i(j)$ if $w_i(j) > 0$ and $q_i(j) = 0$ if $w_i(j) < 0$. In addition, let K_{agg}, K_{asym}, and K_{cyclic} be the characteristics of the amended participation problem (N, q, c). Then, $V = n \cdot c - K_{agg} - K_{asym} + K_{cyclic} + \sum_{(i,j) \in D} |w_i(j)|$ when $D = \{(i, j) \mid w_i(j) < 0 \ s.t. \ i, j \in N\}$.

7 Group Identity and Selection

In this section we demonstrate our model as a special case in which externalities assume the values 0 and 1. We interpret it as an environment in which an agent either benefits from the participation of his peer or gains no benefit. We provide three examples of group identity in which the society is partitioned into two groups and agents have hedonic preferences over members in these groups:

(1) **Segregation** - agents benefit from participating with their own group's members and enjoy no benefit from participating with members from the other group. More specifically, consider the two groups B_1 and B_2 such that for each $i, j \in B_k$, $k = 1, 2$, we have $w_i(j) = 1$. Otherwise, $w_i(j) = 0$.

(2) **Desegregation**[12] - agents benefit from participating with the other group's members and enjoy no benefit from participating with members of their own group. More specifically, consider the two groups B_1 and B_2 such that for each $i, j \in B_k$, $k = 1, 2$, we have $w_i(j) = 0$. Otherwise, $w_i(j) = 1$.

(3) **Status** - the society is partitioned into two status groups, high and low. Each member of the society benefits from participating with each member of the high status group and enjoys no benefit from participating with members of the low status group. Formally, let B_1 be the high status group and set $w_i(j) = 1$ if and only if $j \in B_1$ (otherwise $w_i(j) = 0$).

We show that in a segregated environment the principal cost of incentives is increasing with the level of mixture, while in the anti-segregation case the principal's cost is declining with mixture. In the Status case the cost is declining with the number of agents recruited from B_1.

Proposition 9 Let (N, w, c) be a participation problem. Let n_1 and n_2 be the number of agents selected from groups B_1 and B_2 respectively such that $n_1 + n_2 = n$. Denote by $v(n_1, n_2)$ the principal cost of incentivizing agents under the optimal mechanism given that the group composition is n_1 and n_2. The following holds:

1) under Segregation $v(n_1, n_2)$ is declining in $\mid n_1 - n_2 \mid$.

2) under Desegregation $v(n_1, n_2)$ is increasing in $\mid n_1 - n_2 \mid$.

3) under Status $v(n_1, n_2)$ is increasing in n_1.

Proof. In both segregated and desegregated environments the externality structure is symmetric and, following Corollary 5.1, all rankings are optimal. Let's consider first the segregated environment. Since all rankings are optimal, a possible optimal mechanism is $v = (c, ..., c-(n_1-1), c, ..., c-(n_2-1))$. Hence, the optimal payment for the principal is $v(n_1, n_2) = n \cdot c - \sum_{l=1}^{n_1-1} l - \sum_{k=1}^{n_2-1} k = n \cdot c - \frac{n_1(n_1-1)}{2} - \frac{(n-n_1)(n-n_1-1)}{2}$. Assuming that $v(n_1, n_2)$ is continuous with n_1 then $\frac{\partial v(n_1,n_2)}{\partial n_1} = -1/2(2n_1 - 1 - 2(n - n_1) + 1) = 0$ and since $\frac{\partial^2 v(n_1,n_2)}{\partial n_1^2} < 0$ the maximum is achieved when $n_1^* = \frac{n}{2}$, which is the worst-case scenario for the principal in the segregated environment, and therefore the cost of incentivizing is declining with $\mid n_1 - n_2 \mid$. With respect to the desegregation example, a possible optimal mechanism is $v = (c, ..., c, c-n_1, ..., c-n_1)$.

[12] Like a singles party.

Therefore, the principal's payment is $v(n_1, n_2) = n \cdot c-\ (n - n_1) \cdot n_1$. Again, let's assume that $v(n_1, n_2)$ is continuous with n_1, in which case solving $\frac{\partial v(n_1,n_2)}{\partial n_1} = -n + 2n_1 = 0$ results that since $\frac{\partial^2 v(n_1,n_2)}{\partial n_1^2} > 0$ the minimum payment for the principal in the desegregated environment is received at $n_1^* = \frac{n}{2}$, and the cost of incentivizing is increasing with $|n_1 - n_2|$. Finally, in a status environment, due to the many externalities' equalities among the agents there may be numerous optimal rankings. However, since group B_1 is the more esteemed group, all agents from B_1 beat all agents from B_2; therefore agents from B_1 should precede the agents from B_2 in the optimal ranking. A possible optimal ranking is $\varphi = \{i_1, ., , , i_{n_1}, j_1, ..., j_{n_2}\}$ when $i_k \in B_1$, $j_m \in B_2$ and $1 \leq k \leq n_1$, $1 \leq m \leq n_2$. Therefore, a possible optimal mechanism is $v = (c, ..., c-(n_1-1), c-n_1, ..., c-n_1)$. The principal's payment is $v(n_1, n_2) = n \cdot c- \sum_{l=1}^{n_1-1} l - n_2 \cdot n_1 = n \cdot c - \frac{n_1(n_1-1)}{2} - (n - n_1)n_1$. Again, assuming that $v(n_1, n_2)$ is continuous with n_1, then $\frac{\partial v(n_1,n_2)}{\partial n_1} = n_1 + \frac{1}{2} - n = 0$ and since $\frac{\partial^2 v(n_1,n_2)}{\partial n_1^2} > 0$ the minimal payment for the principal is achieved at $n_1^* = n - \frac{1}{2}$. Note that $V(n_1 = n) = V(n_1 = n - 1) = n \cdot c - \frac{n(n-1)}{2}$. Therefore, we can say that the best scenario for the principal is when $n_1 = n$. Alternatively, the cost of incentivizing is increasing with n_1. ■

8 Endogenous Externalities

In this paper we analyzed a model of multi-agent initiatives with exogenous externalities, i.e., i's level of attraction of j, $w_i(j)$ is fixed and exogenous. As we saw, the matrix of bilateral externalities affects agents' payoffs. This may suggest some preliminary game in which agents invest effort to increase the positive externalities that they induce on others. For example, agents can invest in their social skills to make themselves more attractive invitees to social events. A firm may invest to increase its market share in order to improve its tournament's position in an acquisition game. Indeed, under certain circumstances such an investment may turn out to be extremely attractive as we have seen that a slight change in externalities may result in a substantial change in rewards. The preliminary game on externalities can be thought of as a network formation game similar to the ones discussed in the network formation literature (see Jackson 2003 for a comprehensive survey).

Specifically, consider a selection[13] of an optimal mechanism function that maps each matrix of externalities to a payoff vector $\Gamma : w \to \pi$ (payoffs for agents include both the transfer from the principal as well as the intrinsic benefits from participation). One can think of the matrix of externalities as a generalized network in the sense that it specifies the intensity[14] of links, in contrast to standard networks which only specify whether a link exists. If we assume that agents can increase bilateral externalities according to a given cost function we will have that the externalities become endogenous. The new game will now have two stages. The first one is a network formation game (that determines the externalities) and the second stage is the participation game. The analysis of such a game is beyond the scope of this paper but we find this task to be a fascinating next step.

9 Conclusion

In this paper we explored the implications of the type-dependent externalities in a regular principal multiple agents environment. We exposed an interesting relation between the participation problem and sports tournaments and illustrated the importance of externality asymmetry in enabling a reduction of the incentives paid by the principal. In addition, we have shown that contrary to the intuition, providing the highest incentive to the most popular agent would not necessarily produce the best results. Moreover, an increase in the positive externalities in the group would not always provide a reduction in the necessary incentives. Conditions under which such changes are efficient were given. In addition, we provided the general terms under which agents would be interested in affecting the externalities that they induce on others in order to increase their future rewards. But, also the principal can significantly benefit from such changes. In this case, the principal can affect these relations already in his selection stage of the initiative.

This model is related to a growing branch of literature - two sided markets. In this literature, a platform principal's aim is to attract buyers and sellers to use his platform (such as a credit card company) by taking advantage

[13]We refer to selection because the optimal mechanism may not be unique.

[14]See Calvo, Lasaga, and van den Nouweland (1999), Calvo-Armengol and Jackson (2001, 2001b), Goyal and Moraga (2001), and Page, Wooders, and Kamat (2001) for such models.

of the externalities they impose on each other and gain from their trades. One can easily use our framework to describe a multi sided markets, by characterizing the externalities between the sides and solving for the optimal incentive mechanism that induce the sides to participate.

Finally, we have assumed that agents make their participation decisions simultaneously (i.e., in ignorance of the participation decisions of the others). An alternative track of modeling would be to assume that the principal approaches agents sequentially and makes participation decisions publicly known. A model of this sort may potentially be able to address some other interesting issues concerning joint initiatives, including the optimal sequencing and the way it is affected by the structure of externalities. Related to this is Winter (2006), which addresses similar issues in a moral hazard model of incentives in organizations.

10 Appendix

Consider a more general participation problem in which the targeted set of participants is determined endogenously by weighting the principal's benefits from participation with the cost of providing incentives. Specifically, the principal is facing a (finite) universe U of agents from which she has to select a group $N \subset U$ for the initiative. The principal's gross benefit from the participation of N is denoted by $u(N)$. The incentive mechanism v offers each agent $i \in U$ a payoff v_i if i participates and zero if he doesn't. For a mechanism v we denote by $E(v)$ the collection of all groups that can participate in equilibrium under v, i.e., for $N \in E(v)$ if and only if there exists an equilibrium of $G(v)$ with N being the group of participants. We also denote by $C(v) = \sum_{i \in U} v_i$ the total cost of the mechanism v. Under the worst-case scenario the mechanism v will generate a net benefit of $\min_{N \in E(v)}[u(N) - C(v)]$. Hence, a principal who wishes to maximize his net benefit under the worst-case scenario will attempt to choose a mechanism v that solves $\max_v \min_{N \in E(v)}[u(N) - C(v)]$. Proposition A1 asserts that to solve this problem the principal needs to design the optimal mechanism for group N (i.e., sustaining the participation of N as a unique equilibrium as we did earlier in the paper) and then select the optimal group. Formally:

Proposition 10 For each $N \subset U$ let $v^*|N$ denote the optimal mechanism sustaining full participation as a unique equilibrium when the set of agents is in N (as in our model in Section 3). Then

$$\max_v \min_{N \in E(v)}[u(N) - C(v)] = \max_N [u(N) - C(v^*|N)]$$

Proof. Let $d(N|v) = u(N) - C(v)$ and consider $\hat{v}$ to be the mechanism that maximizes $\min_{N \in E(v)}[u(N) - C(v)]$. In addition, Let $\hat{N}$ be a group of participants in an equilibrium that has the minimal value under mechanism $\hat{v}$; hence $\hat{N} \in \arg\min_{N \in E(\hat{v})} d(N|\hat{v})$. Let $\underline{N}$ be the smallest subset of $\hat{N}$ that belongs to $E(\hat{v})$. We will demonstrate that $d(\hat{N}|\hat{v}) = d(\underline{N}|\hat{v})$.[15] If $\hat{N} = \underline{N}$ then $d(\hat{N}|\hat{v}) = d(\underline{N}|\hat{v})$ and we're done. Let $\underline{N} \subset \hat{N}$ and assume by contradiction that $d(\hat{N}|\hat{v}) > d(\underline{N}|\hat{v})$. Since $\underline{N}$ is the smallest subset of $\hat{N}$ there is no equilibrium $A \in E(\hat{v})$ such that $A \subset \underline{N}$, and therefore the equilibrium that results from the payments $\hat{v}_i$ for every $i \in \underline{N}$ must be

[15] Actually, we show that $d(\hat{N}|\hat{v}) \geq d(\underline{N}|\hat{v})$, but since $\hat{N} \in \arg\min_{N \in E(\hat{v})} d(N|\hat{v})$, the equality must hold.

unique. Since $v^*|\underline{N}$ is an optimal mechanism that sustains a unique equilibrium of $\underline{N}$ we have by definition $C(v^*|\underline{N}) \leq C(\hat{v})$ and therefore $d(\underline{N}|v^*) \geq d(\underline{N}|\hat{v})$. Note that since $E(v^*) = \{\underline{N}\}$ then $d(\underline{N}|v^*) = \min_{N\in E(v^*)} d(N|v^*)$. Therefore, $\min_{N\in E(v^*)} d(N|v^*) \geq d(\underline{N}|\hat{v}) > d(\hat{N}|\hat{v}) = \min_{N\in E(\widehat{v})} d(N|\hat{v})$, in contradiction to the assumption that $\hat{v}$ is the mechanism that maximizes $\min_{N\in E(v)} d(N|v)$. Thus, $d(\underline{N}|\hat{v}) = d(\hat{N}|\hat{v})$ and $d(\underline{N}|\hat{v}) = \min_{N\in E(\hat{v})} d(N|\hat{v})$. Moreover, since $\hat{v}$ is optimal $C(v^*|\underline{N}) = C(\hat{v})$ then v^* also maximizes $\min_{N\in E(v)} d(N|v)$. Hence, the optimization problem $\max_v \min_{N\in E(v)}[u(N) - C(v)]$ is solved with the pair (N, v) that brings the maximal value $d(N|v)$ such that v is the optimal mechanism that induces N to be a unique equilibrium, which is equivalent to $\max_N[u(N) - C(v^*|N)]$. ■

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