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Centro de Investigación Operativa
Universidad Miguel Hernández de Elche
Avda. de la Universidad s/n
03202 Elche (Alicante)

TRABAJOS I+D

Strategic absentmindedness in finitely repeated games*

Ignacio García-Jurado[1], Natividad Llorca[2], Ana Meca[2], Manuel A. Pulido[3], Joaquín Sánchez-Soriano[2]

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Abstract

In this paper we consider finitely repeated games in which players can unilaterally commit to behave in an absentminded way in some stages of the repeated game. We prove that the standard conditions for folk theorems can be substantially relaxed when players are able to make this kind of compromises, both in the Nash and in the subgame perfect case. We also analyze the relation of our model with the repeated games with unilateral commitments studied, for instance, in García-Jurado et al. (2000).

Key words: Repeated games, absentminded players, folk theorems, unilateral commitments.

2000 AMS Subject classification: 91A20.

JEL classification: C72, C73.

1 Introduction

The concept of absentminded agent was introduced in Piccione and Rubistein (1997) in the context of extensive decision problems, i.e., extensive games with only one player. The authors illustrated this concept by means of what they call "the paradox of the absentminded driver". This example can be summarized as follows. After a long party, an individual is sitting late at night planning his midnight trip home. To reach home he has to take the highway and get off at the second exit. Turning at the first exit leads him into a very dangerous area (let us say payoff 0). Turning at the second exit yields him the highest reward (i.e., payoff 4). If he continues beyond the second exit, he cannot go back and at the end of the highway he will find a motel where he can spend the night (with payoff

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[1]Department of Statistics and Operations Research, Santiago de Compostela University, Spain.

[2]Operations Research Center, Miguel Hernández University, Spain.

[3]Department of Statistics and Operations Research, Murcia University, Spain.

1). Assume that the driver is absentminded and that he is aware of this fact. By absentminded we mean that, at an intersection, he cannot know whether it is the first or the second one. So, before entering the highway, all he can do is to decide whether or not to exit at an intersection. This example provides a situation in which an agent cannot distinguish between two histories on the same path; it illustrates that extensive games with information sets intersecting a path more than once can make sense and deserve to be studied. The reader is referred to volume 20, issue 1, of the journal *Games and Economic Behavior*, specially devoted to imperfect recall games, for a complete discussion about this topic.

A different issue treated by game theory is how repetition generates cooperation in the context of strategic games. It is a very well-known fact that the unique Nash equilibrium path of the finitely repeated prisoners dilemma consists of both players defeating in all the stages. This result, which is in contrast with experimental evidence, can be overcome by the introduction of some drops of irrationality. Several models of bounded rationality have been used to explain the experimentally observed cooperative behavior in this context. For instance, Dilger (2006) shows that cooperative equilibria can be theoretically obtained when players exhibit absentmindedness in some parts of the finitely repeated prisoners dilemma. In this setting, absentmindedness means that at least one of the agents forgets in which round is he playing, perhaps by not counting the rounds.

In the present paper we go further than Dilger (2006) and consider the following issue: assume that in a finitely repeated game the players have the ability to "become absentminded" in some parts of the game, and that they use this ability strategically. How does this affect the conditions of the folk theorems?

One can wonder how is it possible that players have this ability. For instance, they have it when they can make unilateral commitments. There is also a literature dealing with commitments as a strategic option. The great strategic importance of an ability to make firm commitments was first pointed out in Schelling (1960). Faiña-Medín et al. (1989) modify the finitely repeated prisoners dilemma by adding an initial round in which the players have the option of committing themselves to a subset of their strategies. The main result of that paper is that, if the prisoners dilemma is repeated a large enough number of times and players can restrict their strategy sets in a preliminary round of the game, then there is a symmetric subgame perfect equilibrium in which both players act cooperatively throughout the post-commitment stages of the game. García-Jurado et al. (2000) take a more general framework and consider finitely repeated games in which players can make unilateral commitments regarding the possible restriction of their sets of strategies. They prove that every outcome which is strictly preferred to the minimax outcome by all players can be supported by a Nash equilibrium when the basic game is sufficiently repeated. Other papers like García-Jurado and González-Díaz (2006) and Renou (2008) explore how the players' ability of making unilateral commitments to some of their strategies affect equilibrium payoffs.

What we prove in this paper is that the standard conditions for folk theorems can be substantially relaxed when players can make absentmindedness compromises, both when dealing with Nash and with subgame perfect equilibrium concepts. We also show that these absentmindedness compromises can be seen as a special case of unilateral commitments regarding the deletion of strategies, as the ones considered for instance in García-Jurado and González-Díaz (2006).

The paper is organized as follows. Section 2 formally describes and analyzes our model of finitely repeated games with strategic absentmindedness. In section 3 we present two Nash folk theorems for these games. Finally, in section 4 two subgame perfect folk theorems are provided.

2 Strategic absentmindedness

A finite strategic game G is defined by a triple (N, A, π) where $N = \{1, \dots, n\}$ is the set of players, $A = \prod_{i \in N} A_i$ is the set of strategy profiles (A_i being the finite strategy set of each player $i \in N$), and $\pi = (\pi_1, \dots, \pi_n)$ is the payoff function profile, where $\pi_i : A \longrightarrow \mathbb{R}$ is the payoff function of player i, for all $i \in N$. The minimax vector of G, denoted by v, is the vector $(v_1, \dots, v_n)$ given by:

$$v_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} \pi_i(a_{-i}, a_i),$$

for all $i \in N$, where $A_{-i} := \prod_{j \in N \setminus \{i\}} A_j$ and a_{-i} is a generic element of A_{-i}.

For a finite strategic game G, the $m-$times repetition of G with discount parameter δ ($\delta \in (0, 1]$) is the strategic game

$$G^{m,\delta} = (A_1^H, \dots, A_n^H, \pi_1^{m,\delta}, \dots, \pi_n^{m,\delta})$$

where:

- A history at time q, for all $q \in \{2, \dots, m\}$, is an element of A^{q-1}. We assume that there is a unique history at time 1 which we denote by 0. We write $A^0 := \{0\}$. The set of all histories, $\bigcup_{q=1}^{m} A^{q-1}$, is denoted by H.

- For every $i \in N$, A_i^H is the set of maps from H to A_i. A strategy profile in the repeated game $\sigma \in \prod_{i \in N} A_i^H$ describes, for all players $i \in N$, which actions players are going to play for every history $h \in H$. Every σ determines a path $p(\sigma) = (p^1(\sigma), ..., p^m(\sigma))$ given by:

$$\begin{aligned}
&p^1(\sigma) = \sigma(0), \\
&p^2(\sigma) = \sigma\left(p^1(\sigma)\right), \\
&p^3(\sigma) = \sigma\left(p^1(\sigma), p^2(\sigma)\right), \\
&... \\
&p^m(\sigma) = \sigma\left(p^1(\sigma), ..., p^{m-1}(\sigma)\right).
\end{aligned}$$

- The payoff function of each player $i \in N$ in the repeated game is given by:

$$\pi_i^{m,\delta}(\sigma) = \frac{1-\delta}{1-\delta^m} \sum_{q=1}^{m} \delta^{q-1} \pi_i(p^q(\sigma))$$

  for all $\sigma \in \prod_{i \in N} A_i^H$.

Let us now introduce the concept of unilateral commitment which has been used, for example, in García-Jurado et al. (2000) and García-Jurado and González-Díaz (2006). In those papers folk theorems for finitely repeated games with unilateral commitments are obtained. In one of these games the players are able to delete some of their strategies. More particularly, the repeated game has a previous stage in which players announce their unilateral commitments specifying which of their strategies they are not going to use. Then, the repeated game is played and the players have to respect the commitments they have announced.

Formally, given a finite strategic game G, its m-times repetition with discount parameter δ and with unilateral commitments $U(G^{m,\delta})$ is the strategic game

$$(S_1^{UC}, \ldots, S_n^{UC}, \gamma_1^{m,\delta}, \ldots, \gamma_n^{m,\delta})$$

such that, for all $i \in N$,

- $S_i^{UC} = \left\{ (\omega_i, f_i) \;\middle|\; \begin{array}{l} \omega_i \subset A_i^H, \omega_i \neq \varnothing, f_i : \prod_{j \in N} (2^{A_j^H} \backslash \varnothing) \longrightarrow A_i^H \\ \text{with } f_i(\omega) \in \omega_i, \forall \omega \in \prod_{j \in N} (2^{A_j^H} \backslash \varnothing) \end{array} \right\}$, and

- $\gamma_i^{m,\delta}(\omega, f) = \pi_i^{m,\delta}(f_1(\omega), \ldots, f_n(\omega))$.

García-Jurado et al. (2000) and García-Jurado and González-Díaz (2006) prove that the standard conditions for folk theorems can be substantially relaxed when players can make unilateral commitments.

Now we focus on our main model. In this paper we provide general results for a formal model of repeated games with strategic absentmindedness, i.e., repeated games with a previous stage in which players can announce that they will be absentminded in some parts of the repeated game (in other words, that they will make the same choice in some of their information sets). Dilger (2006) already showed that when absentminded players play the finitely repeated prisoners dilemma, the efficient payoffs can be sustained in equilibrium. However, here we go much further and consider the following issue: assume that in a repeated game the players have the ability to "become absentminded" in some parts of the game, and that they use this ability strategically. How does this ability affect the conditions of the folk theorems in this context?

Let us start giving a formal definition of absentmindedness. Take $h, h' \in H$; we say that h is a subhistory of h' (henceforth $h < h'$) if h is a history at time q, h' is a history at q' with $q < q'$, and $h' = (h, g)$ where $g \in A^{q'-q}$.

Definition 2.1 *Let $G^{m,\delta}$ be a finitely repeated game. An agent $i \in N$ exhibits absentmindedness in $G^{m,\delta}$ if there exist $h, h' \in H, h < h'$ such that both belong to the same information set (and so $\sigma_i(h) = \sigma_i(h')$ for every possible strategy of player i).*

Notice that, in a standard repeated game, every history can be identified with an information set of each player. In repeated games with strategic absentmindedness, an information set I is a collection of histories such that, if $h, h' \in I$, then $h < h'$ or $h' < h$.

Given a finitely repeated game $G^{m,\delta}$ we can now consider the repeated game with strategic absentmindedness and with discount parameter δ $A(G^{m,\delta})$. It is a new game with a a previous stage where the players announce (simultaneously and independently) in what parts of the repeated game they are going to exhibit absentmindedness; these announces are what we call absentmindedness compromises. Afterward, the repeated game is played and the absentmindedness compromises have to be respected.

Formally $A(G^{m,\delta})$ is the game $(S_1^A, \ldots, S_n^A, \gamma_1^{m,\delta}, \ldots, \gamma_n^{m,\delta})$ where, for all $i \in N$:

- S_i^A is player i's strategy set, containing the pairs (α_i, f_i) satisfying that:
  - α_i is a partition of H such that, for every class $I_i \in \alpha_i$, I_i is an information set, i.e., if $h, h' \in I_i$, then $h < h'$ or $h' < h$. We denote by $\mathcal{P}(H)$ the set of such partitions of H.
  - f_i is a map which associates to every $\alpha \in \mathcal{P}(H)^N$ an $f_i(\alpha) \in A_i^H$ which satisfies that $f_i(\alpha)(h) = f_i(\alpha)(h')$, for all $h, h' \in I_i$ and all $I_i \in \alpha_i$.
- $\gamma_i^{m,\delta}(\alpha, f) = \pi_i^{m,\delta}(f_1(\alpha), \ldots, f_n(\alpha))$, for all $(\alpha, f) \in \prod_{i \in N} S_i^A$.

Finitely repeated games with strategic absentmindedness and finitely repeated games with unilateral commitments differ in the strategy sets of the players. In the absentmindedness setting a player announces a partition of the set of all histories, so that he is going to choose the same action in every history of a class of the partition. However, in the unilateral commitments setting a player announces a subset of his strategy set in the original repeated game. In some sense, in the unilateral commitments setting some possible histories are removed, whereas this is not the case in the absentmindedness setting.

Our first result deals with the relationship between unilateral commitments and absentmindedness compromises in finitely repeated games. We start introducing a notation which will be useful for our purposes. Let G be a finite game and consider its finite repetition with with strategic absentmindedness $A(G^{m,\delta})$. Take now α_i an absentmindedness compromise of player i and denote by $\omega_i(\alpha_i)$ the following subset of A_i^H

$$\omega_i(\alpha_i) = \{\sigma_i \in A^H \mid h, h' \in I_i \text{ implies that } \sigma_i(h) = \sigma_i(h'), \text{ for all } I_i \in \alpha_i\}.$$

Proposition 2.2 *Every absentmindedness compromise can be seen as a unilateral commitment.*

Proof. Let $A(G^{m,\delta})$ be a finitely repeated game with absentminded compromises. If player $i \in N$ chooses the absentminded compromise α_i, in practice he is committing to use a strategy of $\omega_i(\alpha_i)$ in the repeated game. ■

Remark 2.3 *In view of Proposition 2.2, a possible way of implementing in practice absentminded compromises is through unilateral commitments, in the sense of García-Jurado et al. (2000). In González-Díaz (2006) and Renou (2008), for instance, the reader can find discussions about the feasibility and economic interest of these commitments.*

Remark 2.4 *In Proposition 2.2 we have proved that an absentmindedness compromise can be identified with a unilateral commitment. However, a repeated game with strategic absentmindedness can be seen as an extensive game having some information sets which contain nodes in the same path, which is not the case for repeated games with unilateral commitments.*

The following example shows that the reciprocal of Proposition 2.2 is not true, i.e., that there exist unilateral commitments which cannot be imitated by absentmindedness compromises.

Example 2.5 *Let $G = (N, A, \pi)$ be a finite game with $N = \{1, 2\}$ and $A_1 = \{U, D\}$, $A_2 = \{L, R\}$, and take $G^{2,\delta}$ for a given $\delta \in (0, 1]$. Now consider the following unilateral commitment for player one:*

$$\omega_1 = \{\sigma_1 \in A_1^H \mid \sigma_1(h) = U \text{ for all } h \in H\}.$$

It is easy to check that player one has just five possible absentmindedness compromises in $A(G^{2,\delta})$; moreover for every such a compromise α_1 it holds that $\omega_1(\alpha_1) \neq \omega_1$.

3 Nash Folk Theorems

The main objective of this paper is to study the effect of strategic absentmindedness on the appearing of constructive behavior in repeated games. The issue of how repetition generates cooperation has been approached in the game theoretical literature through the so-called folk theorems[4]. The classical Nash folk theorem for finitely repeated games states that if G is a strategic game such that for each player there is a Nash equilibrium that gives him a payoff strictly greater than his minimax payoff, then every feasible payoff vector greater than the minimax vector can be approximated by a Nash equilibrium of $G^{m,\delta}$ for m and δ large enough.

García-Jurado et al. (2000) shows that when players can make unilateral commitments every feasible payoff vector greater than the minimax vector can

[4]Refer to Benoit and Krishna (1998) for a survey on the topic.

be approximated by a Nash equilibrium of $G^{m,\delta}$, for m and δ large enough, without requiring any extra condition for game G.

Next we prove that a similar result holds for repeated games with strategic absentmindedness. Notice that, according to Proposition 2.2 and Example 2.5, the absentmindedness setting can be seen as a special case of the unilateral commitments setting. Thus, in some sense, the results presented here are stronger than the results in García-Jurado et al. (2000).

More precisely, we provide two folk theorems. The first one asserts that every outcome of a finite strategic game G which is strictly greater than the minimax vector can be supported by a Nash equilibrium of $G^{m,\delta}$ for m and δ large enough. The second one states that every convex combination of outcomes of G which is strictly greater than the minimax vector can be approximated by a Nash equilibrium of $G^{m,\delta}$ for m and δ large enough.

Theorem 3.1 *Let G be a finite strategic game with minimax vector v. Suppose that there exists $u \in \{\pi(a) \mid a \in A\}$ such that $u > v$. Then the game $A(G^{m,\delta})$ has a Nash equilibrium whose associated payoff is u, for m and δ large enough.*

Proof. Take a such that $u = \pi(a)$ and denote by I_a the following collection of histories of $A(G^{m,\delta})$:

$$I_a = \{0, a, (a, a), ..., (a, \overset{m-1}{\cdots}, a)\}.$$

Now, take a strategy profile $(\bar{\alpha}, \bar{f})$ satisfying, for each $i \in N$, that:

- $\bar{\alpha}_i = \{I_a, \{h\} \text{ with } h \notin I_a\}$,
- $\bar{f}_i(\bar{\alpha})(h) = a_i$, for all $h \in I_a$.
- $\bar{f}_i(\bar{\alpha})(h) = (p_{-j})_i$, for all h history at time q in which all players have always played according to a and only player j ($j \neq i$) has deviated at stage $q-1$, where

$$p_{-j} \in \arg \min_{a_{-j} \in A_{-j}} \{\max_{a_j} \{\pi_j(a_{-j}, a_j)\}\}.$$

- $\bar{f}_i(\bar{\alpha}_{-j}, \alpha_j)(h) = (p_{-j})_i$, for all $h \in H$, all $\alpha_j \neq \bar{\alpha}_j$, and all $j \neq i$.

Let us check that such an $(\bar{\alpha}, \bar{f})$ is a Nash equilibrium if m and δ are large enough. Indeed, if one player j deviates from it then he will be punished for all others players by playing p_{-j}. Obviously, j will not gain if he deviates from $\bar{\alpha}$. The only other deviations allowed for j include that he deviates at stage one, in which case he will obtain a payoff smaller than or equal to

$$\frac{1-\delta}{1-\delta^m}(\pi_j(a_{-j}, a'_j) + \sum_{t=2}^{m} \delta^{t-1} v_j).$$

Therefore, if m and δ are large enough j will not gain with his deviation. ■

Theorem 3.2 *Let G be a finite strategic game with minimax vector v. Take $u \in F := conv\{\pi(a) \mid a \in A\}$ with $u > v$. Then, for each $\varepsilon > 0$ there exists $\delta_0 \in (0,1)$ such that: for each $\delta \in [\delta_0, 1]$ there is an $m_0 \in \mathbb{N}$ satisfying that, for each $m \geq m_0$, the game $A(G^{m,\delta})$ has a Nash equilibrium with payoff vector w such that* [5] $\|w - u\| < \varepsilon$.

Proof. Take $\varepsilon > 0$ small enough so as to ensure that $u_i - \varepsilon > v_i$ for each $i \in N$. Since $u \in F$, $u = \sum_{j=1}^{r} p_j \pi(a^j)$ with $p_j > 0$, for all j, and $\sum_{j=1}^{r} p_j = 1$. Then, it is clear that there exists a collection of r positive integer numbers $\{z_1, \ldots, z_r\}$ such that, if we denote by z the sum $z_1 + \cdots + z_r$, it holds that

$$\|\frac{1}{z}\sum_{j=1}^{r} z_j \pi(a^j) - u\| < \frac{\varepsilon}{2}.$$

Let us now denote by $\{b^j\}_{j=1}^{z}$ the collection of action profiles

$$\{b^j\}_{j=1}^{z} = \{a^1, \overset{z_1}{\ldots}, a^1, a^2, \overset{z_2}{\ldots}, a^2, \ldots, a^r, \overset{z_r}{\ldots}, a^r\}.$$

It is clear that

$$\lim_{\delta \to 1^-} \frac{1-\delta}{1-\delta^z} \sum_{j=1}^{z} \delta^{j-1} \pi(b^j) = \frac{1}{z} \sum_{j=1}^{z} \pi(b^j).$$

Hence, there exists $\delta_1 \in (0,1)$ such that, for each $\delta \in [\delta_1, 1]$, it holds that

$$\left\| \frac{1}{z} \sum_{j=1}^{z} \pi(b^j) - \frac{1-\delta}{1-\delta^z} \sum_{j=1}^{z} \delta^{j-1} \pi(b^j) \right\| < \frac{\varepsilon}{2}.$$

Therefore, we obtain

$$\left\| \frac{1-\delta}{1-\delta^z} \sum_{j=1}^{z} \delta^{j-1} \pi(b^j) - u \right\| <$$

$$\left\| \frac{1}{z} \sum_{j=1}^{z} \pi(b^j) - \frac{1-\delta}{1-\delta^z} \sum_{j=1}^{z} \delta^{j-1} \pi(b^j) \right\| + \left\| \frac{1}{z} \sum_{j=1}^{z} \pi(b^j) - u \right\| < \varepsilon.$$

Consider now a strategy profile f^* of $G^{m,\delta}$, for a given $m \in \mathbb{N}$, where players successively play the action profiles in $\{b^j\}_{j=1}^{z}$ (m needs not to be a multiple of z; if $m = m_1 z + m_2$, with $m_2 < z$, then after being played m_1 times the action profiles in $\{b^j\}_{j=1}^{z}$, the first m_2 action profiles of $\{b^j\}_{j=1}^{z}$ are played). It is clear that, in this situation, there is an $m_0 \in \mathbb{N}$ such that, for each $m \geq m_0$,

$$\left\|\pi^{m,\delta}(f^*) - u\right\| < \varepsilon.$$

Next, we define the following r collections of histories for $G^{m,\delta}$ (assume, for instance, that $m = m_1 z + m_2$ with $m_2 < z_1$):

[5] For every $x \in \mathbb{R}$, we write $\|x\|$ for its infinite norm, which is defined by $\|x\| := \max\{|x_1|, \ldots, |x_n|\}$.

- $I_{a^1} = \{0, (a^1), \dots, (a^1, \overset{z_1-1}{\dots}, a^1), (\{b^j\}_{j=1}^z),$
  $(\{b^j\}_{j=1}^z, a^1), \dots, (\{b^j\}_{j=1}^z, a^1, \overset{z_1-1}{\dots}, a^1), \dots,$
  $(\{b^j\}_{j=1}^z, \overset{m_1}{\dots}, \{b^j\}_{j=1}^z), (\{b^j\}_{j=1}^z, \overset{m_1}{\dots}, \{b^j\}_{j=1}^z, a^1), \dots,$
  $(\{b^j\}_{j=1}^z, \overset{m_1}{\dots}, \{b^j\}_{j=1}^z, a^1, \overset{m_2-1}{\dots}, a^1)\}$

  $\vdots$

- $I_{a^r} = \{(a^1, \overset{z_1}{\dots}, a^1, \dots, a^{r-1}, \overset{z_{r-1}}{\dots}, a^{r-1}),$
  $(a^1, \overset{z_1}{\dots}, a^1, \dots, a^{r-1}, \overset{z_{r-1}}{\dots}, a^{r-1}, a^r), \dots,$
  $(a^1, \overset{z_1}{\dots}, a^1, \dots, a^{r-1}, \overset{z_{r-1}}{\dots}, a^{r-1}, a^r, \overset{z_r-1}{\dots}, a^r),$
  $(\{b^j\}_{j=1}^z, a^1, \overset{z_1}{\dots}, a^1, \dots, a^{r-1}, \overset{z_{r-1}}{\dots}, a^{r-1}),$
  $\{b^j\}_{j=1}^z, a^1, \overset{z_1}{\dots}, a^1, \dots, a^{r-1}, \overset{z_{r-1}}{\dots}, a^{r-1}, a^r), \dots,$
  $(\{b^j\}_{j=1}^z, \overset{m_1-1}{\dots}, \{b^j\}_{j=1}^z, a^1, \overset{z_1}{\dots}, a^1, \dots, a^{r-1}, \overset{z_{r-1}}{\dots}, a^{r-1}, a^r, \overset{z_r-1}{\dots}, a^r)\}$

Now consider, for $\delta \in (0,1]$ and $m \in \mathbb{N}$, a strategy profile $(\bar{\alpha}, \bar{f})$ in $A(G^{m,\delta})$ satisfying, for each $i \in N$, that:

- $\bar{\alpha}_i = \{I_{a^1}, I_{a^2}, \dots, I_{a^r}, \{h\}$ for all $h \notin \cup_{j=1}^r I_{a^j}\}$.
- $\bar{f}_i(\bar{\alpha})(h) = a_i^j$, for all $h \in I_{a^j}$ and all j.
- $\bar{f}_i(\bar{\alpha})(h) = (p_{-k})_i$, for all h history at time q in which all players have played according to $\{b^j\}_{j=1}^z$ and only player k ($k \neq i$) has deviated at stage $q-1$.
- $\bar{f}_i(\bar{\alpha}_{-k}, \alpha_k)(h) = (p_{-k})_i$, for all $h \in H$, all $\alpha_k \neq \bar{\alpha}_k$, and all $k \neq i$.

For $(\bar{\alpha}, \bar{f})$ we already know the following: if we take $\varepsilon > 0$ small enough so as to ensure that $u_i - \varepsilon > v_i$ (for each $i \in N$), there exists $\delta_1 \in (0,1)$ such that, for every $\delta \in [\delta_1, 1]$, we can choose $m_0 \in \mathbb{N}$ satisfying that, for every $m \geq m_0$, it holds that

$$\left\| \pi^{m,\delta}(\bar{f}(\bar{\alpha})) - u \right\| < \varepsilon.$$

Now if only one player k deviates from $(\bar{\alpha}, \bar{f})$ in $A(G^{m,\delta})$, then he will be punished by all others players playing p_{-k}. Obviously, k will not gain if he deviates from $\bar{\alpha}$. The only other deviations allowed for him includes that he deviates at stage one or at one of the stages $t = z_j + 1$, with $j \in \{1, \dots, r-1\}$, in which case he will have a payoff smaller than or equal to

$$\frac{1-\delta}{1-\delta^m} \left(\sum_{j=1}^{t-1} \delta^{j-1} \pi_i(b^j) + \delta^{t-1} \pi_i(b_{-i}^t, b_i') + \sum_{j=t+1}^{m} \delta^{j-1} v_i \right). \tag{1}$$

Notice that, for every $\delta \in [\delta_1, 1]$, we can choose m large enough to ensure that the payoff in (1) is as close to v_i as we like. In particular, we can obtain that it is below $u_i - \varepsilon$, and then $(\bar{\alpha}, \bar{f})$ is a Nash equilibrium of $A(G^{m,\delta})$, and the proof is concluded. ■

4 Subgame Perfect Folk Theorems

It is a very well-known feature that, when dealing with imperfect information games, the Nash equilibrium concept has important drawbacks (see, for intance Selten (1975)). This is the reason why the subgame perfect equilibrium and other refinements have been introduced for extensive games. Roughly speaking, the subgame perfect equilibrium concept disregards those Nash equilibria which are only possible if some players give credit to irrational plans of others, and it is a very appropriate concept to analyze repeated games. That is the reason why there is a large literature on subgame perfect folk theorems (which is also surveyed in Benoit and Krishna (1998)).

In García-Jurado and González-Díaz (2006) it is studied how the standard conditions for subgame perfect folk theorems in finitely repeated games change when players can make unilateral commitments. Surprisingly enough, they notice that, roughly speaking, finitely repeated games with unilateral commitments do not have subgame perfect equilibria (in pure strategies). So, they need to use a variation of this concept, the so-called virtually subgame perfect equilibrium concept, for which they prove a folk theorem.

In this section we show that many finitely repeated games with strategic absentmindedness do have subgame perfect equilibria, and prove two subgame perfect folk theorems which require of the one-shot game similar conditions to the ones in García-Jurado and González-Díaz (2006). Like in the Nash case we provide an exact subgame perfect folk theorem and an approximated one.

Theorem 4.1 *Let G be a finite strategic game. Take $u \in \{\pi(a) \mid a \in A\}$ such that, for every $i \in N$, there exists a Nash equilibrium x^i with $u_i > \pi_i(x^i)$. Then the game $A(G^{m,\delta})$ has a subgame perfect equilibrium whose associated payoff is u, for m and δ large enough.*

Proof. Take a such that $u = \pi(a)$ and consider the strategy $(\bar{\alpha}, \bar{f})$ defined, for each $i \in N$, as follows:

$$\bar{\alpha}_i = \{I_a, \{h\} \text{ with } h \notin I_a\},$$

and

- if all players play according to $\bar{\alpha}$ then
  - $\bar{f}_i(\bar{\alpha})(h) = a_i$, for all $h \in I_a$,
  - $\bar{f}_i(\bar{\alpha})(h) = x_i^j$, for all h history such that the first player who deviates from a is just j,
  - $\bar{f}_i(\bar{\alpha})(h) = x_i^1$ for all other h,
- if only player $j \in N$ deviates from $\bar{\alpha}$ then
  - $\bar{f}_i(\bar{\alpha}_{-j}, \alpha_j)(h) = x_i^j$ for all h and all $\alpha_j \neq \bar{\alpha}_j$;

- otherwise, we define
  - $\bar{f}_i(\alpha) = x_i^1$ for all h.

It is easy to check that $(\bar{\alpha}, \bar{f})$ is a subgame perfect equilibrium of $A(G^{m,\delta})$ for m and δ large enough, and that its associated payoff is u. ■

To complete the paper, we present the approximated version of the subgame perfect folk theorem for finitely repeated games with strategic absentmindedness.

Theorem 4.2 *Let G be a finite strategic game. Take $u \in F = conv\{\pi(a) \mid a \in A\}$ such that, for every $i \in N$, there exists a Nash equilibrium x^i with $u_i > \pi_i(x^i)$. Then, for each $\varepsilon > 0$ there exists $\delta_0 \in (0,1)$ such that: for each $\delta \in [\delta_0, 1]$ there is an $m_0 \in \mathbb{N}$ satisfying that, for each $m \geq m_0$, the game $A(G^{m,\delta})$ has a subgame perfect equilibrium with payoff vector w such that $\|w - u\| < \varepsilon$.*

Proof. The proof is an easy adaptation of the proof of Theorem 3.2, taking into account the proof of Theorem 4.1. ■

The main conclusions of this paper are the following. An absentmindedness compromise can be identified with a unilateral commitment, but the reciprocal is not true. So, the class of finitely repeated games with strategic absentmindedness can be seen as a special case of the class of finitely repeated games with unilateral commitments. Besides, we prove that the standard conditions for folk theorems can be substantially relaxed when players can make absentmindedness compromises, both when dealing with Nash and with subgame perfect equilibrium concepts. This relaxations are of similar quality (maybe better) than the relaxations achieved when dealing with unilateral commitments.

References

Benoit J.P. and Krishna V. (1998). "The Folk Theorems for Repeated Games: A Synthesis". EconWPA-9902001.

Dilger A. (2006). "The Absent-Minded Prisoner". Spanish Economic Review 8, 301-315.

Faíña-Medín A., García-Jurado I., Méndez-Naya J. and Méndez-Naya L. (1998). "Unilateral Commitments in the Finitely Repeated Prisoners' Dilemma". Annals of Operations Research 84, 187-194.

García-Jurado I., Méndez-Naya L., and Patrone F. (2000). "Unilateral Commitments in Finitely Repeated Games". International Game Theory Review 2, 129-139.

García-Jurado I. and González-Díaz J. (2006). "The Role of Commitment in Repeated Games". Optimization 55, 541-553.

Piccione M. and Rubinstein A. (1997). "On the Interpretation of Decision Problems with Imperfect Recall". Games and Economic Behavior 20, 3-24.

Renou L. (2008). “Commitment games”. To appear in Games and Economic Behavior.
Schelling T. (1960). “The strategy of conflict”. Harvard University Press.
Selten R. (1975). “Reexamination of the perfectness concept for equlibrium points in extensive games”. International Journal of Game Theory 4, 25-55.