A NEW COALITIONAL VALUE

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Abstract

We propose a modification of the Shapley value for monotonic games with a coalition structure. The resulting coalitional value is a twofold extension of the Shapley value in the following sense: (1) the amount obtained by any union coincides with the Shapley value of the union in the quotient game; and (2) the players of the union share this amount proportionally to their Shapley value in the original game (i.e., without unions). We provide axiomatic characterizations of this value close to those existing in the literature for the Owen value and include applications to coalition formation in bankruptcy and voting problems.

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1 Introduction

The assessment of the strategic position of each player in any game is a main objective of the cooperative game theory, since it can be applied to e.g. sharing costs or profits in economic problems or measuring the power distribution in voting systems. The Shapley value (Shapley [21], also in Roth [20]) is the best known concept in this respect, and its axiomatic presentation introduced a new, elegant style in game theory and opened a fruitful research line.

Forming coalitions is a most natural behavior in cooperative games, and the evaluation of the consequences that derive from this action is of great interest to game theorists and practitioners. Games with a coalition structure were first considered by Aumann and Drèze [6], who extended the Shapley value to this new framework in such a manner that the game really splits into subgames played by the unions

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isolatedly from each other, and every player receives the payoff allocated to him by the Shapley value in the subgame he is playing within his union.

A second approach was adopted by Owen [18] (also in Owen [19]), when introducing and characterizing axiomatically a coalitional value called now Owen value. In this case, the unions play a quotient game among themselves, and each one receives a payoff which, in turn, is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by the Shapley value. In addition to the initial one, many other axiomatic characterizations of the Owen value can be found in the literature (Hart and Kurz [14], Winter [26], Amer and Carreras [4] and [5], Vázquez–Brage et al. [24], Hamiache [12], or Albizuri [1], among others).

We shall restrict our attention to monotonic games, which form a central class of cooperative games. For each finite player set N, the set of monotonic games on N possesses a cone structure, i.e., it is closed under the linear operations $v + w$ and λv for $\lambda \geq 0$, and a lattice structure with respect to the standard operations $v \vee w$ and $v \wedge w$, and it is closed under the passing to quotient games and subgames. The class includes, among others: the null game; all simple games (where values are interpreted as power indices); specifically, all unanimity games, which form a useful basis of the space of all games on each given player set; all nonnegative superadditive games; and hence all nonnegative convex games, where the Shapley value belongs to the core due to convexity. The class shows a satisfactory regular behavior with respect to many notions of the cooperative game theory. For example, any value exclusively based on marginal contributions and/or internal proportionalities is nonnegative on this class; in particular, it is ready to act as a power index on the subclass of simple games.

In this paper, we consider another extension of the Shapley value, only for monotonic games with a coalition structure. As in the case of the Owen value, the Shapley value is used twice: first, in the quotient game, where each union receives its Shapley value in this game (this former step being therefore common to the Owen value and the new value proposed here); second, within each union, since we contend that all players of each union should profit equally by joining it and hence they obtain payoffs proportional to their Shapley value in the original game.

The organization is then as follows. In Section 2, the framework of the paper is stated and a minimum of preliminaries is provided. In Section 3, we motivate and define the proportional coalitional Shapley value and discuss its properties. The axiomatic approach is presented in Section 4, where we supply several axiomatic characterizations for this new coalitional value. In Section 5, the value is applied to discuss the coalition formation in bankruptcy and voting problems, including a numerical instance in each case. Finally, Section 6 collects some discussion and the conclusions.

2 Preliminaries

Although the reader is assumed to be generally familiar with the cooperative game theory, we recall here some basic notions. If S is any finite set, $s = |S|$ will usually denote its cardinality.

2.1 Games and values

Let $N = \{1, 2, \ldots, n\}$ be a finite set of *players* and 2^N be the set of its *coalitions* (subsets of N). A *cooperative game* (from now on, simply a *game*) on N is a function $v : 2^N \to \mathbb{R}$, which assigns a real number $v(S)$ to each coalition $S \subseteq N$ and satisfies $v(\emptyset) = 0$. [1] A player i is a *null player* in v if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$. Two players i, j are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T \subseteq N$. As was announced in the introduction, in the sequel we will restrict our attention mainly to this class of games: $\mathcal{G}_N^+$ will denote the cone of monotonic games on a given N and $\mathcal{G}^+$ the class of all monotonic games (arbitrary N). A monotonic game v is *simple* if $v(S) = 0$ or $v(S) = 1$ for every $S \subseteq N$. The (sub)lattice of simple games on a given N will be denoted by $\mathcal{S}G_N$, and by $\mathcal{S}G$ the class of all simple games (arbitrary N). In particular, a simple game v is called a *weighted majority game* if there are nonnegative *weights* $w_1, w_2, \ldots, w_n$ and a *quota* $q > 0$ such that $v(S) = 1$ iff $\sum_{i \in S} w_i \geq q$, and $v(S) = 0$ otherwise. We then write $v \equiv [q; w_1, w_2, \ldots, w_n]$.

For every nonempty coalition $T \subseteq N$, the *unanimity game* $u_T \in \mathcal{S}G_N$ is defined by $u_T(S) = 1$ if $T \subseteq S$ or else $u_T(S) = 0$. Every game $v \in \mathcal{G}_N^+$ can be uniquely written as a linear combination of unanimity games using the Harsanyi dividends [13]:

$$v = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T u_T, \quad \text{where} \quad \alpha_T = \sum_{S \subseteq T} (-1)^{t-s} v(S). \tag{1}$$

By a *value* on $\mathcal{G}^+$ we will mean a map f that assigns to every game $v \in \mathcal{G}_N^+$ a vector $f[v] \in \mathbb{R}^N$ with components $f_i[v]$ for all $i \in N$. For example, the *Shapley value* φ (Shapley [21]) is defined, if $v \in \mathcal{G}_N^+$ and $i \in N$, by

$$\varphi_i[v] = \sum_{S \subseteq N \backslash \{i\}} \frac{1}{n\binom{n-1}{s}} [v(S \cup \{i\}) - v(S)].$$

The Shapley value is the only value on $\mathcal{G}^+$ that satisfies the following properties:

- *Efficiency:* $\sum_{i \in N} \varphi_i[v] = v(N)$ for all $v \in \mathcal{G}_N^+$.
- *Null player property:* if i is null in v then $\varphi_i[v] = 0$. [2]
- *Symmetry:* if i, j are symmetric in v then $\varphi_i[v] = \varphi_j[v]$.
- *Additivity:* $\varphi[v + w] = \varphi[v] + \varphi[w]$ for all $v, w \in \mathcal{G}_N^+$.

The proof follows the same guidelines as Shapley's [21] original proof [3] by applying, when using additivity in the uniqueness proof, the following relationship among monotonic games derived from (1):

[1] When mentioning a game v we will often refer implicitly to the corresponding set of players N on which it is defined. This will allow us to alleviate the notation .

[2] In fact, if v is monotonic then $\varphi_i[v] > 0$ iff i is a nonnull player in v, so that the null player property could be stated as an "iff" condition .

[3] Referred to φ as a value on $\mathcal{G}_N$, the vector space of all cooperative games on N.

$$v + v^- = v^+ \quad \text{if} \quad v^+ = \sum_{T:\, \alpha_T > 0} \alpha_T u_T \quad \text{and} \quad v^- = \sum_{T:\, \alpha_T < 0} -\alpha_T u_T. \tag{2}$$

2.2 Games with a coalition structure

Let $N = \{1, 2, \ldots, n\}$. Each partition $P = \{P_1, P_2, \ldots, P_m\}$ of N is a *coalition structure* on N, each P_k will be a *union*, and $P(N)$ will denote the set of all coalition structures on N. The so–called *trivial coalition structures* are $P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$, where each union is a singleton, and $P^N = \{N\}$, where the grand coalition forms. If $j \in P_k \in P$ then P_{-j} will denote the partition obtained from P when player j leaves union P_k and becomes isolated, i.e.,

$$P_{-j} = \{P_h \in P : h \neq k\} \cup \{P_k \backslash \{j\}, \{j\}\}.$$

A *game with a coalition structure* is a pair $[v; P]$ where $v \in \mathcal{G}_N$ and $P \in P(N)$ for some N. We denote by $\mathcal{G}_N^{+cs}$ the set of monotonic games with a coalition structure on a given N and by $\mathcal{G}^{+cs}$ the class of all monotonic games with a coalition structure (arbitrary N).

If $[v; P] \in \mathcal{G}_N^{+cs}$ the *quotient game* $v^P \in \mathcal{G}_M^+$ is the cooperative game played by the unions, or, rather, by the set $M = \{1, 2, \ldots, m\}$ of their representatives, as follows [4]:

$$v^P(R) = v(\bigcup_{r \in R} P_r) \quad \text{for all} \quad R \subseteq M.$$

(In the trivial case $P = P^n$ we will identify $M = N$, $k = i$ if $P_k = \{i\}$, and $v^P = v$.) A union P_k will be a *null union in* $[v; P]$ if k is a null player in v^P, [5] and two unions P_k, P_h will be *symmetric unions in* $[v; P]$ if k, h are symmetric players in v^P.

By a *coalitional value* on $\mathcal{G}^{+cs}$ we will mean a map g that assigns to every pair $[v; P] \in \mathcal{G}_N^{+cs}$ a vector $g[v; P] \in \mathbb{R}^N$ with components $g_i[v; P]$ for all $i \in N$. For example, the *Owen value* Φ (Owen [18]) is defined, if $[v; P] \in \mathcal{G}_N^{+cs}$ and $i \in N$, by

$$\Phi_i[v; P] = \sum_{R \subseteq M \backslash \{k\}} \sum_{T \subseteq P_k \backslash \{i\}} \frac{1}{mp_k} \frac{1}{\binom{m-1}{r}} \frac{1}{\binom{p_k-1}{t}} [v(Q \cup T \cup \{i\}) - v(Q \cup T)],$$

where P_k is the union such that $i \in P_k$ and $Q = \cup_{r \in R} P_r$. The Owen value is the only coalitional value on $\mathcal{G}^{+cs}$ that satisfies the following properties:

- *Efficiency:* $\sum_{i \in N} \Phi_i[v; P] = v(N)$ for all $[v; P] \in \mathcal{G}_N^{+cs}$.
- *Null player property:* if i is null in v then $\Phi_i[v; P] = 0$ for all $P \in P(N)$. [6]
- *Symmetry within unions:* if $i, j \in P_k$ are symmetric in v then $\Phi_i[v; P] = \Phi_j[v; P]$.

[4] Recall footnote 1. While v refers to N, v^P will refer to M.

[5] For example, if all $i \in P_k$ are null in v then P_k is null in $[v; P]$. The converse is not true.

[6] Although the game is assumed to be monotonic, it is not true in general that $\Phi_i[v; P] \neq 0$ when i is a nonnull player in v (cf. footnotes 2 and 9).

- *Symmetry in the quotient game:* if P_k, P_h are symmetric unions in $[v;P]$ then

$$\sum_{i\in P_k}\Phi_i[v;P]=\sum_{j\in P_h}\Phi_j[v;P].$$

- *Additivity:* $\Phi[v+w;P]=\Phi[v;P]+\Phi[w;P]$ for all $v,w\in\mathcal{G}_N^+$ and all $P\in P(N)$.

The proof is similar to Owen's [18] original proof [7] using again relationship (2).

A coalitional value g on $\mathcal{G}^{+cs}$ is a *coalitional Shapley value* if $g[v;P^n]=\varphi[v]$ for all $v\in\mathcal{G}_N^+$. For example, so is the Owen value. Then the Owen value is also the only *coalitional Shapley value* on $\mathcal{G}^{+cs}$ that satisfies the following properties:

- *Quotient game property:* for all $[v,P]\in\mathcal{G}^{+cs}$ and all $P_k\in P$

$$\sum_{i\in P_k}\Phi_i[v;P]=\Phi_k[v^P;P^m].$$

- *Balanced contributions within unions:* for all $[v,P]\in\mathcal{G}^{+cs}$, $P_k\in P$ and $i,j\in P_k$

$$\Phi_i[v;P]-\Phi_i[v;P_{-j}]=\Phi_j[v;P]-\Phi_j[v;P_{-i}].$$

This second axiomatic characterization was first discovered by Vázquez–Brage et al. [24] when dealing with coalitional values on $\mathcal{G}_N^{cs}$. In this case, the proof for monotonic games with a coalition structure is completely analogous to theirs.

At this point we would like to add a comment, quoted from Vázquez–Brage et al. [24], about the property of balanced contributions within unions:

> *This property states that, if two different players i and j are in the same a priori union, the loss (or gain) that player i inflicts on player j when he leaves the union is the same as the loss (or gain) inflicted on player i when player j leaves the union. This property reflects the idea that all players in a union should profit equally by joining the union and hence that, in particular, it cannot be the case that one specific player receives all the benefits generated by the formation of the union.*

3 A new value for monotonic games with a coalition structure

In this section we define a new coalitional value for monotonic games. We will call to it "proportional coalitional Shapley value". [8]

Let $[v;P]\in\mathcal{G}^{+cs}$ and $P=\{P_1,P_2,\dots,P_m\}$. We derive the payoff that a player $i\in P_k$ will obtain from the following two–step procedure. First, the unions are

[7] Referred to Φ as a coalitional value on $\mathcal{G}_N^{cs}$, the set of all cooperative games with a coalition structure on N.

[8] Incidentally, we wish to mention that this value was first defined and applied, only as a power index (that is, for simple games), by one of us in the early eighties (Carreras and Martín Rioja [9] and [10]). Independently re–discovered by Alonso–Meijide recently, it will be given here a complete treatment and a theoretical foundation.

assumed to play the quotient game v^P and get the payoff allocated by the Shapley value to each one of them in this game: union P_k gets $\varphi_k[v^P]$. Then, we assume that this amount is shared among the players of P_k proportionally to their Shapley value in the original game v. Thus, the allocation that player $i \in P_k$ receives is

$$\varphi_k[v^P]\frac{\varphi_i[v]}{\sum_{j\in P_k}\varphi_j[v]} \ .$$

Of course, due to the monotonicity of v, we have $\sum_{j\in P_k}\varphi_j[v] > 0$, and hence the procedure works, unless all players of P_k are null in v (cf. footnotes 2 and 5). In this exceptional case, k is null in v^P whence $\varphi_k[v^P] = 0$ and the allocation is undefined. Since, more generally, $\varphi_k[v^P] = 0$ iff k is null in v^P (even though not all players of P_k are null in v), the exceptional case can be consistently solved by allocating 0 to all $i \in P_k$ whenever k is null in v^P. We formalize the entire procedure as follows.

Definition 3.1 The *proportional coalitional Shapley value* π is the coalitional value on $\mathcal{G}^{+cs}$ defined, if $[v; P] \in \mathcal{G}^{+cs}$ and $i \in P_k$, by

$$\pi_i[v;P] = \begin{cases} \varphi_k[v^P]\dfrac{\varphi_i[v]}{\sum_{j\in P_k}\varphi_j[v]} & \text{if } P_k \text{ is nonnull in } [v;P], \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Notice that, since we are dealing with monotonic games only, if neither i (in v) nor P_k (in $[v; P]$) are null then $\pi_i[v; P] > 0$. More generally, $\pi[v; P] \geq 0$ for all $[v; P] \in \mathcal{G}^{+cs}$. We will call *nonnegativity* to this property, which is shared by the Shapley value ($\varphi[v] \geq 0$ for all $v \in \mathcal{G}^+$) and also by the Owen value ($\Phi[v; P] \geq 0$ for all $[v; P] \in \mathcal{G}^{+cs}$).

The proportional coalitional Shapley value reflects the assumption that, once a union has obtained its payoff, the players of such union should benefit equally from joining it, in the sense that their payoffs should keep the proportions between the allocations given by the Shapley value in the original game: technically, if $i, j, \ldots$ are nonnull players in a union P_k then

$$\frac{\pi_i[v;P]}{\varphi_i[v]} = \frac{\pi_j[v;P]}{\varphi_j[v]} = \cdots = c_k, \tag{4}$$

c_k being a constant such that $c_k = 0$ if P_k is null in $[v; P]$ and $c_k > 0$ otherwise.

In particular, the proportional coalitional Shapley value leaves invariant important strength relationships between players in the original game. Indeed, let $i, j \in P_k$. If they are symmetric in v (so that $\varphi_i[v] = \varphi_j[v]$), they will get equal payoffs at the end of the process, i.e., $\pi_i[v; P] = \pi_j[v; P]$; if, instead, they are asymmetric, in the sense that $\varphi_i[v] \neq \varphi_j[v]$, they will still get different payoffs: $\pi_i[v; P] \neq \pi_j[v; P]$ (unless union P_k is null in $[v; P]$, in which case $\pi_i[v; P] = 0 = \pi_j[v; P]$).

From a practical viewpoint, it is worthy mentioning that the calculus of $\pi[v; P]$ requires only computing $\varphi[v]$ and $\varphi[v^P]$. The following example will illustrate this.

Example 3.2 Let us take $n = 6$ and consider $[v; P]$ where $v \equiv [9; 4, 3, 3, 3, 2, 2]$ is a weighted majority game and $P = \{\{1, 2\}, \{3, 5\}, \{4, 6\}\}$. Then, $v^P \equiv [9; 7, 5, 5]$,

$$\varphi[v] = (18/60, 10/60, 10/60, 10/60, 6/60, 6/60),$$
$$\varphi[v^P] = (1/3, 1/3, 1/3),$$

and therefore

$$\pi[v; P] = (3/14, 5/42, 5/24, 5/24, 1/8, 1/8).$$

Notice that, e.g., players 3 and 5 take joint profit from forming a union in presence of the other two unions, but their final payoffs keep the ratio 5:3 found when evaluating the original game.

The main properties of the proportional coalitional Shapley value will play an essential role in the next section, within the comprehensive framework of an axiomatic approach. However, we find it convenient to state and discuss them here and delaying their proofs until that section.

The first property is that the proportional coalitional Shapley value coincides with the Shapley value under both trivial coalition structures (and is, in particular, a coalitional Shapley value):

$$\pi[v; P^n] = \varphi[v] \quad \text{and} \quad \pi[v; P^N] = \varphi[v] \qquad \text{for all } v \in \mathcal{G}_N^+.$$

Both properties are straightforward to check.

Moreover, the proportional coalitional Shapley value satisfies standard properties already recalled for the Owen value in Section 2: efficiency, symmetry within unions, symmetry in the quotient game, and the quotient game property.

A difference arises in the case of the null player property. While it is a simple implication for the Owen value, it can be strengthened for the proportional coalitional Shapley value: the allocation to a player belonging to a union vanishes iff the player is null (in the original game) or the union is null (in the game with a coalition structure, i.e., as a player in the quotient game). [9]

The balanced contributions property is not satisfied by the proportional coalitional Shapley value. It must be replaced by a "proportionality within unions": for all $[v; P] \in \mathcal{G}_N^{+cs}$, $P_k \in P$ and $i, j \in P_k$,

$$\pi_i[v; P]\pi_j[v; P^n] = \pi_j[v; P]\pi_i[v; P^n].$$

It is interesting to note that, since π is a coalitional Shapley value and the null player property for φ is, actually, an "iff" condition (footnote 2), the proportionality within unions gives, if $i, j \in P_k$ are nonnull in v,

$$\frac{\pi_i[v; P]}{\pi_i[v; P^n]} = \frac{\pi_j[v; P]}{\pi_j[v; P^n]}. \tag{5}$$

[9] The Owen value satisfies the "if" but not the "only if" part of this stronger property: in Example 3.2 we find $\Phi[v; P] = (1/3, 0, 1/6, 1/6, 1/6, 1/6)$, so that player 2 is nonnull in v and union P_1 is nonnull in $[v; P]$ but $\Phi_2[v; P] = 0$.

In fact, since π is a coalitional Shapley value, Eq. (5) leads us back to Eq. (4). This property also admits a comparison with the balanced contributions property of the Owen value: while this latter is written in an additive form, the proportionality within unions adopts a multiplicative aspect. Thus, both properties share the spirit already expressed in our quotation at the end of the previous section: they state that the allocations obtained by any two players of a given union are in some manner related to their payoffs in the original game. [10]

Finally, neither the classical additivity holds for the proportional coalitional Shapley value. It must be replaced by a more cumbersome statement, which needs some previous explanation. Let us define [11], for all $i \in N$, if $i \in P_k \in P$,

$$h_i[v;P] = \begin{cases} \pi_i[v;P] \sum_{j \in P_k} \pi_j[v;P^n] \;/\; \pi_k[v^P;P^m] & \text{if } P_k \text{ is nonnull in } [v;P], \\ \pi_i[v;P^n] & \text{otherwise.} \end{cases}$$

Since π is a coalitional Shapley value, the coefficient of $\pi_i[v;P]$ in this expression when P_k is nonnull becomes

$$\frac{\sum_{j \in P_k} \pi_j[v;P^n]}{\pi_k[v^P;P^m]} = \frac{\sum_{j \in P_k} \varphi_j[v]}{\varphi_k[v^P]}$$

and therefore this fraction might be considered as a (multiplicative) *profit inverse index*, always positive, for any nonnull union P_k (it does not make sense for null

[10] It is worthy mentioning here that the Owen value satisfies a similar property of "proportionality for reduced games within unions" that could replace the property of balanced contributions within unions, thus providing a *new characterization of the Owen value*. Indeed, following Owen [18], let $[v;P]$ and $P_k \in P$ be given. For each $S \subseteq P_k$, let $\overline{v}_S^P$ be the game on M defined by

$$\overline{v}_S^P(R) = v\Big[\big(\bigcup_{r \in R} P_r\big) \backslash (P_k \backslash S)\Big] \quad \text{for each} \quad R \subseteq M.$$

This game is the modification of the standard quotient game v^P when S replaces union P_k, as if the players of $P_k \backslash S$ were temporarily inactive. The *reduced game* of v on P_k, denoted by w_k, is then given by

$$w_k(S) = \varphi_k[\overline{v}_S^P] \quad \text{for each} \quad S \subseteq P_k.$$

If, moreover, P^{p_k} is the trivial partition of P_k into singletons, the property of proportionality for reduced games within unions states that, for each P_k and all $i, j \in P_k$,

$$\Phi_i[v;P]\Phi_j[w_k;P^{p_k}] = \Phi_j[v;P]\Phi_i[w_k;P^{p_k}],$$

which recalls our proportionality within unions. In fact, the Owen value admits being defined similarly to the proportional coalitional Shapley value:

$$\Phi_i[v;P] = \begin{cases} \varphi_k[v^P] \dfrac{\varphi_i[w_k]}{\sum_{j \in P_k} \varphi_j[w_k]} & \text{if } P_k \text{ is nonnull in } [v;P], \\ 0 & \text{otherwise.} \end{cases}$$

[11] Notice that the expression for the nonnull case is well defined since π is a coalitional Shapley value and hence $\pi_k[v^P;P^m] = \varphi_k[v^P] > 0$ by footnote 2.

unions, but neither it is interesting in this case). A profit inverse index equal to 1 means that forming the union does not change the payoffs of its members in the original game (for example, whenever $P = P^n$). If the index is lesser than 1 then the players of the union get profit from joining, whereas an index greater than 1 implies that the union damages its members and is therefore not interesting to them.

Then it holds that, for all $v, w \in \mathcal{G}_N^{+cs}$ and all $P \in P(N)$,

$$h[v+w;P] = h[v;P] + h[w;P], \tag{6}$$

and we call to this property "weighted additivity". The reason is that, by using the definition of function h and the fact that π is a coalitional Shapley value, if P_k is nonnull in $[v;P]$ and $[w;P]$, and hence in $[v+w;P]$ [12], then Eq. (6) yields, for all $i \in P_k$,

$$\pi_i[v;P]\frac{\sum_{j\in P_k}\varphi_j[v]}{\varphi_k[v^P]} + \pi_i[w;P]\frac{\sum_{j\in P_k}\varphi_j[w]}{\varphi_k[w^P]} = \pi_i[v+w;P]\frac{\sum_{j\in P_k}\varphi_j[v+w]}{\varphi_k[(v+w)^P]}, \tag{7}$$

a kind of "weighted" additivity where the weights are the profit inverse indices. Instead, if P_k is null in $[v;P]$ and $[w;P]$, and hence in $[v+w;P]$, we simply get

$$\pi_i[v;P^n] + \pi_i[w;P^n] = \pi_i[v+w;P^n], \tag{8}$$

classical —i.e., not weighted— additivity for $P = P^n$, equivalent in turn to

$$\varphi_i[v] + \varphi_i[w] = \varphi_i[v+w].$$

Thus, Eqs. (7) and (8) enable us to gain an insight into the meaning of the weighted additivity property stated in Eq. (6).

4 Axiomatic approach

We first provide in this section two quite different axiomatic characterizations of the proportional coalitional Shapley value on G^{+cs}. We state and prove them and show, next, the independence of both axiomatic systems. Later on, we discuss the simple game case.

Let us consider the following properties for a coalitional value g defined on $\mathcal{G}^{+cs}$.

- A0. *Nonnegativity:* $g[v;P] \geq 0$ for all $[v;P] \in \mathcal{G}^{+cs}$.
- A1. *Coalitional Shapley value property:* $g[v;P^n] = \varphi[v]$ for all $v \in \mathcal{G}_N^{+}$.
- A2. *Efficiency:* $\sum_{i\in N} g_i[v;P] = v(N)$ for all $[v;P] \in \mathcal{G}_N^{+cs}$.

[12] For monotonic games, a player is null in $v+w$ iff so is in v and w. Since $(v+w)^P = v^P + w^P$, a union P_k is then null in $[v+w;P]$ iff it is null in both $[v;P]$ and $[w;P]$.

A3. *Null player strong property:* if $[v; P] \in \mathcal{G}^{+cs}$ and $i \in P_k$ then $g_i[v; P] = 0$ iff i is null in v or P_k is null in $[v; P]$. [13]

A4. *Symmetry within unions:* if $i, j \in P_k$ are symmetric in v then $g_i[v; P] = g_j[v; P]$.

A5. *Symmetry in the quotient game:* if P_k, P_h are symmetric in $[v; P]$ then

$$\sum_{i \in P_k} g_i[v; P] = \sum_{j \in P_h} g_j[v; P].$$

A6. *Weighted additivity:* for all $v, w \in \mathcal{G}_N^+$ and all $P \in P(N)$

$$h[v + w; P] = h[v; P] + h[w; P]$$

where, for all $i \in N$, if $i \in P_k \in P$,

$$h_i[v; P] = \begin{cases} g_i[v; P] \sum_{j \in P_k} g_j[v; P^n] \; / \; g_k[v^P; P^m] & \text{if } P_k \text{ is nonnull in } [v; P], \\ g_i[v; P^n] & \text{otherwise,} \end{cases}$$

and analogous definitions hold for $h_i[w; P]$ and $h_i[v + w; P]$. [14].

A7. *Quotient game property:* for all $[v; P] \in \mathcal{G}_N^{+cs}$ and $P_k \in P$

$$\sum_{i \in P_k} g_i[v; P] = g_k[v^P; P^m].$$

A8. *Proportionality within unions:* for all $[v; P] \in \mathcal{G}_N^{+cs}$, $P_k \in P$ and $i, j \in P_k$

$$g_i[v; P] g_j[v; P^n] = g_j[v; P] g_i[v; P^n].$$

Theorem 4.1 *There is one and only one coalitional value on $\mathcal{G}^{+cs}$ that satisfies efficiency (A2), the null player strong property (A3), symmetry within unions (A4), symmetry in the quotient game (A5), and weighted additivity (A6). It is the proportional coalitional Shapley value π.*

Proof. (Existence) It suffices to check that the proportional coalitional Shapley value satisfies the properties. Let $[v; P] \in \mathcal{G}_N^{+cs}$.

A2. (Efficiency) From Eq. (3) it follows that, for all $P_k \in P$,

$$\sum_{i \in P_k} \pi_i[v; P] = \varphi_k[v^P].$$

[13] In particular, k is null in v^P iff $g_k[v^P; P^m] = 0$. (The qualifying "strong" is included in the name of the property in order to distinguish it from the usual one.)

[14] According to footnote 13, function h is well defined in the nonnull case since $g_k[v^P; P^m] \neq 0$

By the efficiency of φ,

$$\sum_{i\in N}\pi_i[v;P]=\sum_{k\in M}\sum_{i\in P_k}\pi_i[v;P]=\sum_{k\in M}\varphi_k[v^P]=v^P(M)=v(N).$$

A3. (Null player strong property) According again to Eq. (3) and using footnote 2, it is clear that $\pi_i[v;P]$ vanishes iff i is null in v or k is null in v^P.

A4. (Symmetry within unions) Let $i,j\in P_k$ be symmetric players in v. If P_k is nonnull in $[v;P]$, Eq. (3) and the symmetry of the Shapley value yield $\pi_i[v;P]=\pi_j[v;P]$. If, instead, P_k is null in $[v;P]$ then $\pi_i[v;P]=0=\pi_j[v;P]$.

A5. (Symmetry in the quotient game) Let P_k,P_h be null in $[v;P]$. Then

$$\sum_{i\in P_k}\pi_i[v;P]=0=\sum_{j\in P_h}\pi_j[v;P].$$

Now let P_k,P_h be nonnull but symmetric in $[v;P]$. Then, using Eq. (3) and the symmetry of the Shapley value in the quotient game,

$$\sum_{i\in P_k}\pi_i[v;P]=\varphi_k[v^P]=\varphi_h[v^P]=\sum_{j\in P_h}\pi_j[v;P].$$

A6. (Weighted additivity) Let $i\in P_k$. (a) If P_k is nonnull in $[v;P]$ then, using that π is a coalitional Shapley value, it follows that

$$h_i[v;P]=\pi_i[v;P]\,\frac{\sum\limits_{j\in P_k}\pi_j[v;P^n]}{\pi_k[v^P;P^m]}=\varphi_k[v^P]\frac{\varphi_i[v]}{\sum\limits_{j\in P_k}\varphi_j[v]}\,\frac{\sum\limits_{j\in P_k}\varphi_j[v]}{\varphi_k[v^P]}=\varphi_i[v].$$

(b) If P_k is null in $[v;P]$ we also have $h_i[v;P]=\varphi_i[v]$.

Now, let $[w;P]\in\mathcal{G}_N^{+cs}$ and $i\in P_k$. Independently of whether P_k is or is not null in $[v;P]$, $[w;P]$ and/or $[v+w;P]$ [15], the relationship $h[v+w;P]=h[v;P]+h[w;P]$ follows from our previous result on the coincidence of $h[v;P]$ and $\varphi[v]$ for all v and all P and the additivity of the Shapley value.

(Uniqueness) Let g be a coalitional value on $\mathcal{G}^{+cs}$ satisfying A2–A6. We will show that it is uniquely determined on each $[v;P]\in\mathcal{G}_N^{+cs}$ (arbitrary N). Several steps will be needed. We remark the axioms used at each point.

(a) If $v=0$ then $g_i[v;P]=0$ for all $i\in N$ and all $P\in P(N)$ by **A3**.

(b) Let $v=cu_T$ with $c>0$ and $\emptyset\neq T\subseteq N$. Let $Q=\{k\in M: T\cap P_k\neq\emptyset\}$ and $T_k=T\cap P_k$ for each $k\in Q$. By **A3** again, if $i\notin T$ then $g_i[cu_T;P]=0$. Since $(cu_T)^P=cu_Q$, all $k,h\in Q$ are symmetric in $(cu_T)^P$. Then, by **A5**,

$$\sum_{i\in T_k}g_i[cu_T;P]=\sum_{j\in T_h}g_j[cu_T;P]. \tag{9}$$

[15] Recall footnote 12.

By **A2**,

$$\sum_{i \in T} g_i[cu_T; P] = cu_T(N) = c.$$

Using Eq. (9) it follows that

$$\sum_{i \in T_k} g_i[cu_T; P] = \frac{c}{q} \quad \text{for all} \quad k \in Q \quad (q = |Q|). \tag{10}$$

Since all $i, j \in T_k$ are symmetric in cu_T, from **A4** we get $g_i[cu_T; P] = g_j[cu_T; P]$ and, using Eq. (10), we conclude that

$$g_i[cu_T; P] = \frac{c}{qt_k} \quad \text{for all} \quad i \in T_k \quad (t_k = |T_k|).$$

Summing up, if $i \in P_k$ then

$$g_i[cu_T; P] = \begin{cases} \dfrac{c}{qt_k} & \text{if } i \in T, \\ 0 & \text{otherwise,} \end{cases}$$

and g is uniquely determined on each game of the form cu_T with $c > 0$ and $\emptyset \neq T \subseteq N$.

(c) Now let $P = P^n$. If $i \in P_k$, so that $k = i$ and $P_k = \{i\}$, then, for every $v \in \mathcal{G}_N^+$ we have

$$h_i[v; P^n] = \begin{cases} g_i[v; P^n]\dfrac{g_i[v; P^n]}{g_i[v; P^n]} = g_i[v; P^n] & \text{if } i \text{ is nonnull in } v, \\ g_i[v; P^n] & \text{otherwise,} \end{cases}$$

that is, $h_i[v; P^n] = g_i[v; P^n]$ for all $i \in N$ and all $v \in \mathcal{G}_N^+$. Then, for any $v, w \in \mathcal{G}_N^+$, **A6** yields

$$g[v + w; P^n] = g[v; P^n] + g[w; P^n].$$

By combining it with **A2**–**A4**, this additivity of g as a sole function of the game once $P = P^n$ has been fixed, i.e., as a value on $\mathcal{G}_N^+$, the uniqueness of the Shapley value stated in Subsection 2.1 gives

$$g[v; P^n] = \varphi[v] \quad \text{for all} \quad v \in \mathcal{G}_N^+.$$

(d) Property **A6** can be inductively extended to any finite sum of games. Then, let

$$v = v_1 + v_2 + \cdots + v_r = \sum_{\ell=1}^{r} v_\ell \quad \text{in} \quad \mathcal{G}_N^+,$$

$P \in P(N)$ and $P_k \in P$. If P_k is null in $[v; P]$ then $g_i[v; P] = 0$ for all $i \in P_k$ by **A3**. If P_k is nonnull in $[v; P]$, let $R = \{1, 2, \ldots, r\}$, R^+ be the subset of indices $\ell \in R$ such that P_k is nonnull in $[v_\ell; P]$ and $R^0 = R \backslash R^+$. By **A6** we have, for each $i \in P_k$,

$$g_i[v; P]\frac{\sum_{j \in P_k} g_j[v; P^n]}{g_k[v^P; P^m]} = \sum_{\ell \in R^+} g_i[v_\ell; P]\frac{\sum_{j \in P_k} g_j[v_\ell; P^n]}{g_k[v_\ell^P; P^m]} + \sum_{\ell \in R^0} g_i[v_\ell; P^n]. \tag{11}$$

Not all $j \in P_k$ are null in v (otherwise k would be null in v^P according to footnote 5). Then $\sum_{j \in P_k} g_j[v; P^n] > 0$ [16] and hence, from Eq. (11), $g_i[v; P]$ can be expressed [17] in terms of $g_i[v_\ell; P]$ for $\ell \in R^+$ and expressions concerning only the Shapley value of games v, v^P, v_ℓ and v_ℓ^P for different values of $\ell = 1, 2, \ldots, r$. The conclusion is that if g was uniquely determined (for P) on $v_1, v_2, \ldots, v_r$ then it is uniquely determined on $[v; P]$.

(e) Finally, let $[v; P] \in \mathcal{G}_N^{+cs}$ and $v + v^- = v^+$ according to Eq. (2). Using (d) it follows that g is completely determined on $[v^+; P]$ and $[v^-; P]$ since both games are sums of games of the form considered in (b). Using (d) again, it follows that g is completely determined on $[v; P]$, and this concludes the uniqueness proof. □

Theorem 4.2 *There is one and only one coalitional value on $\mathcal{G}^{+cs}$ that is a nonnegative (A0) coalitional Shapley value (A1) and satisfies the quotient game property (A7) and proportionality within unions (A8). It is the proportional coalitional Shapley value π.*

Proof. (Existence) It suffices again to check that the proportional coalitional Shapley value satisfies the properties. Let $[v; P] \in \mathcal{G}_N^{+cs}$.

A0. (Nonnegativity) It directly follows from Eq. (3) and the nonnegativity of the Shapley value.

A1. (Coalitional Shapley value property) As was said above, this is straightforward to check from Eq. (3).

A7. (Quotient game property) Again from Eq. (3) and having in mind A1, it readily follows that, for all $k \in M$ and independently of whether P_k is or is not null in $[v; P]$,

$$\sum_{i \in P_k} \pi_i[v; P] = \varphi_k[v^P] = \pi_k[v^P; P^m].$$

A8. (Proportionality within unions) Let $i, j \in P_k \in P$. Once more from Eq. (3) and using A1, if P_k is nonnull in $[v; P]$ we get

$$\pi_i[v; P]\pi_j[v; P^n] = \varphi_k[v^P] \frac{\varphi_i[v]\varphi_j[v]}{\sum_{\ell \in P_k} \varphi_\ell[v]} \tag{12}$$

and hence

$$\pi_i[v; P]\pi_j[v; P^n] = \pi_j[v; P]\pi_i[v; P^n]$$

[16] Note that, using (c), we have for v (and analogous relationships for each v_ℓ)

$$g[v; P^n] = \varphi[v] \quad \text{and} \quad g[v^P; P^m] = \varphi[v^P].$$

.

[17] Namely,

$$g_i[v; P] = \sum_{\ell \in R^+} \frac{\varphi_k[v^P]}{\varphi_k[v_\ell^P]} \frac{\sum_{j \in P_k} \varphi_j[v_\ell]}{\sum_{j \in P_k} \varphi_j[v]} g_i[v_\ell; P] + \sum_{\ell \in R^0} \frac{\varphi_k[v^P]}{\sum_{j \in P_k} \varphi_j[v]} \varphi_i[v_\ell].$$

by the symmetry of the second term in Eq. (12) with respect to i, j; and, if P_k is null in $[v;P]$,

$$\pi_i[v;P]\pi_j[v;P^n] = 0 = \pi_j[v;P]\pi_i[v;P^n].$$

(Uniqueness) Let g^1 and g^2 be nonnegative (A0) coalitional Shapley values (A1) on $\mathcal{G}^{+cs}$ satisfying A7 and A8. We will show that $g^1 = g^2$. We will proceed again by steps and remark the axioms used in the proof.

(a) Let $[v;P] \in \mathcal{G}_N^{+cs}$ and assume that i is a null player in v. Let $i \in P_k$. If there is some $j \in P_k$ that is nonnull in v then we have, by **A1** and **A8**,

$$g_i^r[v;P]\varphi_j[v] = g_j^r[v;P]\varphi_i[v] \quad \text{for} \quad r = 1, 2$$

and therefore $g_i^1[v;P] = g_i^2[v;P] = 0$. If, instead, all players of P_k are null in v then P_k is null in $[v;P]$ and, by **A7** and **A1**,

$$\sum_{j \in P_k} g_j^r[v;P] = g_k^r[v^P;P^m] = \varphi_k[v^P] = 0 \quad \text{for} \quad r = 1, 2.$$

From **A0** it follows that $g_j^r[v;P] = 0$ for all $j \in P_k$ and, in particular, for $j = i$. Summing up, we have $g_i^1[v;P] = g_i^2[v;P] = 0$ for every player i null in v and any P.

(b) Using **A1** and **A7** it follows that g^1, g^2 coincide on unions: for each P_k

$$\sum_{i \in P_k} g_i^1[v;P] = g_k^1[v^P;P^m] = \varphi_k[v^P] = g_k^2[v^P;P^m] = \sum_{j \in P_k} g_j^2[v;P].$$

(c) Let us assume for a while that $g_i^1[v;P] \neq g_i^2[v;P]$ for some $[v;P]$ and some i. Let $i \in P_k$ for some P_k. According to (b), i cannot be the only player in P_k for which the values do not coincide on $[v;P]$. Moreover, if e.g. $g_i^1[v;P] > g_i^2[v;P]$ then there will be some $j \in P_k$ with $g_j^1[v;P] < g_j^2[v;P]$. By step (a), i, j must be nonnull in v. Let us apply **A8** and **A1** to these players. Then

$$\begin{aligned} &(\alpha) \quad g_i^1[v;P]\varphi_j[v] = g_j^1[v;P]\varphi_i[v] \quad \text{and} \\ &(\beta) \quad g_i^2[v;P]\varphi_j[v] = g_j^2[v;P]\varphi_i[v]. \end{aligned} \tag{13}$$

It follows that none of $g_i^1[v;P]$, $g_j^1[v;P]$, $g_i^2[v;P]$ and $g_j^2[v;P]$ can vanish: otherwise, their signs would contradict (α) or (β) in Eqs. (13). [18] In fact, they are all positive.

Hence, by dividing term by term the equalities of (13) we get

$$\frac{g_i^1[v;P]}{g_i^2[v;P]} = \frac{g_j^1[v;P]}{g_j^2[v;P]} \ .$$

However, the former fraction is > 1 while the latter is < 1, a contradiction. The conclusion is that $g^1 = g^2$. □

[18] E.g., if $g_i^1[v;P]$=0 then (α) implies $g_j^1[v;P] = 0$, but therefore, according to our initial assumptions on the behavior of i, j with respect to g^1, g^2, we get $g_i^2[v;P] < 0$ and hence $g_j^2[v;P] > 0$, thus contradicting (β) since $\varphi_i[v], \varphi_j[v] > 0$.

Remark 4.3 (*Independence of the axiomatic system in Theorem 4.1*) The axiom system A2–A6 is independent. Indeed:

(i) Let β denote the Banzhaf value, that is, the extension of the Banzhaf power index for nonnull simple games (Banzhaf [7]) to a value for all cooperative games (Owen [17]; also in Owen [19]). The coalitional value given for $i \in P_k$ by

$$g_i[v;P] = \begin{cases} \beta_k[v^P]\dfrac{\beta_i[v]}{\sum_{j \in P_k} \beta_j[v]} & \text{if } P_k \text{ is nonnull in } [v;P], \\ 0 & \text{otherwise} \end{cases}$$

satisfies A3–A6 but not A2.

(ii) The coalitional value defined by

$$g_i[v;P] = \frac{v(N)}{mp_k} \quad \text{for all} \quad i \in P_k$$

satisfies A2, A4, A5 and A6 but not A3.

(iii) Let $\omega = (\omega_1, \omega_2)$ be a weighting vector such that $\omega_1 \neq \omega_2$ and φ^ω be the corresponding weighted Shapley value ((Kalai and Samet [15]). The coalitional value defined by

$$g[v;P] = \begin{cases} \varphi^\omega[v] & \text{if } n = 2 \text{ and } P = P^N, \\ \pi[v;P] & \text{otherwise} \end{cases}$$

satisfies A2, A3, A5 and A6 but not A4.

(iv) Using the same weighted Shapley value as in the previous item, the coalitional value defined by

$$g[v;P] = \begin{cases} \varphi^\omega[v] & \text{if } n = 2 \text{ and } P = P^n, \\ \pi[v;P] & \text{otherwise} \end{cases}$$

satisfies A2, A3, A4 and A6 but not A5.

(v) If $v \in \mathcal{G}_N$ and $i \in N$, let

$$\mu_i(v) = \max_{S \subseteq N \setminus \{i\}} \{v(S \cup \{i\}) - v(S)\}$$

denote the maximum of the marginal contributions of player i in v. Then, the coalitional value given for $i \in P_k$ by

$$g_i[v;P] = \begin{cases} \dfrac{\mu_i(v)}{\sum_{j \in P_k} \mu_j(v)} \dfrac{\mu_k(v^P)}{\sum_{h \in M} \mu_h(v^P)} v(N) & \text{if } P_k \text{ is nonnull in } [v;P], \\ 0 & \text{otherwise} \end{cases}$$

satisfies A2–A5 but not A6.

Remark 4.4 (*Independence of the axiomatic system in Theorem 4.2*) The axiom system formed by A0, A1, A7 and A8 is independent. Indeed:

(i) As $N = \{1, 2, \ldots, n\}$, for any nonempty subset $S \subseteq N$ we can consider the minimum and maximum members of S according to the ordering of natural numbers. Let us consider the coalitional value g defined on $\mathcal{G}_N^{+cs}$, for each N, as follows. Given $[v; P]$ and $P_k \in P$,

(a) if $|P_k| > 1$ and all $i \in P_k$ are null in v then, for each $i \in P_k$,

$$g_i[v; P] = \begin{cases} -1 & \text{if } i \text{ is the minimum of } P_k, \\ 1 & \text{if } i \text{ is the maximum of } P_k, \\ 0 & \text{otherwise;} \end{cases}$$

(b) in any other case, $g_i[v; P] = \pi_i[v; P]$ for all $i \in P_k$.

The coalitional value so defined satisfies A1, A7 and A8 but it fails to be nonnegative.

(ii) Let $\alpha > 0$. The coalitional value defined by $g_i[v; P] = \alpha / p_k$ if $[v; P] \in \mathcal{G}^{+cs}$ and $i \in P_k$ is not a coalitional Shapley value but satisfies A0, A7 and A8.

(iii) The coalitional value given by $g[v; P] = \varphi[v]$ for all $[v; P] \in \mathcal{G}^{+cs}$ is a nonnegative coalitional Shapley value that satisfies A8 but not A7.

(iv) The Owen value Φ is a nonnegative coalitional Shapley value that satisfies A7 but not A8.

Remark 4.5 (*Simple games*) As usual, given a simple game v we denote by

$$W(v) = \{S \subseteq N : v(S) = 1\}$$

the set of *winning* coalitions and by

$$W^m(v) = \{S \in W(v) : T \subset S \Rightarrow T \notin W(v)\}$$

the set of *minimal* winning coalitions. Note that, if

$$W^m(v) = \{S_1, S_2, \ldots, S_r\},$$

we have a decomposition of v as union of unanimity games:

$$v = u_{S_1} \vee u_{S_2} \vee \cdots \vee u_{S_r}.$$

Let us denote by $\mathcal{SG}_N^{cs}$ the set of simple games with a coalition structure on a given N and by $\mathcal{SG}^{cs}$ the class of all simple games with a coalition structure (arbitrary N). Almost all properties A0–A8 can be restated for a coalitional value on simple games just by restricting to this class of games the domain of validity of each one (having in mind, for A7, that the class is closed under the passing to quotient games). The only exception is A6 (weighted additivity) since, as was noticed by Dubey [11], additivity does not work for simple games. He replaced it with a "transfer property" in order to get an axiomatic characterization of the Shapley value on these games (the so–called *Shapley–Shubik power index* due to the seminal paper by Shapley and Shubik [22]); correspondingly, we will replace A6 with the following property:

A9. *Weighted transfer property:* for all $v, w \in \mathcal{SG}_N$ and all $P \in P(N)$

$$h[v \vee w; P] + h[v \wedge w; P] = h[v; P] + h[w; P]$$

where, for all $i \in N$, if $i \in P_k \in P$,

$$h_i[v;P] = \begin{cases} g_i[v;P] \displaystyle\sum_{j \in P_k} g_j[v;P^n] \;/\; g_k[v^P;P^m] & \text{if } P_k \text{ is nonnull in } [v;P], \\ g_i[v;P^n] & \text{otherwise,} \end{cases}$$

and analogous definitions hold for $h_i[w;P]$, $h_i[v \vee w;P]$ and $h_i[v \wedge w;P]$ (see footnote 14).

Then Theorems 4.1 and 4.2 give rise to axiomatic characterizations of the proportional coalitional Shapley value on simple games with a coalition structure (i.e., as a *coalitional* power index).

Corollary 4.6 *There is a unique coalitional value on $\mathcal{SG}^{cs}$ that satisfies efficiency (A2), the null player strong property (A3), symmetry within unions (A4), symmetry in the quotient game (A5), and the weighted transfer property (A9). It is (the restriction of) the proportional coalitional Shapley value.*

Proof. (Existence) It suffices to check that the proportional coalitional Shapley value π satisfies all these properties on $\mathcal{SG}^{cs}$. For A2–A5, the proof is the same as in Theorem 4.1. As for A9, the basic point is that

$$v \vee w + v \wedge w = v + w \quad \text{for all} \quad v, w \in \mathcal{SG}_N.$$ [19]

Using A6 it follows that if $g = \pi$ then

$$h[v \vee w; P] + h[v \wedge w; P] = h[v; P] + h[w; P] \quad \text{for all} \quad P \in P(N).$$

(Uniqueness) Let g be a coalitional value on $\mathcal{SG}^{cs}$ satisfying A2–A5 and A9. We will show that it is uniquely determined on each $[v;P] \in \mathcal{SG}^{cs}_N$ (arbitrary N). The steps will be similar to those of Theorem 4.1 and even a bit simpler.

(a) If $v = 0$ then $g_i[v;P] = 0$ for all $i \in N$ and all $P \in P(N)$ by **A3**.

(b) Let $v = u_T$ for some nonempty $T \subseteq N$. As in Theorem 4,1, from **A2**, **A3**, **A4** and **A5** it follows that $g[u_T;P]$ is uniquely determined for all P: if $i \in P_k$ then

$$g_i[u_T;P] = \begin{cases} \dfrac{1}{qt_k} & \text{if } i \in T, \\ 0 & \text{otherwise,} \end{cases}$$

[19] In fact, it is straightforward to show that this holds for all $v, w \in \mathcal{G}_N$, so that A9 is true for π on the set of all monotonic games.

where $Q = \{k \in M : T \cap P_k \neq \emptyset\}$, $T_k = T \cap P_k$ for each $k \in Q$, $q = |Q|$ and $t_k = |T_k|$.

(c) Now, let $P = P^n$. As in Theorem 4.1, we find that $h[v; P^n] = g[v; P^n]$ for all $v \in \mathcal{SG}_N$. From **A9** it follows that

$$g[v \vee w; P^n] + g[v \wedge w; P^n] = g[v; P^n] + g[w; P^n].$$

Therefore, by combining with **A2**–**A4** this property of g as a sole function of the game once $P = P^n$ has been fixed, i.e., as a value on $\mathcal{SG}_N$, the uniqueness of the Shapley value on $\mathcal{SG}_N$ stated by Dubey [11] yields

$$g[v; P^n] = \varphi[v] \quad \text{for all} \quad v \in \mathcal{SG}_N.$$

(d) Property **A9** can be inductively extended to any finite union of games. Then, if $W^m(v) = \{S_1, S_2, \ldots, S_r\}$ with $r \geq 2$, from $v = u_{S_1} \vee u_{S_2} \vee \cdots \vee u_{S_r}$, we get, for all P,

$$h[v; P] = \sum_{j=1}^{r} (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq r} h[u_{S_{i_1}} \vee u_{S_{i_2}} \vee \cdots \vee u_{S_{i_j}}; P]. \text{ [20]} \tag{14}$$

(e) We are ready to show that $g[v; P]$ is uniquely determined on all $[v; P] \in \mathcal{SG}^{cs}$. Let $P_k \in P$. If P_k is null in $[v; P]$ then $g_i[v; P] = 0$ for all $i \in P_k$ by **A3**. If P_k is nonnull, let $R = \{1, 2, \ldots, r\}$, R^+ be the subset of indices $\ell \in R$ such that P_k is nonnull in $[v_\ell; P]$, and $R^0 = R \backslash R^+$. Using Eq. (14) we get, for each $i \in P_k$,

$$g_i[v; P] \frac{\sum_{j \in P_k} g_j[v; P^n]}{g_k[v^P; P^m]} = \sum_{\ell \in R^+} g_i[v_\ell; P] \frac{\sum_{j \in P_k} g_j[v_\ell; P^n]}{g_k[v_\ell^P; P^m]} + \sum_{\ell \in R^0} g_i[v_\ell; P^n].$$

Not all $j \in P_k$ are null in v (otherwise P_k would be null in $[v; P]$). Then $\sum_{j \in P_k} g_j[v; P^n] > 0$ and we have

$$g_i[v; P] = \sum_{\ell \in R^+} \frac{\varphi_k[v^P]}{\varphi_k[v_\ell^P]} \frac{\sum_{j \in P_k} \varphi_j[v_\ell]}{\sum_{j \in P_k} \varphi_j[v]} g_i[v_\ell; P] + \sum_{\ell \in R^0} \frac{\varphi_k[v^P]}{\sum_{j \in P_k} \varphi_j[v]} \varphi_i[v_\ell].$$

Thus, $g_i[v; P]$ can be expressed in terms of $g_i[v_\ell; P]$ for $\ell \in R^+$ and expressions concerning only the Shapley value of games v, v^P, v_ℓ and v_ℓ^P for different values of $\ell = 1, 2, \ldots, r$. The conclusion is that, as g was uniquely determined (for P) on $u_{S_1}, u_{S_2}, \ldots, u_{S_r}$ according to (b), it is uniquely determined on $[v; P]$. $\square$

Corollary 4.7 *There is a unique coalitional value on $\mathcal{SG}^{+cs}$ that is a nonnegative (A0) coalitional Shapley value (A1) and satisfies the quotient game property (A7) and proportionality within unions (A8). It is (the restriction of) the proportional coalitional Shapley value π.*

Proof. Existence follows at once from existence in Theorem 4.2, and the uniqueness proof is completely analogous. $\square$

[20] For example, if $n = 3$

$h[u \vee v \vee w; P] = h[u; P] + h[v; P] + h[w; P] - h[u \wedge v; P] - h[u \wedge w; P] - h[v \wedge w; P] + h[u \wedge v \wedge w; P]$.

5 Two examples

In this section we present two examples of application of the proportional coalitional Shapley value. The first one refers to the classical bankruptcy problem. The second refers to a voting problem.

Example 5.1 (*The bankruptcy problem*) The bankruptcy problem arises when the estate E of a given company does not suffice to cover the sum of demands of the set $N = \{1, 2, \dots, n\}$ of its creditors. Let $d = (d_1, d_2, \dots, d_n)$ be the *demand vector*. Of course, E and all d_i are positive numbers and it is assumed that

$$E < \sum_{i \in N} d_i.$$

We denote a bankruptcy problem by $[N, E, d]$. Different rules can be found in the literature that associate with each bankruptcy problem a division of the estate E among the creditors. [21] One of the bankruptcy rules is the *random arrival rule* (O'Neill [16]). Suppose that the creditors arrive one at a time and they are honored until estate runs out. This means that, given a particular ordering π of the creditors, the demand received by each one of them is: (i) fully satisfied, if the creditors preceding him in π can receive their complete demands and the remaining part of the state is greater than, or equal to, his demand; (ii) partially satisfied, if this remaining part is greater than 0 but lesser than his demand; or (iii) 0, if the sum of demands of the creditors preceding him in π is greater than, or equal to, the state. To obtain independence of the ordering, the arithmetic average over the set Π of all possible orderings is taken. Then, the random arrival rule allocates to each creditor $i \in N$ the amount

$$ra_i[N, E, d] = \frac{1}{n!} \sum_{\pi \in \Pi} \min \Big\{ d_i, \max\{0,\ E - \sum_{\substack{j \in N: \\ \pi(j) < \pi(i)}} d_j\} \Big\}.$$

Based on a problem of rights arbitration taken from the Talmud, O'Neill [16] introduced a cooperative game that reflects the strategic possibilities of the creditors with respect to both the estate and the demand vector in a bankruptcy problem.

More precisely, given a bankruptcy problem $[N, E, d]$, he defined a (monotonic) cooperative game $v^{E,d}$ on the set N of creditors as follows:

$$v^{E,d}(S) = \max\{0, E - \sum_{j \notin S} d_j\} \quad \text{for each} \quad S \subseteq N.$$

In words, the worth that game $v^{E,d}$ assigns to each coalition is the remaining (if any) of the estate when all the other creditors went to try to fully receive their demands (note that $\sum_{i \in S} d_i > E - \sum_{j \notin S} d_j$). If anything of the estate is not yet available, the coalition gets 0. There are interesting links between certain bankruptcy rules and solution concepts of the cooperative game theory. For our proposal, the important relation is the following (O'Neill [16]):

[21] A good review of the topic is Thomson [23].

The random arrival rule of a bankruptcy problem $[N,E,d]$ provides the same allocations as the Shapley value of the corresponding cooperative game $v^{E,d}$ on N.

If the norms do not prevent the creditors from forming coalitions and presenting joint demands, we are led to consider all coalition structures on the creditor set in order to decide whether some of them should go together or not and which are their possibilities to force such a situation.

We wish then to apply the proportional coalitional Shapley value to this problem and analyze the possibilities to find a *stable* coalition structure in the sense that will be explained below. Using this coalitional value, the unions of creditors share first the estate E according to the application of the Shapley value to the quotient game and, then, the payoff allocated to each union is shared among its members proportionally to their Shapley value in the original game $v^{E,d}$. The advantages of the method are: (a) the Shapley value (the most important solution for cooperative games) is the only rule used to divide the state and (b) the final sharing is based on proportionality, a simple method that most people can understand easily. [22]

Given a coalition structure $P = \{P_1, P_2, \ldots, P_m\}$ on the set of creditors, it is easy to see that the proportional coalitional Shapley value of $[v^{E,d}; P]$ assigns to a creditor $i \in P_k$ an amount related with the random arrival rule as follows: if $\overline{d} = (\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_m)$ is the demand vector of the unions given by $\overline{d}_k = \sum_{j \in P_k} d_j$ for each k, then

$$\pi_i[v^{E,d}; P] = ra_k[M, E, \overline{d}] \frac{ra_i[N,E,d]}{\displaystyle\sum_{j \in P_k} ra_j[N,E,d]} \ ,$$

We will consider the numerical case taken from the Talmud by O'Neill. Then, $n = 4$, $E = 120$ and $d = (120, 60, 40, 30)$, so that

$$v^{E,d} = 50u_{\{1,2\}} + 30u_{\{1,3\}} + 20u_{\{1,4\}} + 10u_{\{1,2,3\}} + 10u_{\{1,2,4\}} + 10u_{\{1,3,4\}} - 10u_{\{1,2,3,4\}}.$$

By applying the proportional coalitional Shapley value to $[v^{E,d}; P]$ for all P we get the results given in Table 1.

In order to discuss this results, let us first compute the payoff that each creditor can be sure to get by his own and independently of the behavior of the others, that is, in the five coalition structures where the creditor stays alone. It is clear that these "sure" payoffs are: 57.5000 for creditor 1, 28.3333 for 2, 18.3333 for 3, and 13.3333 for 4 (these amounts sum up to 117.5000). Therefore any coalition structure where a creditor joins other creditors and gets less than his sure payoff should be disregarded. This leaves us with the possibilities displayed in Table 2.

The creditors' preferences on the five structures shown in Table 2 are the following:

[22] Therefore, no creditor should be expected to say "Don't play with my money!", "No adventures in business!", or something similar.

coalition structure P	1	2	3	4
{{1},{2},{3},{4}}	57.5000	29.1667	19.1667	14.1667
{{1,2},{3},{4}}	56.3942	28.6058	20.0000	15.0000
{{1,3},{2},{4}}	56.2500	30.0000	18.7500	15.0000
{{1,4},{2},{3}}	56.1628	30.0000	20.0000	13.8372
{{2,3},{1},{4}}	58.3333	29.1667	19.1667	13.3333
{{2,4},{1},{3}}	58.3333	29.1667	18.3333	14.1667
{{3,4},{1},{2}}	58.3333	28.3333	19.1667	14.1667
{{1,2},{3,4}}	56.3942	28.6058	20.1250	14.8750
{{1,3},{2,4}}	56.2500	30.2885	18.7500	14.7115
{{1,4},{2,3}}	56.1628	30.1724	19.8276	13.8372
{{1,2,3},{4}}	57.0472	28.9370	19.0157	15.0000
{{1,2,4},{3}}	57.0248	28.9256	20.0000	14.0496
{{1,3,4},{2}}	56.9725	30.0000	18.9908	14.0367
{{2,3,4},{1}}	60.0000	28.0000	18.4000	13.6000
{N}	57.5000	29.1667	19.1667	14.1667

Table 1: The value π in the Talmud bankruptcy problem

coalition structure P	1	2	3	4
{{1},{2},{3},{4}}	57.5000	29.1667	19.1667	14.1667
{{2,3},{1},{4}}	58.3333	29.1667	19.1667	13.3333
{{2,4},{1},{3}}	58.3333	29.1667	18.3333	14.1667
{{3,4},{1},{2}}	58.3333	28.3333	19.1667	14.1667
{N}	57.5000	29.1667	19.1667	14.1667

Table 2: Feasible coalition structures in the Talmud bankruptcy problem

- 1: $\{2,3\} \equiv \{2,4\} \equiv \{3,4\} > P^n \equiv P^N$, although this creditor does not control the formation of these two–player coalitions because he does not intervene in any of them.
- 2: $P^n \equiv \{2,3\} \equiv \{2,4\} \equiv P^N > \{3,4\}$.
- 3: $P^n \equiv \{2,3\} \equiv \{3,4\} \equiv P^N > \{2,4\}$.
- 4: $P^n \equiv \{2,4\} \equiv \{3,4\} \equiv P^N > \{2,3\}$.

Therefore, we can conclude that creditor 1 is an undesired coalition partner and the *stable* coalition structure that could be expected to form would be one among

- $\{\{2,3\},\{1\},\{4\}\}$
- $\{\{2,4\},\{1\},\{3\}\}$

- $\{\{3,4\},\{1\},\{2\}\}$

or

- $P^n = \{\{1\},\{2\},\{3\},\{4\}\}$, the singleton trivial structure.

Which couple (if any) would form might finally depend on the personality of the involved creditors (2, 3 and 4). Each one of these couples improves the payoff of its members with respect to their sure level but also that of creditor 1, whereas the other isolated creditor gets just his sure payoff. After all, maybe P^n would be a most likely solution, since it does not require any agreement. (We disregard P^N because it gives the same result as P^n but its "formation cost"—an agreement of all creditors—would be indubitably high.)

Let us illustrate the meaning attached to the vector of profit inverse indices by considering, e.g., the coalition structure $P = \{\{2,3\},\{1\},\{4\}\}$, for which the vector is $(1.00, 0.98, 1.06)$. This vector reflects that 2 and 3 are indifferent between forming coalition $\{2,3\}$ or remaining alone while, if this sole coalition forms, 1 would get profit and 4 would be damaged.

Example 5.2 (*The Catalonia Parliament, Legislature 2003–2007*) Independent nation until 1714, Catalonia is nowadays one of the seventeen regions (called "autonomous communities") that shape the federal–like administrative structure of Spain since the early eighties. Specific traditions, culture and language have been zealously preserved by the Catalan people during centuries, and a strong nationalist feeling is still shared by a majority of its inhabitants.

The Statute of Autonomy grants Catalonia a great deal of freedom to organize itself and to decide on a series of matters concerning it. In particular, Catalonia possesses a legislative body: the Catalonia Parliament. One of its main tasks is to elect among the parliamentarians, at the beginning of each four–year legislature, the president of the Catalonia Government, who in turn chooses the *consellers* (ministers) that will form this autonomous (regional) government.

Five parties or pre–electoral coalitions elected members to the Catalonia Parliament (135 seats) in the elections held on 16 November 2003. Their names and abbreviations and the seat distribution are:

1. *Convergència i Unió (CiU)*, Catalan nationalist middle–of–the–road coalition of two federated parties: 46 seats.

2. *Partit dels Socialistes de Catalunya (PSC)*, moderate left–wing socialist party, federated to the national Partido Socialista Obrero Español: 42 seats.

3. *Esquerra Republicana de Catalunya (ERC)*, radical Catalan nationalist left–wing party: 23 seats.

4. *Partit Popular de Catalunya (PPC)*, conservative party, Catalan delegation of the national Partido Popular: 15 seats.

5. *Iniciativa per Catalunya–Verds (ICV)*, coalition of Catalan euro–communist parties, federated to the national Izquierda Unida, and ecologist groups ("Verds"): 9 seats.

Due to the party voting discipline—held even within CiU, considered a single party from now on—, there are just five agents in this situation, which will be referred to by their number or their abbreviation, indistinctly. The straight majority rule usually applied in the Parliament for passing a bill enables us to describe the situation by means of a weighted majority game where N is the above set of parties and $v \equiv [68; 46, 42, 23, 15, 9]$. The set of minimal winning coalitions of this simple game is

$$W^m = \{\{1,2\}, \{1,3\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}\}.$$

We wish to analyze the coalition structures in this game since no party got the absolute majority and hence a coalition government might be expected to form. To this end, we will use the proportional coalitional Shapley value. At least two reasons can be adduced to support this choice.

First, when acting as a power index, the proportional coalitional Shapley value is mainly based on the Shapley–Shubik index, an elegant mathematical concept whose characteristic properties (efficiency, symmetry, null player property and transfer property) can be easily accepted by politicians. Moreover, this index allocates to any party a weighed sum of the times it is crucial for all coalitions of each given size from 1 to n, and hence it measures the basic strategic capability of this party (i is *crucial* for $S \subseteq N$ in a simple game v iff $S \in W(v)$ but $S \backslash \{i\} \notin W(v)$).

Second, the proportional coalitional Shapley value applies the Shapley–Shubik index to the quotient game, where the unions act, but then, within each union, the fraction of power obtained this way is efficiently and simply shared proportionally to the power of each member of the union in the original game. This combination, of efficiency, simplicity and use of the power indices in the quotient game (among unions) and in the original game, seems to us a most natural procedure from a practical political viewpoint.

Therefore, from the essential intervention of the original game in the final result, it follows that this game should reflect the *actual* strategic possibilities of each party. Taking v as starting game, we are assuming that all possible coalitions are fully likely and no party disregards entering any union because of ideological constraints. In practice, this is not always so. Then we will first modify the original game and replace it with a more realistic *restricted game* prior to attack the coalition formation problem.

We will follow Carreras [8], where two basic procedures are shown for taking into account ideological constraints in simple games. The first one is based on affinities, the second—more radical—on incompatibilities. The grounds for these procedures can be found in the reference. The political structure of the current Catalonia Parliament—see below—strongly suggests that the incompatibility method will be the most suitable here. [23]

[23] In other real–world examples, and depending on the underlying political structure, the affinity

The method is follows. Given I, an irreflexive and symmetric binary relation on N, a coalition S is called I–*admissible* iff no pair of incompatible players i, j (that is, with iIj) can be found in S. Then we take

$$(W_I)^m = \{S \in W^m : S \text{ is } I\text{–admissible}\}$$

as new set of minimal winning coalitions and v_I, the simple game so defined, as restricted game.

In our case, we just consider the incompatibilities given by $3I4$ and $4I5$. Indeed, all parties would be more or less able to join any other, if necessary; the only—but radical—exceptions are the full direct antagonisms between ERC and PPC on one hand and PPC and ICV on the other, both due to ideological reasons that encompass very different positions on the left–to–right axis as well as, in the case of ERC, sound divergences with respect to the Catalan nationalism. The incompatibility relation introduced in these terms gives rise to a restricted game v_I that will be used as starting point and is defined by

$$(W_I)^m = \{\{1, 2\}, \{1, 3\}, \{2, 3, 5\}\}.$$

Let us first analyze a bit this game and discover which has been the effect of passing from v to v_I. By comparing the Shapley–Shubik power index of games v and v_I we get

$$\varphi[v] = (0.4000, 0.2333, 0.2333, 0.0667, 0.0667),$$
$$\varphi[v_I] = (0.4167, 0.2500, 0.2500, 0.0000, 0.0833).$$

The effects of the incompatibilities (imposed by parties 3 and 5 rather than by party 4) are clear. Parties 3 and 5 get more power by declaring their incompatibility with party 4, and this party becomes a null player. Nevertheless, simultaneously, parties 1 and 2 get more power too—an effect perhaps not completely desired by parties 3 and 5. The symmetry of 2 and 3 in the original game is preserved, whereas the symmetry between 4 and 5 is broken in favor of party 5. Summing up, it seems fully justified, from an ideological but also strategic viewpoint, the introduction of the incompatibility relation I and the use of v_I, instead of v, as starting game in the analysis of the coalition formation. From now on, we will remove party 4 from the game and consider the fifteen coalition structures arising in the new player set $N = \{1, 2, 3, 5\}$.

The application of the proportional coalitional Shapley value π gives the results displayed in Table 3.

The sure payoff of each party, i.e., the power it can get independently of the behavior of the remaining parties, is 0. No coalition structure can then be disregarded for this reason. If now we look at the best result that each party can obtain, we find:

procedure might be better suited in order to define the (more realistic) restricted game. Moreover, perhaps not all links should be of the same intensity. More accuracy in this sense, that maybe would imply some degree of subjective component, could be carried out using more sophisticated restriction methods such as those presented in Amer and Carreras [3] and [4], based on *cooperation indices*. As a first approach, we have preferred here to keep ourselves within the more elementary—i.e., more qualitative, less quantitative—approach of Carreras [8].

coalition structure P	CiU (1)	PSC (2)	ERC (3)	ICV (5)
{{1},{2},{3},{5}}	0.4167	0.2500	0.2500	0.0833
{{1,2},{3},{5}}	0.6250	0.3750	0.0000	0.0000
{{1,3},{2},{5}}	0.6250	0.0000	0.3750	0.0000
{{1,5},{2},{3}}	0.5556	0.1667	0.1667	0.1111
{{2,3},{1},{5}}	0.1667	0.3333	0.3333	0.1667
{{2,5},{1},{3}}	0.3333	0.2500	0.3333	0.0833
{{3,5},{1},{2}}	0.3333	0.3333	0.2500	0.0833
{{1,2},{3,5}}	0.6250	0.3750	0.0000	0.0000
{{1,3},{2,5}}	0.6250	0.0000	0.3750	0.0000
{{1,5},{2,3}}	0.4167	0.2500	0.2500	0.0833
{{1,2,3},{5}}	0.4545	0.2727	0.2727	0.0000
{{1,2,5},{3}}	0.5556	0.3333	0.0000	0.1111
{{1,3,5},{2}}	0.5556	0.0000	0.3333	0.1111
{{2,3,5},{1}}	0.0000	0.4286	0.4286	0.1429
{N}	0.4167	0.2500	0.2500	0.0833

Table 3: The value π for the coalition structures in v_I

- 0.6250 for CiU in the coalition structures where it joins either PSC or ERC.
- 0.4286 for PSC and also for ERC when they join together ICV.
- 0.1667 for ICV when just PSC and ERC join.

Thus, no coalition structure can be distinguished this way. Nevertheless, in a simple game one is mainly interested in the formation of a winning coalition, and this is a new criterion under which Table 3 can be reduced to a shorter Table 4 (where the singleton trivial structure is maintained as a *disagreement point*).

Now, all is clear. The best result for each party is:

- 0.6250 for CiU in the coalition structures where it joins either PSC or ERC.
- 0.4286 for PSC and also for ERC when they join together ICV.
- 0.1429 for ICV when joining both PSC and ERC.

The conclusion is obvious. In spite of the interest of CiU in joining PSC or ERC, the expected result, that is, the only *stable* scenario, is the formation of a coalition among PSC, ERC and ICV, which possesses a vote majority and can then form a government free to act with the support of this majority in the Parliament. Since each one of its members gets its best value in this coalition (in a proportion of $3 : 3 : 1$), the coalition structure that it defines enjoys stability in the very sense of the Nash equilibrium,

coalition structure P	CiU (1)	PSC (2)	ERC (3)	ICV (5)
{{1},{2},{3},{5}}	0.4167	0.2500	0.2500	0.0833
{{1,2},{3},{5}}	0.6250	0.3750	0.0000	0.0000
{{1,3},{2},{5}}	0.6250	0.0000	0.3750	0.0000
{{1,2},{3,5}}	0.6250	0.3750	0.0000	0.0000
{{1,3},{2,5}}	0.6250	0.0000	0.3750	0.0000
{{1,2,3},{5}}	0.4545	0.2727	0.2727	0.0000
{{1,2,5},{3}}	0.5556	0.3333	0.0000	0.1111
{{1,3,5},{2}}	0.5556	0.0000	0.3333	0.1111
{{2,3,5},{1}}	0.0000	0.4286	0.4286	0.1429
{N}	0.4167	0.2500	0.2500	0.0833

Table 4: The value π for the "winning" coalition structures in v_I

since no member of the union wishes to leave it. [24]

It is worthy of mention that, after a short period of negotiations, PSC, ERC and ICV signed an agreement (*Pacte del Tinell*) and formed the Catalonia Government. Thus, our result fully concurs with the actual behavior of the involved parties. [25]

Remark 5.3 (*The proportional coalitional Shapley value for level coalition structures*) Level coalition structures were first considered by Owen [18], who showed that the Owen value adapts perfectly to them (see also Winter [25]). Let us discuss a related question, interesting from both the theoretical and the practical side, raised by the fact that a three–member coalition arises as the only stable one in Example 5.2. Which way will coalition $\{2, 3, 5\}$ form: entering all at once or after forming some binary pre–coalition? Using a simpler notation there are four possibilities, namely:

- $2 + 3 + 5$

[24] An analogous analysis, carried out by taking v instead of v_I as starting game, that is, leaving aside the ideological constraints provided by the incompatibility relation I, would have led us to conclude that the formation of either $\{1, 4, 5\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$ gives the only stable coalition structures among the " minimal winning" ones.

The result is interesting for two reasons. (a) On the theoretical side, it is worthy of mention that party 1 gets its best payoff (fraction of power) in joining both weakest parties and forming a ternary coalition, instead of joining just one of the other—stronger—parties (2 or 3). Maybe this is so because, in terms of strategic strength (i.e., according to the Shapley–Shubik index of game v), 2 and 3 are individually stronger than 4 and 5 together. (b) On the political side, $\{1, 4, 5\}$ and $\{2, 3, 4\}$ would be hardly accepted as good solutions to the coalition formation problem just because PPC and ICV in the first case, and ERC and PPC in the second, hold the only antagonistic ideological positions. This would leave us with $\{2, 3, 5\}$ as the only politically reasonable solution. Although it just confirms our previous analysis using v_I as initial game, we still claim that the ideological constraints have to be incorporated to the starting point, as we did, because they deeply influence the bargaining.

[25] Strong divergences between ERC and the remaining members of the tripartite government (PSC and ICV) were increasingly apparent until May 2006, when ERC finally left the government, the Parliament was dissolved, anticipated elections on 1 November 2006 were called for, and Legislature 2003–2007 finished prematurely.

- (2 + 3) + 5
- (2 + 5) + 3
- 2 + (3 + 5)

Then, a previous question is: does the proportional coalitional Shapley value apply when the coalition structure has two or more levels? The answer is positive. For instance, let us show how to proceed with (2 + 3) + 5 or, more formally, $\mathcal{P} = \{\{1\},\{\{2,3\},\{5\}\}\}$. Let $P = \{\{1\},\{2,3\},\{5\}\}$ in N, $M = \{1,2,3\}$, $P' = \{\{1\},\{2,3\}\}$ in M and $M' = \{1,2\}$. From v_I we get $(v_I)^P$ and, from this latter, $((v_I)^P)^{P'}$. The Shapley value in each game gives

$$\begin{aligned}
&\varphi[v_I] = (5/12, 3/12, 3/12, 1/12),\\
&\varphi[(v_I)^P] = (1/6, 4/6, 1/6),\\
&\varphi[(v_I)^P)^{P'}] = (0, 1).
\end{aligned}$$

The Shapley value 1 allocated to player 2 in $((v_I)^P)^{P'}$ is shared among coalitions $\{2,3\}$ and $\{5\}$ proportionally to 4 and 1, and hence it allocates 4/5 to $\{2,3\}$ and 1/5 to party 5. Next, 4/5 is equally shared among parties 2 and 3 and gives 2/5 each. Thus, the proportional coalitional Shapley value corresponding to the two–level structure $\mathcal{P}$, informally denoted by (2 + 3) + 5, gives

$$\pi[v_I;\mathcal{P}] = (0, 2/5, 2/5, 1/5).$$

We can then discuss which of the four possibilities would be the preferred one. Table 5 provides the details.

coalition structure	PSC (2)	ERC (3)	ICV (5)
2 + 3 + 5	0.4286	0.4286	0.1429
(2 + 3) + 5	0.4000	0.4000	0.2000
2 + (3 + 5)	0.5000	0.3750	0.1250
(2 + 5) + 3	0.3750	0.5000	0.1250

Table 5: The value π for the level coalition structures among 2, 3 and 5 in v_I

Each party would prefer to be called to enter the last (the best position for ICV in structure $\{\{2,3\},\{1\},\{5\}\}$ found in Table 3 was already announcing this fact). As this is impossible to satisfy, we conclude that the very way to form the coalition should be all at once, that is, 2 + 3 + 5.

Remark 5.4 (*Why computing power?*) To which end and how should our numerical evaluation of the coalition formation be used? Well, once a coalition government is formed, it is essential to share presidencies, cabinet ministers, parliamentary positions and many other political responsibilities among the members of the coalition. For instance, the main positions in the Catalonia case are the following (we are not trying to be exhaustive):

- Presidency of the Government
- Presidency of the Parliament
- Vice–presidency of the Government (*Conseller en Cap*)
- Ministry of Institutional Relations
- 14 more ministries (*conselleries*), all of them of a similar category

If the parties that join agree to assign a numerical value to each one of these positions, say A, B, C, D and $(14)d$, respectively, they should share them in such a way that the total values got by the three parties would satisfy the proportion 3:3:1 that corresponds to the proportional coalitional Shapley value of the finally formed coalition structure. More precisely, since, actually, PSC got A, ERC got B and C, and ICV got D, which should have been the sharing of the 14 remaining ministries? The answer is easy: if α, β and γ are the number of ministries attached to PSC, ERC and ICV, respectively, the conditions would be

$$\alpha + \beta + \gamma = 14 \qquad \text{and} \qquad \frac{A + \alpha d}{3} = \frac{B + C + \beta d}{3} = D + \gamma d.$$

For example, if $A = 10$, $B = 7$, $C = 7$, $D = 4$ and $d = 1$ we would get $\alpha = 8$, $\beta = 4$ and $\gamma = 2$. Instead, the actual sharing was as follows: 8 for PSC, 5 for ERC and 1 for ICV. From this distribution we can infer that the "values" of the positions agreed by the colligated parties were satisfying, if we take $d = 1$,

$$B + C = A + 3 \qquad \text{and} \qquad D = \frac{A + 5}{3}.$$

For instance, something like $A = 10$, $B + C = 13$ and $D = 5$ would fulfill the above conditions.

6 Discussion and conclusions

We have proposed and studied a new coalitional value for monotonic games with coalition structure defined on an arbitrary finite player set. This value, which is a coalitional Shapley value, combines the Shapley value and the proportional rule [26] in a two–step procedure like that of the Owen value. Its heuristics is clear and its calculus is simple, even in the case of level coalition structures, to which it perfectly adapts. We feel that it would be of a great interest in the future an experimental work devoted to checking practitioners' opinion about this value.

Two clearly different axiomatic characterizations of this new value, using in each case logically independent sets of properties, have been stated and adapted, moreover, to the subclass of simple games. In addition, we have developed applications to the

[26] The use of this rule is the main reason for restricting the domain of the value to monotonic games.

analysis of the coalition formation in two contexts: the bankruptcy problem and a voting problem, getting in both cases very reasonable results.

Our definition of the new value has been compared with that of the Owen value: curiously, the Owen value admits a definition that strongly recalls ours (cf. footnote 10). Also our axiomatizations can be compared with some of the existing ones for the Owen value. [27] They are quite similar and share the spirit of the balanced contributions property, although we state it in a multiplicative form whereas for the Owen value it adopts an additive form.

The main difference lies in the replacement in Theorem 4.1 for our value of the additivity (resp., transfer property in case of simple games) of the Owen value with a *weighted* additivity (resp., *weighted* transfer property), where weights can be interpreted as *inverse profit indices.* A "perverse law", according to which the simpler is a definition of a concept the more complicated is its axiomatic characterization, seems to hold here, but this does not apply in the case of Theorem 4.2.

We would like to quote here the following ideas from Alonso et al. [2]:

> *For any value, understood as a solution concept for cooperative conflicts, it is always interesting to have, in both theory and practice, not only an explicit formula but also a list of properties of the value. In particular, axiomatic systems are not only mathematically elegant: they also provide a most convenient and economic tool to decide on the use of one or other value. Of course, there may be many criteria to decide which is the most suited value/index to apply in a given situation. Then, we feel that one should strongly avoid being dogmatic at this point. Probably, there is no value able to cover all situations. For example, there is no unanimous criterion to choose among using either the Shapley value φ or the Banzhaf value β as power index in all cases. We contend that pure and applied game theorists should be flexible at most in this respect. Really, only a few properties found in the literature can really be considered absolutely compelling, i.e., almost no axiom is compelling in vacuo but only inserted in the framework of a given, specific cooperative conflict. Even those that appear as the best placed in this sense might well be conditioned by the characteristics of the problem where we pretend to use the value they define. The conclusion is that all of us should look at axioms with an open mind and without a priori value judgements. The history of science is full of examples of theoretical models that only after a certain period of time have been proven to be useful in practice.*

Finally, let us refer to the examples analyzed in Section 5. In both coalition formation problems the Owen value could have been used instead of our value. In the bankruptcy problem (Example 5.1), the Owen value yields that the formation of coalition $\{2, 3, 4\}$ would be the most interesting scenario for creditor 1, but this coalition is the worst for all its members among the feasible ones. Each one of creditors

[27] At this point we wish to declare that, as professional mathematicians and vocational game theorists, we consider the Shapley value and the Owen value two monuments to the human intelligence.

2, 3 and 4 would prefer forming a two–member coalition with any other of them or go back to the singleton trivial structure. This result is quite similar to ours.

Instead, in the voting problem (Example 5.2), and starting at the restricted game as we did, the Owen value differs from the proportional coalitional Shapley value in, at least, three points. Using the Owen value: (a) the payoff obtained by PSC (resp., ERC) in joining CiU and forming a binary minimal winning coalition increases if, simultaneously, the remaining two parties ERC and ICV (resp., PSC and ICV) form a coalition, a fact that, unfortunately, is not under the control of PSC (resp., ERC); (b) by assuming that these remaining parties do not join, the payoff that PSC and ERC get in the binary coalitions CiU + PSC and CiU + ERC, respectively, equals their payoffs in the ternary coalition PSC + ERC + ICV; and (c) no stable coalition structure arises, and only after disregarding the binary coalitions because of (a) can PSC + ERC + ICV be accepted as a "pseudo–stable" coalition structure, where PSC and ERC do not get their strictly maximum share of power.

As was seen in Example 5.2, according to the proportional coalitional Shapley value acting as power index: (a) the payoff obtained by the members of a *winning coalition* in a simple game can never be affected by the behavior of the remaining players (we feel that this is a very positive feature); (b) the payoffs obtained by PSC and ERC in their binary coalitions with CiU differ from their payoff in PSC + ERC + ICV, so that maybe this value discriminates better than the Owen value; and (c) coalition PSC + ERC + ICV arises as the only stable coalition (structure).

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