Centro de Investigación Operativa

I-2008-10

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August 2008

ISSN 1576-7264
Depósito legal A-646-2000

TRABAJOS I+D

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How to divide a cake when people have different metabolism?*

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Abstract

This paper deals with bankruptcy problems in which the players have different utility functions defined in terms of the quantity of allocated resource. We tackle this kind of situation by means of an NTU-game, which turns out to be ordinally convex and balanced. We introduce the CEA-rule in this context and provide two characterizations of this.

Keywords: bankruptcy problems, non-linear utilities, NTU-games, allocation rules and axiomatic characterizations.

AMS classification: 91A12
JEL classification: C71

*This work has been partially supported by the Government of Spain and FEDER fund under projects MTM2005-09184-C02-02, SEJ2005-07637-C02-02 and TEC2005-08211-C02.

1 Introduction

In a bankruptcy problem, one has to divide an estate amongst a set of agents, each of whom has a claim on it, and the total amount claimed exceeds the estate available, so not all the claims of the agents can be met completely. Bankruptcy problems were introduced by O'Neill (1982) and Aumann and Maschler (1985) and have been applied to a wide array of economic problems such as: taxation problems (Young, 1988), surplus-sharing problems (Moulin, 1987), cost-sharing problems (Moulin, 1988), apportionment of indivisible goods problems (Young, 1994) and priority problems (Moulin, 2000 and Young, 1994). In brief, one can say that the classical bankruptcy model deals with allocation problems in which there is a perfectly divisible estate to be distributed and the demand of each agent involved can be characterized by a single (monetary) claim on it.

Kaminski (2000) and Calleja et al (2005) also consider bankruptcy problems in which the demand of each claimant is not one-dimensional, as is the case in classical bankruptcy models, but multi-dimensional. In the former paper different priorities are assigned to the various components of an agent's claim vector, however, in the latter work, on the contrary, they do not assume any exogenously given priorities on the claim components.

Another extension of the classical model is introduced in Pulido et al (2002, 2008), where two types of information are used. On the one hand, the claims of the different agents and, on the other hand, a vector with additional information -exogenous to the claimants but directly related to the problem. This additional information is considered as a point of reference or starting point to begin with the distribution of the estate.

In all the above-mentioned papers, the estate is considered to be perfectly divisible. Nevertheless, in Herrero and Martínez (2006) bankruptcy problems with indivisible goods are studied and bankruptcy rules, with priorities among the agents, are provided. The paper by Orshan et al (2003) focuses on bankruptcy problems with non-transferable utility (NTU). This kind of problem arises in situations where the estate is not perfectly divisible but a set of all possible allocations.

In this work we consider bankruptcy problems where the estate is perfectly divisible, although the satisfaction derived from receiving one unit is not the same for all agents, i.e. they have different utility functions which can be non-linear. Bearing in mind these circumstances, we introduce an NTU-game and, afterwards, analyse bankruptcy rules which extend the classical ones. This kind of problems can be found in different settings. For instance, Lucas-Estañ et al (2007) and Gozálvez et al (2007) describe the channel assignment problem in wireless telecommunications networks through bankruptcy techniques, taking into account that the users in the system do not have the same utility functions and these are non-linear. In their work, the Constrained Equal Awards (CEA) and the Constrained Equal Losses (CEL) rules are adjusted to such situations where the estate (number of channels to be assigned to) is countable and bounded[1]. However, this idea can be adapted for communications networks in which the allocated resource -the bandwidth or channels which can be shared- is perfectly divisible. Allocating resources (money, for instance) for buying the equipment for several laboratories is another case where this approach can be meaningful. Since not all laboratories have the same features, it is understandable that the same amount of money does not provide the same level of contentment. Finally, and in order to complete a possible list of examples, we can consider the situation in which several agents have to share some water resources. It is glaringly obvious that the financial return of the water resources assigned to each agent can be completely different, although they all receive the same amount. Moreover, this return can be non-linear with the assigned amount of water.

Therefore, the model proposed in this paper can be used to tackle situations in which there is an estate (assumed to be perfectly divisible) and a set of agents who have demands on it, but the same quantity assigned to different individuals produces different levels of satisfaction. In each particular case, the utility function translates the resources assigned to an agent as the return that they perceive. In the case of channel assignment, for example, a user receives a certain number of channels which represents a particular transmission rate or quality of service (QoS). In the example of buying equipment for laboratories, the claimants obtain money which is converted into a certain functionality of the laboratory. And finally, in the problem of allocating water resources, the water assigned to each landowner is translated into earnings. Thus, we analyse the problem of allocating a scarce resource not from the resource point of the view, but from the perspective of how much profit a claimant can make from the resources he obtains. As a result, since agents are different, drawing an analogy with the problem of sharing a cake, we are talking about how to share one when people have different metabolism, so we do not look at the portion of the cake that an agent receives, but focus on the satisfaction it gives them.

The paper is organized as follows. In Section 2 we introduce an NTU-

[1]These works concentrates in algorithms that allow to obtain the CEA and the CEL rules.

game associated with a bankruptcy problem in which the players have different utility functions defined over the quantity of allocated resources. We also analyze its properties and the non-emptiness of the core. In Section 3 we define the CEA rule for this kind of bankruptcy problems and provide two characterizations of this rule in Section 4.

2 The model

A standard bankruptcy problem can be described by a triple (N, E, d), where $N = \{1, ..., n\}$ is the finite set of agents, $E \geq 0$ is the estate to be divided and $d \in \mathbb{R}^N_+$, the vector of claims, is such that $\sum_{i \in N} d_i \geq E$. To deal with a bankruptcy problem (N, E, d) we can derive a classical bankruptcy TU-game (N, v), where the value of a coalition $S \subset N$ is given by

$$v(S) = \max\{E - \sum_{i \in N \setminus S} d_i, 0\}.$$

Therefore, it represents what is left for the players in S after the demands of the players in $N \setminus S$ have been satisfied.

As it was motivated in Section 1, we are interested in bankruptcy problems where the demands of the agents are based on a certain amount of the resource involved and they will represent the corresponding level of satisfaction or utility for them. Thus, in our case, each agent $i \in N$ has their own bounded utility function $u_i : \mathbb{R}_+ \to [0, k_i]$, which translates resources into degree of contentment. We assume that these utility functions are non-decreasing, right-continuous and such that $u_i(x) = k_i$ for all $x \geq m_i$, where

$$m_i = \min\left\{\arg\ \max_x\ u_i(x)\right\}, \tag{1}$$

for all $i \in N$. We represent a bankruptcy problem with non-linear utilities as a triple (N, E, d), where N is the finite set of agents, $E \geq 0$ is the estate to be divided and $d = (m, u)$ with $E \leq \sum_{i \in N} m_i$. In our context, associated with a bankruptcy problem of this type, we can introduce a cooperative bankruptcy game with non-transferable utility $(N, V_{(N,E,d)}, H)$, whose elements are: N is the set of players, $V_{(N,E,d)}$ is the map which assigns to each coalition $S \subset N, S \neq \varnothing$ a subset $V_{(N,E,d)}(S) \subset \mathbb{R}^S$ with

$$V_{(N,E,d)}(S) = \left\{ k \in \mathbb{R}^S \;\middle|\; \begin{array}{l} \exists x \in \mathbb{R}^S_+, \sum_{i \in S} x_i \leq v(S) \text{ and} \\ u(x) = (u_1(x), ..., u_s(x)) \geq k \end{array} \right\}, \tag{2}$$

where $v(S) = \max\{E - \sum_{i\in N\setminus S} m_i, 0\}$, and $H = \prod_{i=1}^{n} [0, u_i(m_i)]$ is a compact subset of $\mathbb{R}^N_+$. For the sake of brevity, we usually identify the game $(N, V_{(N,E,d)}, H)$ with the pair $(N, V_{(N,E,d)})$.

The next result shows that it is actually an NTU-game.

Proposition 1 *Let (N, E, d) be a bankruptcy problem with non-linear utilities, where $d = (m, u)$ and m is defined as in (1), and $(N, V_{(N,E,d)})$ is the corresponding bankruptcy game. Then, it holds that*

(i)	$V_{(N,E,d)}(S) \neq \varnothing, \forall S \subset N.$
(ii)	*If* $k \in V_{(N,E,d)}(S)$ *and* $w_i \leq k_i, \forall i \in S,$ *then* $w \in V_{(N,E,d)}(S)$.
(iii)	*If* $S \cap T = \varnothing,$ *then* $V_{(N,E,d)}(S) \times V_{(N,E,d)}(T) \subset V_{(N,E,d)}(S \cup T)$.
(iv)	$k \in V_{(N,E,d)}(N) \Longrightarrow k \leq w,$ *for some* $w \in H,$

i.e. $(N, V)_{(N,E,d)}$ *is an NTU-game.*

Proof. From (2) it is easy to check that conditions $(i), (ii)$ and (iv) are met. Condition (iii) represents the superadditivity in our context. Consider $(k^S, k^T) \in V_{(N,E,d)}(S) \times V_{(N,E,d)}(T)$ with $S \cap T = \varnothing$. Then, there exist

$$\begin{array}{l} x^S \in \mathbb{R}^S_+ \mid \sum_{i\in S} x_i^S \leq v(S), u\left(x^S\right) \geq k^S, \text{ and} \\ x^T \in \mathbb{R}^T_+ \mid \sum_{i\in T} x_i^T \leq v(T), u\left(x^T\right) \geq k^T. \end{array}$$

Since (N, v) is a classical bankruptcy game, it holds that

$$v(S \cup T) \geq v(S) + v(T) \geq \sum_{i\in S} x_i^S + \sum_{i\in T} x_i^T,$$

with $\left(x^S, x^T\right) \in \mathbb{R}^{S\cup T}_+$. Thus, $u\left(x^S, x^T\right) \geq \left(k^S, k^T\right) \in V_{(N,E,d)}(S \cup T)$, i.e. superadditivity is fulfilled. $\square$

A solution for NTU-games is a correspondence F that allocates to each game (N, V), of a family of NTU-games, a subset $F(N, V) \subset \mathbb{R}^N$. A well-known solution concept for NTU-games is the *core*,

$$C(N, V) = \{x \in V(N) \mid \nexists\, S \subset N,\ S \neq \emptyset,$$

$$\text{and } y \in V(S), \text{ with } y_i > x_i\ \forall i \in S\}.$$

In our setting, Theorem 1 states that these bankruptcy NTU-games are balanced.

Theorem 1 *Let (N, E, d) be a bankruptcy problem with non-linear utilities and $(N, V_{(N,E,d)})$ the corresponding bankruptcy NTU-game. Then the core, $C(N, V_{(N,E,d)})$, is non-empty.*

Proof. Consider a balanced collection $\{S_1, ..., S_m\}$, i.e. suppose that there exist positive numbers $\alpha_1, ..., \alpha_m$ such that $\sum_{j:i\in S_j} \alpha_j = 1, \forall i \in N$. We have to prove that $\bigcap_{j=1}^{m} V_{(N,E,d)}(S_j) \subset V_{(N,E,d)}(N)$. Take $k \in \bigcap_{j=1}^{m} V_{(N,E,d)}(S_j)$, then there exist $x^j \in \mathbb{R}^{S_j}$ such that $\sum_{i\in S_j} x_i^j \leq v(S_j)$ and $u_i\left(x_i^j\right) \geq k_i, \forall i \in S_j$. Thus, $\alpha_j \sum_{i\in S_j} x_i^j \leq \alpha_j v(S_j)$ and, consequently,

$$\sum_{j=1}^{m} \alpha_j \sum_{i\in S_j} x_i^j \leq \sum_{j=1}^{m} \alpha_j v(S_j) \leq v(N) = E,$$

where the last inequality holds because (N, v) is a classical bankruptcy game. Taking into account that $\sum_{i\in N} \sum_{j:i\in S_j} \alpha_j x_i^j = \sum_{j=1}^{m} \alpha_j \sum_{i\in S_j} x_i^j$ and $\sum_{j:i\in S_j} \alpha_j x_i^j \geq \min_{j:i\in S_j} \left\{x_i^j\right\}$, we can derive that $u_i(\sum_{j:i\in S_j} \alpha_j x_i^j) \geq u_i(\min_{j:i\in S_j} \left\{x_i^j\right\}) \geq k_i$. Therefore, $k \in V_{(N,E,d)}(N)$. □

Related with the non-emptiness of the core we obtain the next result.

Theorem 2 *Let (N, E, d) be a bankruptcy problem with non-linear utilities, $(N, V_{(N,E,d)})$ the corresponding bankrupty NTU-game and (N, v) the game with transferable utility associated with the bankruptcy problem (N, E, m). If $x \in C(N, v)$ then $u(x) \in C(N, V)$.*

Proof. If $x \in C(N, v)$[2] then $\sum_{i\in N} x_i = v(N)$ and thus $u(x) \in V(N)$. Now suppose that there exist $S \subset N$ and $k \in V(S)$ such that $k_i > u_i(x_i)$ for all $i \in S$. $k \in V(S)$ implies that there exists $y \in \mathbb{R}^S$ such that $\sum_{i\in S} y_i \leq v(S)$ and $u(y) \geq k$. Thus we have that $u_i(y_i) \geq k_i > u_i(x_i)$ for all $i \in S$ and then $y_i > x_i$ for all $i \in S$ because the functions u_i are non-decreasing. In this way we obtain that $v(S) \geq \sum_{i\in S} y_i > \sum_{i\in S} x_i$, but this is a contradiction with $x \in C(N, v)$. □

Classical bankruptcy games turned out to be convex, so the next step in our analysis of bankruptcy NTU-games should be related to studying

[2] If (N, v) is a game with transferable utilty, the core of the game is defined as $C(N, v) = \{X \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N) \text{ and } \sum_{i\in S} x_i \geq v(S)\ \forall S \subset N\}$.

whether this property also holds in our context or not. Vilkov (1977) and Sharkey (1981) introduced two related concepts for NTU-games, ordinal and cardinal convexity, respectively, mainly based on supermodularity. The reader is referred to Hendrickx et al (2000, 2002) for other marginalistic interpretations and their relationships.

An NTU-game (N, V) is called ordinally convex if, for all coalitions $S, T \subset N$ such that $S \neq \emptyset, T \neq \emptyset$ and for all $k \in \mathbb{R}^N$ such that $k_S \in V(S)$ and $k_T \in V(T)$, we have $k_{S\cap T} \in V(S \cap T)$ or $k_{S\cup T} \in V(S \cup T)$.

An NTU-game is called cardinally convex if

$$V^0(S) + V^0(T) \subset V^0(S \cap T) + V^0(S \cup T)^{3},$$

for all $S, T \subset N, S \neq \varnothing \neq T$.

In this paper we focus on the so-called ordinal convexity, because as the next example shows, although strong conditions on the utility functions are requiered we can not guarantee that bankruptcy NTU-games are cardinally convex.

Example 1 *Let* $(\{1,2,3,4\}, 8, ((1, 1000x), (2, x^2), (3, x^3), (4, x^4)))$ *be a bankruptcy problem with convex utilities.*

In this case, $V^0(1,2) + V^0(1,3) - V^0(1) \nsubseteq V^0(1,2,3)^4$. *Since, we can consider*

$$\begin{array}{ll} \textit{for } \{1,2\}, & (1000, 0) \in V(1,2) \text{ and} \\ \textit{for } \{1,3\}, & (1000, 1) \in V(1,3). \end{array}$$

Furthermore, it is straightforward to prove that $V(1) = \{k \in \mathbb{R} \mid k \geq 0\}$ *and if* $(k_1, k_2, k_3) \in V(1,2,3)$ *then* $k_1 \leq 1000$.

Theorem 3 *Let* (N, E, d) *be a bankruptcy problem with non-linear utilities. The corresponding bankruptcy NTU-game* $(N, V_{(N,E,d)})$ *is ordinally convex.*

Proof. Let $k \in \mathbb{R}^N$ be such that $k_S \in V(S)$ and $k_T \in V(T)$. Then, there exist x^S with $\sum_{i\in S} x_i^S \leq v(S), k_S \leq u(x^S)$, and y^T with $\sum_{i\in T} y_i^T \leq v(T), k_T \leq u(y^T)$.

We can assume without loss of generality that $x_i^S = y_i^T$ for all $i \in S \cap T$, because $k_i^S = k_i^T$ for all $i \in S \cap T$. If $S \cap T = \varnothing$, it is straightforward that $k_{S\cup T} \in V(S \cup T)$. Thus, let us assume that $S \cap T \neq \varnothing$. In this problem, we have $k_{S\cap T} \leq u(x_{S\cap T}^S)$ and two cases can arise:

---

[3]If (N, V) is an NTU-game and $S \subset N$, $V^0(S) = \{x \in \mathbb{R}^N \mid x_S \in V(S) \text{ and } x_i = 0\ \forall i \in N\backslash S\}$.

[4]Instead of $V(\{i\})$, $V(\{i,j\})$, etc. we often write $V(i)$, $V(i,j)$, etc.

1. If $\sum_{i\in S\cap T} x_i^S \leq v(S\cap T)$, then $k_{S\cap T} \in V(S\cap T)$.

2. If $\sum_{i\in S\cap T} x_i^S > v(S\cap T)$, we take $z \in \mathbb{R}_+^{S\cup T}$ such that

$$\begin{array}{ll} z_i = x_i^S, & \text{if } i \in S\backslash T \\ z_i = y_i^T, & \text{if } i \in T\backslash S \\ z_i = x_i^S = y_i^T, & \text{if } i \in S\cap T. \end{array}$$

   Therefore, $\sum_{i\in S\cup T} z_i = \sum_{i\in S} x_i^S + \sum_{i\in T} y_i^T - \sum_{i\in S\cap T} x_i^S \leq v(S) + v(T) - v(S\cap T) \leq v(S\cup T)$, where the last inequality holds because the classical bankruptcy game (N, v) is convex. Moreover, $k_{S\cup T} \leq u(z)$ and, as a result, $k_{S\cup T} \in V(S\cup T)$. $\square$

3 The CEA rule

3.1 Definition

We have focussed our attention on bankruptcy NTU-games so far. In this section we are going to tackle the problem using bankruptcy rules. In particular, we are interested in the role which the Constrained Equal Awards (CEA) rule plays in this setting.

Definition 1 *The Constrained Equal Awards (CEA) rule for bankruptcy problems with non-linear utilities is a nonnegative function which assigns to each (N, E, d) a payoff vector $CEA(N, E, d)$, where*

$$CEA_i(N, E, d) = \min\{m_i, u_i^{-1}(\alpha)\}$$

with $\alpha \in \mathbb{R}_+$ and such that

$$\sum_{i=1}^{n} \min\{m_i, u_i^{-1}(\alpha)\} = E \text{ and } \sum_{i=1}^{n} \min\{m_i, u_i^{-1}(\alpha')\} > E,$$

for all $\alpha' > \alpha$[5]*.*

[5] We would like to point out that we could opt for an alternative definition of the CEA rule in which efficiency was not required, i.e. with

$$\sum_{i=1}^{n} \min\{m_i, u_i^{-1}(\alpha)\} \leq E,$$

but we have preferred to keep as it was introduced in this section in order to preserve such an important feature for a rule.

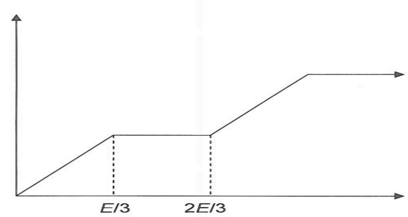
Remark 1 *We use*

$$u_i^{-1}(\alpha) = \begin{cases} \min\{x \in \mathbb{R}_+ \mid u_i(x) \geq \alpha\} & \textit{if } \alpha \leq u_i(m_i) \\ +\infty \textit{ or } m_i & \textit{if } \alpha > u_i(m_i) \end{cases}$$

as the definition of the inverse function of u_i, for all $i \in N$.

As Example 2 illustrates, we need to impose an additional restriction upon the utility functions in order to guarantee that the CEA rule is uniquely determined.

Example 2 *Let $(\{1,2\}, E, d)$ be an NTU bankruptcy problem, where both agents have the same utility function:*



In this case, we cannot find an $\alpha \in \mathbb{R}_+$ such that $\sum_{i=1}^{2} \min\{m_i, u_i^{-1}(\alpha)\} = E$.

However, if we consider strictly increasing utility functions, as in Proposition 2, the CEA rule is well defined.

Proposition 2 *Let (N, E, d) be a bankruptcy problem with non-linear utilities. If, for all $i \in N$, u_i are strictly increasing, then $CEA(N, E, d)$ consists in only one point.*

Proof. First, we are going to prove that every u_i^{-1} is a continuous and non-decreasing function in $(0, u_i(m_i))$.

Let $0 < \alpha < \alpha' < u_i(m_i)$. We have

$$\{x \in \mathbb{R}_+ \mid u_i(x) \geq \alpha'\} \subset \{x \in \mathbb{R}_+ \mid u_i(x) \geq \alpha\},$$

and, therefore

$$\min\{x \in \mathbb{R}_+ \mid u_i(x) \geq \alpha'\} \geq \min\{x \in \mathbb{R}_+ \mid u_i(x) \geq \alpha\},$$

which implies $u_i^{-1}(\alpha) \leq u_i^{-1}(\alpha')$. Hence, u_i^{-1} is a non-decreasing function.

In order to prove that u_i^{-1} is continuous in $(0, u_i(m_i))$, let $\alpha \in (0, u_i(m_i))$. Consider the limits

$$\lim_{y \to \alpha^+} \min\{x \in \mathbb{R}_+ \mid u_i(x) \geq y\} = x_1 \text{ and}$$
$$\lim_{y \to \alpha^-} \min\{x \in \mathbb{R}_+ \mid u_i(x) \geq y\} = x_2.$$

Both exist and coincide with the infimum, respectively supremum, of the function because u_i^{-1} is a non-decreasing and bounded function. Furthermore, we have that $x_1 \geq u_i^{-1}(\alpha) \geq x_2$.

Let us assume that $x_1 > u_i^{-1}(\alpha)$. Thus, we can take x' such that $u_i^{-1}(\alpha) < x' < x_1$. Since u_i is strictly increasing, we obtain

$$\alpha \leq u_i(u_i^{-1}(\alpha)) < u_i(x') < u_i(x_1),$$

and $\min\{x \in \mathbb{R}_+ \mid u_i(x) \geq u_i(x')\} \leq x' < x_1$. But this is a contradiction with the definition of x_1. Hence, it holds that $x_1 = u_i^{-1}(\alpha)$ and, consequently, u_i^{-1} is right-continuous.

If $u_i^{-1}(\alpha) > x_2$, we can consider x' such that $x_2 < x' < u_i^{-1}(\alpha)$. Since u_i is strictly increasing, two cases arise:

1. $u_i(x') \leq \alpha$. Since $x' > x_2$, we obtain a contradiction with the way in which x_2 was chosen.

2. $u_i(x') > \alpha$. This means that $x' \in \{x \in \mathbb{R}_+ \mid u_i(x) \geq \alpha\}$, so $x' \geq \min\{x \in \mathbb{R}_+ \mid u_i(x) \geq \alpha\} = u_i^{-1}(\alpha)$ what contradicts $x' < u_i^{-1}(\alpha)$.

Therefore, we can conclude $u_i^{-1}(\alpha) = x_2$, and this implies that u_i^{-1} is left-continuous and, as a result, it is continuous.

Let us consider the set $\{u_i(m_i), i \in N\}$. From it we can build the corresponding sequence $\alpha_1 < \alpha_2 < ... < \alpha_r$, $r \leq n$, and associated with each α we can derive the sets of players $S_j = \{i \in N \mid u_i(m_i) \geq \alpha_j\}$, $j = 1, 2, ..., r$. Note that $S_1 = N \supsetneq S_2 \supsetneq ... \supsetneq S_r$. We define the function:

$$U^{-1}(\alpha) = \begin{cases} \sum_{i \in S_1} u_i^{-1}(\alpha) & \text{if } 0 \leq \alpha \leq \alpha_1 \\ \sum_{i \in S_1 \setminus S_2} m_i + \sum_{i \in S_2} u_i^{-1}(\alpha) & \text{if } \alpha_1 < \alpha \leq \alpha_2 \\ ... & ... \\ \sum_{i \in S_1 \setminus S_r} m_i + \sum_{i \in S_r} u_i^{-1}(\alpha) & \text{if } \alpha_{r-1} < \alpha \leq \alpha_r \\ \sum_{i \in S_1} m_i & \text{if } \alpha > \alpha_r. \end{cases}$$

This is a continuous function and $\sum_{i\in S_1} m_i > E$. Taking into account the Intermediate Value Theorem, there exists $\alpha \in \mathbb{R}_+$ such that $U^{-1}(\alpha) = E$ and $U^{-1}(\alpha') > E$, for all $\alpha' > \alpha$. Furthermore, this α is unique.

Finally, note that $U^{-1}(\alpha) = \sum_{i=1}^{n} \min\{m_i, u_i^{-1}(\alpha)\} = \sum_{i=1}^{n} CEA_i(N, E, d)$.
□

3.2 Game theoretical rules

In this subsection we relate two possible approaches for the study of bankruptcy problems with non-linear utilities. Following Section 2, the first approach is based on NTU-games. The second, according to the subsection above, is based on the direct definition of rules across the problem. We should note that the main ideas of this subsection are inspired by Curiel et al (1987).

Let (N, V) be an NTU-game. It will be of interest to consider the so-called *utopia vector* $K(V)$ (Borm et al, 1992) that is defined for each $i \in N$ by

$$K_i(V) = \sup\{t \in \mathbb{R} \text{ such that } \exists a \in \mathbb{R}^{N\setminus\{i\}} \text{ with } (a,t) \in V(N),$$
$$\nexists b \in V(N\setminus\{i\}) \text{ with } b > a\}.$$

To attain our objective, first we give a definition.

Definition 2 *A rule f for bankruptcy problems with non-linear utilities is a game theoretical rule if there exists a solution F for NTU-games such that $u(f(N, E, d)) = F(N, V_{(N,E,d)})$, for all (N, E, d).*

We provide a characterization of the game theoretical rules for bankruptcy problems with non-linear utilities that generalizes the well-known result for classical bankruptcy rules.

Theorem 4 *A rule f for bankruptcy problems with non-linear utilities is a game theoretical rule if and only if for every bankruptcy problem with non-linear utilities (N, E, d), $f(N, E, d) = f(N, E, d')$, where $d'_i = (\min\{E, m_i\}, u_i)$ for all $i \in N$.*

Proof. *Necessary condition.* Let f be a game theoretical rule for bankruptcy problems with non-linear utilities and F the corresponding solution for NTU-games. Let (N, E, d) be a bankruptcy problem with non-linear utilities, then:

$$u(f(N, E, d)) = F(N, V_{(N,E,d)}) = F(N, V_{(N,E,d')}) = u(f(N, E, d')).$$

Thus, we have that $f(N,E,d) = f(N,E,d')$.

Sufficient condition. Let f be a rule for bankruptcy problems with non-linear utilities such that for all (N,E,d), $f(N,E,d) = f(N,E,d')$. We will define a solution for NTU-games. Let (N,V) be an NTU-game in which each player $i \in N$ has an utility function u_i. We define

$$F(N,V) = u(f(N,E^V,d^V)),$$

where

$$E^V = \max\{\sum_{i\in N} x_i \ \big| x \in u^{-1}(V(N)_+)\},\ d^V = (m^V, u) \text{ and}$$

$$m_i^V = u_i^{-1}(K_i(V)) + n^{-1}(E^V - \sum_{i\in N} u_i^{-1}(K_i(V)))_+.$$

Let (N,E,d) be a bankruptcy problem with non-linear utilities. We obtain that

$$u(f(N,E,d)) = u(f(N,E,d')) = F(N,V_{(N,E,d)})$$

and, then, the rule f is game theoretical. $\square$

Curiel et al (1987) proved that the CEA rule applied to a classical bankruptcy problem provides an element of the core of the game with transferable utility associated with the bankruptcy problem. In the general setup we are working with, we obtain a similar result.

Theorem 5 *Let (N,E,d) be a bankruptcy problem with non-linear utilities. Then, $u(CEA(N,E,d)) \in C(N,V_{(N,E,d)})$.*

Proof. Suppose that there exists a bankruptcy problem with non-linear utilities (N,E,d) such that $u(CEA(N,E,d)) \notin C(N,V_{(N,E,d)})$. Then there exist a coalition $S \subset N$ and a vector of utilities $k \in V_{(N,E,d)}(S)$ such that

$$k > u(CEA(N,E,d))_S. \tag{3}$$

Because $k \in V_{(N,E,d)}(S)$ and the definition of the NTU-game $(N,V_{(N,E,d)})$ there is a vector $x \in \mathbb{R}^S_+$ such that

$$\sum_{i\in S} x_i \le v(S) \text{ and } u(x) \ge k, \tag{4}$$

where $v(S) = \max\{0, E - \sum_{i\in N\setminus S} m_i\}$, that is, (N, v) is the classical TU bankruptcy game associated with the bankruptcy problem (N, E, m). In this way, by (3) and (4) we have that

$$v(S) \geq \sum_{i\in S} x_i > \sum_{i\in S} CEA_i(N, E, d). \tag{5}$$

We will define a rule, f, for bankruptcy problems. Let (N, E, c) be such a problem, then:

$$f(N, E, c) = \begin{cases} CEA(N, E, c), & \text{if } c \neq m \\ \\ CEA(N, E, d = (m, u)), & \text{if } c = m. \end{cases}$$

It is straightforward to check that for all bankruptcy problems (N, E, c),

$$\sum_{i\in N} f_i(N, E, c) = E \text{ and } 0 \leq f_i(N, E, c) \leq c_i, \text{ for all } i \in N.$$

By using Theorem 6 in Curiel et al (1987), we have that for every bankruptcy problem (N, E, c), $f(N, E, c) \in C(N, v_{(N,E,c)})$ and, then, for all $S \subset N$, $v(S) \leq \sum_{i\in S} CEA_i(N, E, d)$. But this is a contradiction with (5) and, as a result, $u(CEA(N, E, d)) \in C(N, V_{(N,E,d)})$. $\square$

4 Axiomatic characterizations

In this section we provide two axiomatic characterizations of the CEA rule for bankruptcy problems with non-linear utilities. We assume the condition given by Proposition 2. To start with, we introduce some natural properties for a rule in this context.

A *rule for bankruptcy problems with non-linear utilities* is a function f that assigns to every bankruptcy problem with non-linear utilities (N, E, d) a vector $f(N, E, d) \in \mathbb{R}^N$ such that $0 \leq f_i(N, E, d) \leq m_i$ for all $i \in N$ and $\sum_{i\in N} f_i(N, E, d) = E$. Hence a rule provides a possible division of the resource among the users, where the amount $f_i(N, E, d)$ that user i obtains is non-negative and not larger than their claim on the resource m_i.

Equal treatment. A rule f satisfies equal treatment if for all (N, E, d) and for all $i, j \in N$ such that $d_i = d_j$, then $f_i(N, E, d) = f_j(N, E, d)$.

If a rule f satisfies equal treatment, two agents with the same resource claim and the same utility function will obtain the same outcome. This property has a similar flavour to the basic symmetry requirement of equal treatment of equals (cf. O'Neill, 1982).

Invariance under claims truncation. A rule f satisfies invariance under claims truncation if for all (N, E, d) we have

$$f(N, E, d) = f(N, E, d'),$$

where $d'_i = (\min\{E, m_i\}, u_i)$ for all $i \in N$.

This property (cf. Curiel et al, 1987) means that truncating each resource claim by the estate does not influence the outcome.

Weak composition. A rule f satisfies weak composition if, for all (N, E, d) and for all $0 \leq E' \leq E$ when $u_i = u_j$ for all $i, j \in N$, we have

$$f(N, E, d) = f(N, E', d) + f(N, E - E', (m_i - f_i(N, E', d), u_i)_{i \in N}).$$

According to this property (cf. Young, 1988) we can divide the estate using two different procedures, which result in the same outcome when the utilities of the users are equal. In the first, we divide the avaliable resources directly by means of f. In the other procedure, first we divide a part E' of the estate and then the remainder, $E - E'$, on the basis of the remaining claims, both times using f.

Utility consistency of awards. A rule f satisfies utility consistency of awards if, for all (N, E, d) and for all $S \subset N$, it holds

$$f(S, \sum_{i \in S} u_i(f_i(N, E, d)), (u_i(m_i), id)_{i \in S}) = (u_i(f_i(N, E, d)))_{i \in S},$$

where id denotes the identity function defined over the set of the real numbers.

Utility consistency of awards (cf. Aumann and Maschler, 1985) states that if a set of users solve a bankruptcy problem with non-linear utilities and after that a subset of them decide to redivide the utility that they have obtained, then this subset of agents should obtain the same utility in both cases.

When we restrict ourselves to the type of classic bankruptcy problems where agents have the same linear utility function, equal treatment, invariance under claims truncation and weak composition reduce to the classical properties of equal treatment, invariance under claims truncation and composition. Dagan (1996) characterized the CEA rule for bankruptcy problems by using these three properties. We extend that result to the context of bankruptcy problems with non-linear utilities by adding a fourth property.

Theorem 6 *A rule f for bankruptcy problems with non-linear utilities satisfies equal treatment, invariance under claims truncation, weak composition, and utility consistency of awards if and only if*

$$f(N, E, d) = CEA(N, E, d),$$

for all bankruptcy problems with non-linear utilities.

Proof. It is straightforward to see that the CEA rule for bankruptcy problems with non-linear utilities satisfies the four properties of the theorem. Thus, we only have to prove the uniqueness. Let f be a rule that satisfies the four properties and (N, E, d) a bankruptcy problem with non-linear utilities. We will prove that $f(N, E, d) = CEA(N, E, d)$. We consider the bankruptcy problem $(N, \sum_{i=1}^{n} u_i(f_i(N, E, d)), (u_i(m_i), id)_{i \in N})$. Since f satisfies *equal treatment*, *invariance under claims truncation*, and *weak composition*, and using similar arguments to those in the proof by Dagan (1996), we obtain that

$$f(N, \sum_{i=1}^{n} u_i(f_i(N, E, d)), (u_i(m_i), id)_{i \in N})$$

$$= CEA(N, \sum_{i=1}^{n} u_i(f_i(N, E, d)), (u_i(m_i), id)_{i \in N}).$$

Moreover, f satisfies *utility consistency of awards* and taking into account the last result, we can assert that for all $i \in N$

$$u_i(f_i(N, E, d)) = CEA_i(N, \sum_{i=1}^{n} u_i(f_i(N, E, d)), (u_i(m_i), id)_{i \in N}). \quad (6)$$

Applying the definition of the CEA rule we have that, for all $i \in N$,

$$CEA_i(N, \sum_{i=1}^{n} u_i(f_i(N, E, d)), (u_i(m_i), id)_{i \in N}) = \min\{u_i(m_i), \alpha\},$$

with $\alpha \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} \min\{u_i(m_i), \alpha\} = \sum_{i=1}^{n} u_i(f_i(N, E, d)).$$

Taking into account (6) we can deduce that

$$f_i(N, E, d) = \min\{m_i, u_i^{-1}(\alpha)\},$$

where $\alpha \in \mathbb{R}$ and we know that $\sum_{i=1}^{n} f_i(N, E, d) = E$. As a consequence,

$$f_i(N, E, d) = CEA_i(N, E, d), \ \forall i \in N. \ \square$$

Corollary 1 *The CEA rule for bankruptcy problems with non-linear utilities is a game theoretical rule.*

Other natural properties are the following.

Strong equal treatment. A rule f satisfies strong equal treatment if for all (N, E, d), for all $i, j \in N$ such that $u_i(m_i) = u_j(m_j)$, then $u_i(f_i(N, E, d)) = u_j(f_j(N, E, d))$.

If a rule f satisfies strong equal treatment and two agents have resource claims with the same utility, they obtain outcomes with the same utility. This property has also a similar flavour than the basic symmetry requirement of equal treatment of equals (cf. O'Neill, 1982) and it implies equal treatment.

Path independence. A rule f satisfies path independence if for all (N, E, d) and for all $E' \geq E$,

$$f(N, E, d) = f(N, E, d'),$$

where $d = (m, u)$ and $d' = (f(N, E', d), u)$.

If a rule f satisfies path independence (cf. Moulin, 1987) we can divide the total amount of resources using two procedures yielding the same result. The first procedure divides the estate directly through f. In the second, we first divide a bigger amount $E' \geq E$ and, then, use the outcome $f(N, E', d)$ as resource claims to divide the real E, in both cases using f.

Consistency. A rule f satisfies consistency if for all (N, E, d), for all $S \subset N$ and for all $i \in S$,

$$f_i(N, E, d) = f_i(S, \sum_{i \in S} f_i(N, E, d), d_S),$$

where d_S denotes the claims on the resources and the utilities of the users in S.

Consistency (cf. Aumann and Maschler, 1985) says that if a set of users solves a bankruptcy problem with non-linear utilities and, afterwards, a subset of them decides to redivide the resources they have obtained, then they should obtain the same part of the total in both cases.

Exemption for two users. A rule f satisfies exemption for two users if for all (N, E, d) with $N = \{1, 2\}$ and for all $i, j \in N$, $i \neq j$, such that $u_i(m_i) \leq \dfrac{u_i(m_i) + u_j(E - m_i)}{2}$, we have $f_i(N, E, d) = m_i$[6].

[6] A general statement for n users of this property can be given.

This property (cf. Herrero and Villar, 2001) refers to the behaviour of a rule when resource claim utilities are completely different. It implies that when the utility of a resource claim is smaller than the proportional division of the estate, the rule should grant it the full resource claim. This property is a protective criterion for those users with small claims.

Herrero and Villar (2001) axiomatically characterize the CEA rule for bankruptcy problems through path independence, consistency and exemption. The next theorem extends that result.

Theorem 7 *A rule f for bankruptcy problems with non-linear utilities satisfies path independence, consistency, and exemption for two users if and only if*

$$f(N, E, d) = CEA(N, E, d),$$

for every bankruptcy problem with non-linear utilities.

Proof. Again, we omit the part of the existence and focus on the uniqueness. Let f be a rule for bankruptcy problems with non-linear utilities that satisfies the properties of the theorem. Then we have that f satisfies strong equal treatment (the proof follows the same steps that the proof of Claim 4 in Herrero and Villar, 2001). Let (N, E, d) be a bankruptcy problem with non-linear utilities in which N has only two elements. First, suppose that

$$u_1(m_1) \leq \frac{u_1(m_1) + u_2(E - m_1)}{2},$$

then *exemption* implies that $f_1(N, E, d) = m_1, \ f_2(N, E, d) = E - m_1$ and, thus, $f(N, E, d) = CEA(N, E, d)$. If

$$u_1(m_1) > \frac{u_1(m_1) + u_2(E - m_1)}{2},$$

then $u_1(m_1) > u_2(E - m_1)$ and $E < u_2^{-1}(u_1(m_1)) + m_1$. Take $E' = u_2^{-1}(u_1(m_1)) + m_1$. We have that $E' > E$, $E' - m_1 = u_2^{-1}(u_1(m_1))$ and, therefore, $u_2(E' - m_1) = u_1(m_1)$. So we can write $u_1(m_1) = \dfrac{u_1(m_1) + u_2(E' - m_1)}{2}$ and, then, *exemption* implies $f(N, E', d) = (m_1, E' - m_1) = CEA(N, E', d)$. *Path independence* allows us to derive $f(N, E, d) = f(N, E, (f(N, E', d), u))$. Taking into account that $f(N, E', d) = (m_1, E' - m_1)$ and $u_2(E' - m_1) = u_1(m_1)$, if we apply *strong equal treatment,* we can conclude that

$$u_1(f_1(N, E, d)) = u_2(f_2(N, E, d))$$

and finally, $f(N, E, d) = CEA(N, E, d)$. In the general case, when N has more than two elements, we can apply *consistency* and the well-known fact that the consistent extension of a rule is unique (see, for example, Aumann and Maschler, 1985), to achieve the desired result. □

Remark. The properties of Theorem 6 and of Theorem 7 are logically independent. We leave the proof to the reader.

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