Sozialökonomisches Seminar
der Universität Hamburg

Computing Power Indices: Multilinear extensions and new characterizations

José María Alonso-Meijide,
Balbina Casas-Méndez, Manfred J. Holler
and Silvia Lorenzo-Freire

June 2007

Beiträge zur Wirtschaftsforschung Nr. 153

Computing Power Indices: Multilinear extensions and new characterizations

J. M. Alonso-Meijide[1], B. Casas-Méndez[2], M. J. Holler[3], S. Lorenzo-Freire[4]

June 2007

No. 153

Abstract Johnston (1978), Deegan and Packel (1979), and Holler (1982) proposed three power indices for simple games: Johnston index, Deegan-Packel index, and the Public Good Index. In this paper, methods to compute these indices by means of the multilinear extension of the game are presented. Furthermore, a new characterization of the Public Good Index is given. Our methods are applied to two real-world examples taken from the political field.

Keywords: simple game, power index, multilinear extension, axiomatization
MSC (2000) classification: 91A12
JEL classification: C71

Forthcoming in: *European Journal of Operational Research*

[1] Corresponding author. meijide@lugo.usc.es, Department of Statistics and Operations Research, University of Santiago de Compostela

[2] eigradu3@usc.es, Department of Statistics and Operations Research, University of Santiago de Compostela

[3] holler@econ.uni-hamburg.de, Institute of SocioEconomics, University of Hamburg

[4] silvia_lorenzo@uvigo.es, Department of Statistics and Operations Research, University of Vigo

1 Introduction

Simple games can be used to model decision-making processes. Many interesting applications have been carried out in the field of political science. Different power indices have been suggested in order to assess the a priori distribution of power in a voting body. However, we arise far from consensus over the issue of choice of an appropriate power index in a given context. In fact, in many papers, the choice of the power index is not theoretically justified and several power indices are employed.

The main power indices of the literature include the Shapley-Shubik index (Shapley and Shubik, 1954), the Banzhaf index (Banzhaf, 1965), the Johnston index (Johnston, 1978), the Deegan-Packel index (Deegan and Packel, 1979), and the Public Good Index (Holler, 1982). These measures are based on an evaluation of an actor's relative importance to coalition formation. In the context of simple games, a winning coalition is vulnerable when it has at least one member whose removal would cause the resulting coalition to be a losing coalition. An agent is considered critical when his elimination from a winning coalition turns this coalition into a losing coalition. A minimal winning coalition is one such that all its members are critical. In this paper the last three indices are further examined.

In Banzhaf's model, power of an agent is proportional to the number of coalitions in which he is critical. Johnston argued that the Banzhaf index, which is based on the idea of a removal of a critical voter from a winning coalition, does not take into account the total number of critical members in each coalition. Clearly, if a voter is the only critical agent in a coalition, he enjoys far more power than in the case where all agents are critical. This is the main idea underlying the Johnston index.

According to Deegan and Packel, only minimal winning coalitions should be considered in establishing the power of a voter. They proposed an index under the assumptions that all minimal winning coalitions are equiprobable and all the voters belonging to the same minimal winning coalition should obtain the same power.

The Public Good Index is determined by the number of minimal winning coalitions containing the voter divided by the sum of such numbers across all the voters. Holler (1982) proposes that only minimal winning coalitions should be considered when it comes to measuring power and the outcome is a public good. This is not to say that merely minimal winning coalitions will form. However, coalition with surplus players are formed by luck as they invite free-riding if (opportunity) costs are implied with coalition membership (e.g., responsibility). In a recent publication, Braham (2005) postulates that a causal explanation of the coalition outcome necessitates to focus on minimal winning coalitions[1].

[1]Let's assume simple majority voting and a winning coalition {A,B,C} with A controlling 30 per cent of votes

In this paper, it will not be discussed which index is most appropriate as the choice of an index may well depend on the situation under analysis. To facilitate this choice, some desirable properties have been introduced in the context of power indices. In this paper, some of these properties will be mentioned, as well as some characterizations of the main power indices according to them. In Lorenzo-Freire *et al.* (2007), new characterizations of Deegan-Packel and Johnston indices are obtained with monotonicity properties. In this paper, a new characterization of the Public Good Index by means of a property of monotonicity with a similar flavour is provided and the other indices are revisited comparing new and old characterizations.

One of the main difficulties with these indices is that computation generally requires the sum of a very large number of terms. Owen (1972) defined the multilinear extension of a game. It gives the expected utility of a random coalition. The multilinear extension has been used by Owen to compute the Shapley value (Shapley, 1953) and the Banzhaf value (Owen, 1975). Both values are probabilistic (Weber, 1988), that is to say, values that satisfy the property of additivity. The multilinear extensions are useful in computing the power of large games such as the Presidential Election Game and the Electoral College Game studied by Owen (1972). The multilinear extension approach has two advantages: thanks to its probabilistic interpretation, the central limit theorem of probability can be applied, and, further, it is applied to composition of games.

The main objective of this paper is to analyze whether some modification of the multilinear extension technique might be used to calculate the indices of Johnston, Deegan-Packel, and Public Good Index. To the best of our knowledge, the multilinear extension has not been applied to values which are not probabilistic. These three indices are defined on the basis of either vulnerable coalitions (in the case of Johnston index) or minimal winning coalitions (in the case of Deegan-Packel index and Public Good Index). Sometimes, however, it is not known a priori which coalitions are vulnerable or minimal winning, even though the game is known, specially in games with a large number of players. Obviously, in such a case it is very difficult to compute these indices. The advantage of the procedures presented here is that if the multilinear extension of the game is known, an algorithm to easily compute these indices can be provided.

In Section 2, the main notions related with characteristic function games and simple games are given. Section 3 is devoted to a review of the main power indices and their axiomatic characterizations. In Section 4, the procedures to calculate the Johnston index, the Deegan-Packel index, and the Public Good Index by means of the multilinear extensions are introduced. In Section 5, they are applied to the Basque Country Parliament and the Victoria Proposal for Amendments to the Canadian Constitution. Section 6 concludes.

---

and B and C controlling 25 per cent of votes each. Then, the minimum winning coalitions {A,B} and {A,C} are responsible for the outcome of coalition {A,B,C}. Obviously, neither B nor C were ”causal” for the outcome of coalition {A,B,C}.

2 Preliminaries

2.1 Characteristic function games

A characteristic function game is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and v, the characteristic function, is a real function on $2^N = \{S : S \subseteq N\}$ with $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a coalition. $G(N)$ denotes the set of characteristic function games with set of player N. Shorthand notation will be used and will be written $S \cup i$ for the set $S \cup \{i\}$ and $S \backslash i$ for the set $S \backslash \{i\}$.

A **null player** in a game (N, v) is a player $i \in N$ such that $v(S \cup i) = v(S)$ for all $S \subseteq N \backslash i$. Two players $i, j \in N$ are **symmetric** in a game (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$.

Given a family of games $H \subseteq G(N)$, a **solution** (or a value) on H is a function f, which assigns to a game $(N, v) \in H$ a vector

$$(f_1(N, v), \ldots, f_n(N, v)) \in \mathbb{R}^n,$$

where the real number $f_i(N, v)$ is the payoff of the player i in the game (N, v) according to f. It is useful to single out a list of desirable properties of solutions.

A solution f is **additive** if $f(N, v + w) = f(N, v) + f(N, w)$ for every $(N, v), (N, w) \in H$ such that $(N, v + w) \in H$, where $(v + w)(S) = v(S) + w(S)$ for every $S \subseteq N$.

A solution f satisfies the **null player** property if $f_i(N, v) = 0$ for every $(N, v) \in H$ and every null player $i \in N$.

A solution f is **symmetric** if $f_i(N, v) = f_j(N, v)$ for every $(N, v) \in H$ and for every pair of symmetric players $i, j \in N$.

A solution f is **efficient** if $\sum_{i \in N} f_i(N, v) = v(N)$ for every $(N, v) \in H$.

A solution f satisfies the **total power** property if

$$\sum_{i \in N} f_i(N, v) = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \sum_{S \subseteq N \backslash i} [v(S \cup i) - v(S)] \text{ for all } (N, v) \in H.$$

Well-known values for $G(N)$ are the Shapley value (Shapley, 1953) and the Banzhaf value (Owen, 1975). Shapley (1953) characterizes the Shapley value and Feltkamp (1995) the Banzhaf value using some of the properties mentioned above. Those characterizations are recalled here.

- The unique solution f defined on $G(N)$ that satisfies additivity, null player, symmetry, and efficiency is the Shapley value. Given a game $(N, v) \in G(N)$, the Shapley value assigns to each player $i \in N$ the real number:

$$\varphi_i(N, v) = \sum_{S \subseteq N \backslash i} \frac{s!(n - s - 1)!}{n!} [v(S \cup i) - v(S)].$$

- The unique solution f defined on $G(N)$ that satisfies additivity, null player, symmetry, and total power is the Banzhaf value. Given a game $(N,v) \in G(N)$, the Banzhaf value assigns to each player $i \in N$ the real number:

$$\beta_i(N,v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus i} [v(S \cup i) - v(S)].$$

Notice that only one property differentiates both characterizations. The Shapley value satisfies efficiency meanwhile the Banzhaf value satisfies total power.

Young (1985) proposed the strong monotonicity property. A solution f satisfies **strong monotonicity** if $f_i(N,v) \geq f_i(N,w)$ for every pair of games $(N,v),(N,w) \in H$ and for all $i \in N$ such that $v(S \cup i) - v(S) \geq w(S \cup i) - w(S)$ for all $S \subseteq N \setminus i$. Using this property, Young proposed an alternative characterization of the Shapley value.

- The unique solution f defined on $G(N)$ that satisfies strong monotonicity, symmetry, and efficiency is the Shapley value.

Lorenzo-Freire *et al.* (2007) give a corresponding characterization of the Banzhaf value.

- The unique solution f defined on $G(N)$ that satisfies strong monotonicity, symmetry, and total power is the Banzhaf value.

2.2 Simple Games

An important subclass of characteristic function games is the class of simple games. A **simple game** is a characteristic function game (N,v) such that:

- $v(S) = 1$ or $v(S) = 0$ for every $S \subseteq N$.
- v is a monotone function, that is, $v(S) \leq v(T)$, for every $S \subseteq T \subseteq N$.
- $v(N) = 1$.

$SI(N)$ denotes the set of simple games with player set N. In a simple game (N,v), a coalition $S \subseteq N$ is **winning** if $v(S) = 1$, and S is **losing** if $v(S) = 0$. $W(v)$ denotes the set of winning coalitions of the game (N,v). A winning coalition $S \subseteq N$ is a **minimal winning** coalition if every proper subset of S is a losing coalition, that is, S is a minimal winning coalition in (N,v) if $v(S) = 1$ and $v(T) = 0$ for any $T \subset S$. $M(v)$ denotes the set of minimal winning coalitions of the game (N,v) and by $M_i(v)$ the subset of $M(v)$ formed by coalitions $S \subseteq N$ such that $i \in S$.

Given $S \subseteq N, (N,u_S)$ denotes the **unanimity game** of the coalition S, *i.e.*, $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. Note that $|M(u_S)| = 1$.

A winning coalition $S \subseteq N$ is a **quasi-minimal winning** coalition (or vulnerable coalition) if there exists a player $i \in S$ such that $S \backslash i \notin W(v)$. $G(v)$ denotes the set of quasi-minimal winning coalitions of the simple game (N, v). It is clear that for every simple game (N, v),

$$M(v) \subseteq G(v) \subseteq W(v).$$

Given a simple game (N, v), a **swing** for a player $i \in N$ is a coalition $S \subseteq N$ such that $S \backslash i$ is a losing coalition and S is a winning one. $\eta_i(v)$ denotes the set of swings for player $i \in N$. A winning coalition $S \subseteq N$ is a minimal winning coalition if and only if $S \in \eta_i(v)$ for every $i \in S$. A winning coalition $S \subseteq N$ is a quasi-minimal winning coalition if $S \in \eta_i(v)$ for at least one player $i \in S$.

Let $S \subseteq N$, $\chi(S)$ denotes the set of **critical players** of S (a **critical coalition**), *i.e.*, $\chi(S)$ is the set of players i of $S \subseteq N$ such that S is a swing for i. $G(S, v)$ denotes the set of quasi-minimal coalitions such that the set of players $S \subseteq N$ are critical, that it to say, the set of quasi-minimal winning coalitions $T \subseteq N$, such that $\chi(T) = S$. It is important to point out that $\chi(S) = S$ is equivalent to say that S belongs to $M(v)$. For every $i \in N$, $G_i(v)$ denotes the subset of $G(v)$ formed by coalitions $S \subseteq N$ such that $i \in \chi(S)$.

3 An alternative perspective on power measures

Given a family of simple games $H \subseteq SI(N)$, a **power index** on H is a function f, which assigns to a simple game $(N, v) \in H$ a vector

$$(f_1(N, v), \ldots, f_n(N, v)) \in \mathbb{R}^n,$$

where the real number $f_i(N, v)$ is the "power" of the player i in the game (N, v) according to f. The power index of a simple game can be interpreted as a measure of the ability of the different players to turn a losing coalition into a winning one.

A set of independent properties (an axiomatic system) is a convenient tool to decide on the use of an index. It is well-known that the indices of Deegan-Packel, Johnston, and Public Good Index are efficient, symmetric, and assign nothing to null players, and that Shapley-Shubik and Banzhaf indices satisfy transfer property. If the standard axiomatizations of the various indices are considered, the Public Good Index is characterized by mergeability. In order to demonstrate the implication of this property, for the Deegan-Packel and Johnston indices the corresponding versions of this property are specified. In a similar way, the implication of monotonicity properties to compare these three indices are studied, and a new characterization of the Public Good Index with a new property with a similar flavour is provided.

3.1 Shapley-Shubik index

Given a simple game (N, v), the Shapley-Shubik power index assigns to each player $i \in N$ the real number

$$\varphi_i(N, v) = \sum_{S \in \eta_i(v)} \frac{(s-1)!\,(n-s)!}{n!},$$

where s is the number of members in S. Given n players, $n!$ is the number of permutations, $(s-1)!\,(n-s)!$ counts the permutations under the restriction of a coalition S.

In the class of simple games, the additivity property introduced by Shapley (1953) does not apply because the sum of two simple games is not a simple game. Dubey (1975) proposed the transfer property as a substitute of the additivity property and characterized the Shapley value in this class of games.

A power index f defined on $H \subseteq SI(N)$ satisfies the **transfer** property if for all $(N, v), (N, w) \in H$ such that $(N, v \vee w)$, $(N, v \wedge w) \in H$, $f(N, v \vee w) + f(N, v \wedge w) = f(N, v) + f(N, w)$ where for all $S \subseteq N$

$$(v \vee w)(S) = \max\{v(S), w(S)\} \text{ and } (v \wedge w)(S) = \min\{v(S), w(S)\}.$$

The characterization is presented below.

- The unique power index f defined on $SI(N)$ that satisfies transfer, null player, symmetry, and efficiency is the Shapley-Shubik index.

3.2 Banzhaf index

Given a simple game (N, v), the Banzhaf index assigns to each player $i \in N$ the real number:

$$\beta_i(N, v) = \frac{|\eta_i(v)|}{2^{n-1}}.$$

For simple games, the total power property defined previously in the context of characteristic function games can be rewritten, so as to state that power of players adds up to the total number of swings divided by the number of coalitions which can join to player $i \in N$.

A power index f defined on $H \subseteq SI(N)$ satisfies the **total power** property if $\sum_{i \in N} f_i(N, v) = \overline{\eta}(v)/2^{n-1}$, for every simple game $(N, v) \in H$, where $\overline{\eta}(v) = \sum_{i \in N} |\eta_i(v)|$.

Dubey and Shapley (1979) characterized the Banzhaf index as follows.

- The unique power index f defined on $SI(N)$ that satisfies transfer, null player, symmetry, and total power is the Banzhaf index.

3.3 Deegan-Packel index, mergeability, and monotonicity

The power index by Deegan and Packel (1979) assumes that

a) only minimal winning coalitions will emerge victorious,

b) each minimal winning coalition has an equal probability of forming, and

c) players in a minimal winning coalition divide the "spoils" equally.

These assumptions seem reasonable in a wide variety of situations. They define the Deegan-Packel index. Given a simple game (N, v), this index assigns to each player $i \in N$ the real number:

$$\rho_i(N, v) = \frac{1}{|M(v)|} \sum_{S \in M_i(v)} \frac{1}{|S|}. \tag{1}$$

The Deegan-Packel index of a player i is equal to the sum of the inverse of the cardinality of S for the coalitions $S \in M_i(v)$, divided by the cardinality of $M(v)$ in order to achieve normalization.

The Deegan-Packel index does not satisfy transfer property, but it satisfies the property of DP-mergeability. Two simple games (N, v) and (N, w) are **mergeable** if for all pair of coalitions $S \in M(v)$ and $T \in M(w)$, it holds that $S \nsubseteq T$ and $T \nsubseteq S$. The minimal winning coalitions in game $(N, v \vee w)$ are precisely the union of the minimal winning coalitions in games (N, v) and (N, w). If two games (N, v) and (N, w) are mergeable, the mergeability condition guarantees that $|M(v \vee w)| = |M(v)| + |M(w)|$.

A power index f on $H \subseteq SI(N)$ satisfies the **DP-mergeability** property if for any pair of mergeable simple games $(N, v), (N, w) \in H$ such that $(N, v \vee w) \in H$, it holds that for all player $i \in N$:

$$f_i(N, v \vee w) = \frac{|M(v)| f_i(N, v) + |M(w)| f_i(N, w)}{|M(v \vee w)|}.$$

This property states that power in a merged game is a weighted mean of power of the two component games, where the weights come from the number of minimal winning coalitions in each component game, divided by the number of minimal winning coalitions in the merged game. Deegan and Packel (1979) characterized ρ as follows.

- The unique power index f on $SI(N)$ satisfying DP-mergeability, null player, symmetry, and efficiency is the Deegan-Packel power index.

Lorenzo-Freire *et al.* (2007) characterized the Deegan-Packel index replacing the property of DP-mergeability with the property of DP-minimal monotonicity.

A power index f on $H \subseteq SI(N)$ satisfies the property of **DP-minimal monotonicity** if for any pair of games (N, v), $(N, w) \in H$, it holds that for all player $i \in N$ such that $M_i(v) \subseteq M_i(w)$,

$$f_i(N, w)|M(w)| \geq f_i(N, v)|M(v)|.$$

i.e., if the set of minimal winning coalitions containing a player $i \in N$ in game (N, v) is a subset of minimal winning coalitions containing this player in game (N, w), then the power of player i in game (N, w) is not less than power of player i in game (N, v) (first, this power must be normalized by the number of minimal winning coalitions in games (N, v) and (N, w)).

- The unique power index f on $SI(N)$ satisfying DP-minimal monotonicity, null player, symmetry, and efficiency, is the Deegan-Packel power index.

3.4 Public Good Index, mergeability, and monotonicity

For the Public Good Index, introduced in Holler (1982), only minimal winning coalitions are considered relevant when it comes to measuring power. Then, given a simple game (N, v), the Public Good Index assigns to each player $i \in N$ the real number:

$$\delta_i (N, v) = \frac{|M_i (v)|}{\sum_{j \in N} |M_j (v)|}. \tag{2}$$

The Public Good Index of a player i is equal to the total number of minimal winning coalitions containing player i, divided by the sum of these numbers for all players.

An axiomatic characterization of this index can be found in Holler and Packel (1983). This characterization has the same flavour as the characterization of the Deegan-Packel index with the property of DP-mergeability.

A power index f on $H \subseteq SI(N)$ satisfies the **PGI-mergeability** property if for any pair of mergeable simple games $(N, v), (N, w) \in H$ such that $(N, v \vee w) \in H$, it holds that for all player $i \in N$:

$$f_i (N, v \vee w) = \frac{f_i (N, v) \sum_{j \in N} |M_j (v)| + f_i (N, w) \sum_{j \in N} |M_j (w)|}{\sum_{j \in N} |M_j (v \vee w)|}.$$

- The unique power index f defined on $SI(N)$ satisfying PGI-mergeability, null player, symmetry, and efficiency is the Public Good Index.

A new characterization of Public Good Index is provided here, using a property similar to strong monotonicity (Young, 1985) instead of PGI-mergeability. This property is named PGI-minimal monotonicity. It takes into account a relation between two simple games (N, v) and (N, w), that is, given in terms of the cardinality of the sets of minimal winning coalitions.

Definition 1 *A power index f on $H \subseteq SI(N)$ satisfies the property of **PGI-minimal monotonicity** if for any pair of games (N, v), $(N, w) \in H$, it holds that:*

$$f_i(N, w) \sum_{j \in N} |M_j(w)| \geq f_i(N, v) \sum_{j \in N} |M_j(v)| ,$$

for all player $i \in N$ such that $M_i(v) \subseteq M_i(w)$.

This property states that if the set of minimal winning coalitions containing a player i in game (N, v) is a subset of minimal winning coalitions containing this player in game (N, w), then the power of player i in game (N, w) is not less than power of player i in game (N, v) (first, this power must be normalized by the number of minimal winning coalitions of every player in games (N, v) and (N, w)).

For any two simple games (N, v) and (N, w), and for all $i \in N$ such that $|M_i(v)| = |M_i(w)|$, using the PGI-minimal monotonicity property, it holds that

$$f_i(N, w) \sum_{j \in N} |M_j(w)| = f_i(N, v) \sum_{j \in N} |M_j(v)|,$$

that is, a relation between the power index of the player i in the two games is obtained.

In the next result, a new characterization of the Public Good Index is proposed.

Theorem 2 *The unique power index f on $SI(N)$ satisfying PGI-minimal monotonicity, null player, symmetry, and efficiency is the Public Good Index.*

The proof immediately follows from a reasoning similar to Theorem 6 in Lorenzo-Freire *et al.* (2007).

3.5 Johnston Index, mergeability, and monotonicity

The Johnston index (Johnston, 1978) takes into account number of swings in a single voter group and divides the spoils equally among the swingers. A player $i \in N$ in a simple game (N, v) is assigned with the amount given by the expression:

$$\gamma_i(N, v) = \frac{1}{|G(v)|} \sum_{S \in G_i(v)} \frac{1}{|\chi(S)|}. \tag{3}$$

The Johnston index of a player i is equal to the sum of the inverse of the cardinality of $\chi(S)$ for the coalitions $S \in G_i(v)$, divided by the cardinality of $G(v)$.

This index coincides with the Deegan-Packel index when $G(v) = M(v)$ (in this case, $S = \chi(S)$ for all $S \in G(v)$). Johnston assumes that not only minimal winning coalitions but also quasi-minimal winning coalitions are relevant, that is, each quasi-minimal winning coalition has an equal probability of forming and players in a quasi-minimal winning coalition divide the "spoils" equally among the swingers.

In Lorenzo-Freire *et al.* (2007), a characterization of this index is given. In this characterization, a new property is introduced, in which the power of a simple game is identified with the power for the unanimity games of critical coalitions. A power index f on $H \subseteq SI(N)$, satisfies the property of **critical mergeability (or J-mergeability)** if for any game $(N, v) \in H$ such that $M(v) = \{S_1, \ldots, S_m\}$ and $M = \{1, \ldots, m\}$, it holds that for all player $i \in N$:

$$f_i(N, v) = \sum_{S \in \mathcal{F}} \frac{|G(S, v)|}{|G(v)|} f_i(N, u_S),$$

where $\mathcal{F} = \{\cap_{j\in R}S_j : \cap_{j\in R}S_j \neq \emptyset, R \subseteq M\}$. This property states that power in a game (N, v) is a weighted mean of the power of the unanimity games of critical coalitions. The weight of a component unanimity game, corresponding to a critical coalition S, is the proportion of quasi-minimal coalitions where S is critical.

- The unique power index f on $SI(N)$ satisfying critical mergeability, null player, symmetry, and efficiency is the Johnston power index.

The Johnston index satisfies a monotonicity property, with a flavour similar to that of the equivalent versions of the Deegan-Packel index and Public Good Index.

Definition 3 *A power index f on $H \subseteq SI(N)$ satisfies the property of* ***J-critical monotonicity*** *if for any pair of games (N, v), $(N, w) \in H$, it holds that:*

$$f_i(N, w)|G(w)| \geq f_i(N, v)|G(v)|,$$

for all player $i \in N$ such that $G_i(v) \subseteq G_i(w)$ and for all $S \in G_i(v)$, $|\chi(S)_v| \geq |\chi(S)_w|$.[2]

We were not successful in trying to prove that the Johnston index is the unique power index with the above property, null player, symmetry, and efficiency. Then, it is an open problem to obtain a characterization of the Johnston index with the previous property. Lorenzo-Freire *et al.* (2007) provide a characterization for the Johnston index.

Now, in Table 1 all the previous properties are putting together, specifying which of them are satisfied by each one of the five power indices considered in this paper.

[2] $\chi(S)_v$ denotes the set of critical players of S in the game (N, v).

Table 1. Power indices and properties

Property	**Shapley Shubik**	**Banzhaf**	**Deegan Packel**	**Public Good Index**	**Johnston**
Null player	✓	✓	✓	✓	✓
Symmetry	✓	✓	✓	✓	✓
Transfer	✓	✓	–	–	–
Efficiency	✓	–	✓	✓	✓
Total power	–	✓	–	–	–
Strong monotonicity	✓	✓	–	–	–
DP-Mergeability	–	–	✓	–	–
DP-Minimal monotonicity	–	–	✓	–	–
PGI-Mergeability	–	–	–	✓	–
PGI-Minimal monotonicity	–	–	–	✓	–
J-Mergeability	–	–	–	–	✓
J-Critical monotonicity	–	–	–	–	✓

4 The multilinear extensions

One of the main difficulties with the Shapley and Banzhaf values is that computation generally requires the sum of a very large number of terms. Owen (1972) defined the multilinear extension of a game. The multilinear extension of a characteristic function game (N, v) is given by:

$$h(x_1, \ldots, x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j)\ v(S),$$

for $0 \leq x_i \leq 1$, $i = 1, \cdots, n$.

Heuristically, $h(x_1, \cdots, x_n)$ can be thought of as the mathematical expectation of a winning coalition with $v(S) = 1$ being formed. Moreover, it is known that composition of games corresponds to composition of their multilinear extensions.

The multilinear extension of a game has been proven to be useful for computation of values. Indeed, the Shapley value of a game can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0, 1]^N$ (see (Owen, 1972)). In turn, the derivatives of that multilinear extension, evaluated at point $(1/2, 1/2, \ldots, 1/2)$, give the Banzhaf value of the game (Owen, 1975). These results, in conjunction with the above mentioned properties of multilinear extensions, allow to simplify the calculation of Shapley and Banzhaf values for games with a large number of players.

In the same way, this procedure is used in this paper to make possible the computation of the Deegan-Packel index, the Public Good Index, and the Johnston index.

4.1 The multilinear extension of the Deegan-Packel index

In this section, the Deegan-Packel power index of a simple game (N, v) is computed using the multilinear extension procedure. First, the next lemma (Owen (1972)), which provides a simple method to compute the multilinear extension of a game as a linear combination of unanimity games, is considered.

Lemma 4 *(Owen, 1972). If a characteristic function game (N, v) can be written in the way $v = \sum_{S \subseteq N} c_S u_S$, then, the multilinear extension of this game is:*

$$h(x_1, \ldots, x_n) = \sum_{S \subseteq N} c_S \prod_{i \in S} x_i,$$

for $0 \leq x_i \leq 1$, $i = 1, \cdots, n$, where c_S is a constant for every $S \subseteq N$ and u_S is the corresponding unanimity game.

It is well known that every game $(N, v) \in G(N)$ can be written as a linear combination of unanimity games. Moreover, if the game (N, v) is a simple game, it holds that:

$$v = \sum_{\substack{S \subseteq N \\ S \in W}} c_S u_S.$$

By the previous lemma, the multilinear extension of a simple game (N, v) is:

$$h(x_1, \ldots, x_n) = \sum_{\substack{S \subseteq N \\ S \in W}} c_S \prod_{i \in S} x_i,$$

where it holds that $c_S = 1$ if $S \in M(v)$.

Next, the procedure (using the multilinear extension of the game) to obtain the Deegan-Packel power index of a simple game is described. The advantage of this result resides in the fact, that the set of minimal winning coalition is not always known a priori, also this procedure gives a method to efficient compute this index, specially in case the number of players is large or when the game can be written as a composition of several games (see Owen (1995)).

Theorem 5 *Let (N, v) be a simple game. The Deegan-Packel power index for every player $i \in N$ can be computed by the following procedure:*

1. Obtain the multilinear extension $h(x_1, \ldots, x_n)$ of the game (N, v).

2. In the previous expression, eliminate the monomials $c_S \prod_{i \in S} x_i$ where $S \subseteq N$ and $c_S \neq 1$. A new multilinear function $l(x_1, \ldots, x_n)$ is obtained.

3. Let p be the minimum degree of the monomials $\prod_{i\in S} x_i$ of the function l. From $k = p+1$ to $k = n$, eliminate those monomials of degree k which are divisible by some monomials of the function l with degree from p to $k-1$. Then, a function g is obtained.

4. Finally, to obtain the Deegan-Packel power index (1), of a player $i \in N$, compute

$$\rho_i (N,v) = \frac{1}{g(1,\ldots,1)} \int_0^1 \frac{\partial g}{\partial x_i}(t,\ldots,t)\, dt.$$

Proof.

Let be $(N,v) \in SI(N)$ and $i \in N$. In steps 2 and 3, those terms corresponding to winning coalitions that are not minimal winning ones are eliminated. Then, it is clear that function g after step 3 is

$$g(x_1,\ldots,x_n) = \sum_{S\in M(v)} \prod_{k\in S} x_k.$$

It holds that $g(1,\ldots,1) = \sum_{S\in M(v)} 1 = |M(v)|$.

Further note that,

$$\int_0^1 \frac{\partial g}{\partial x_i}(t,\ldots,t)\, dt = \int_0^1 \sum_{S\in M_i(v)} t^{|S|-1} dt = \sum_{S\in M_i(v)} \frac{1}{|S|},$$

which completes the proof. ■

4.2 The multilinear extension of the Public Good Index

The next procedure gives a method to compute the Public Good Index.

Theorem 6 *Let (N,v) be a simple game. The Public Good Index for every player $i \in N$ can be computed by the following procedure:*

1, 2, and 3. Same as in Theorem 5.

4. Obtain functions g_i, $i = 1,\ldots,n$ follows:

$$g_i(x_i) = g(1,\ldots,1,x_i,1,\ldots 1), \text{ for } 0 \le x_i \le 1.$$

5. Finally, compute the derivatives, $g_i'(x_i)$, of the above functions to obtain the Public Good Index (2), for player $i \in N$, as follows

$$\delta_i(N,v) = \frac{g_i'(x_i)}{\sum_{j\in N} g_j'(x_j)},$$

with $0 \le x_j \le 1$ for every $j \in N$.

Proof. Let be $(N,v) \in SI(N)$. In a similar way to Theorem 5, the function g after step 3 is

$$g(x_1,\ldots,x_n) = \sum_{S\in M(v)} \prod_{k\in S} x_k.$$

Moreover, the functions g_i for every $i \in N$, after Step 4 are:

$$g_i(x_i) = |M_i(v)|\, x_i + k_i, \text{ where } k_i \in \mathbb{R}.$$

Taking into account that

$$g_i'(x_i) = |M_i(v)|,$$

the result is established. ■

4.3 The multilinear extension of the Johnston index

Next, a procedure to obtain the Johnston power index of a simple game is described. In this case, the advantage of this approach resides in the fact that the set of quasi-minimal winning coalitions and its corresponding critical coalitions are not known a priori, and thus, expression (3) could not be used to obtain the index.

Theorem 7 *Let (N, v) be a simple game. The Johnston power index for every player $i \in N$ can be computed by the following procedure:*

1 and 2. The same as in Theorem 5.

3. Consider p as the minimum degree of the monomials $\prod_{i \in S} x_i$ in the function l. Take $r = 1$.

3.1. If $p + r > n$, denote the relevant function by g and go to step 4.

3.2. Eliminate all the monomials of degree $p + r$ in l if all its divisors with $p + r - 1$ factors and degree $p + r - 1$ are in l.

3.3. Add (with a positive sign) all the monomials of degree $p + r$ and $p + r$ factors in case that they are not in l and only a strict subset of its divisors with p, $p + 1$, $\cdots$, $p + r - 1$ factors and degree p, $p + 1$, $\cdots$, $p + r - 1$, respectively, are in l.

3.4. If there are no monomials of degree $p + r$ in this function, go to step 3.6.

3.5. Replace each monomial of degree $p + r$ by the highest common factor of the set of its divisors in l whose degree is between p and $p + r - 1$.

3.6. Take $r = r + 1$ and go to step 3.1.

4. Finally, to obtain the Johnston power index (3), of a player $i \in N$, compute

$$\gamma_i(N, v) = \frac{1}{g(1, \ldots, 1)} \int_0^1 \frac{\partial g}{\partial x_i}(t, \ldots, t)dt.$$

Proof.

Consider $(N, v) \in SI(N)$ and $i \in N$. In steps 2 and 3, the terms corresponding to the critical coalitions of all quasi-minimal winning coalitions are obtained. Then, it is clear that function g after step 3 is

$$g(x_1, x_2, \ldots, x_n) = \sum_{S \in G(v)} \prod_{k \in \chi(S)} x_k.$$

Moreover, $g(1, \ldots, 1) = \sum_{S \in G(v)} 1 = \mid G(v) \mid$.

Taking into account that

$$\int_0^1 \frac{\partial g}{\partial x_i}(t, \ldots, t)dt = \int_0^1 \sum_{S \in G_i(v)} t^{|\chi(S)|-1} dt = \sum_{S \in G_i(v)} \frac{1}{\mid \chi(S) \mid},$$

the result is established. ■

5 Examples

In this section, the procedures derived above are applied to two examples. In the first one, the Basque Country Parliament, how our algorithms work is demonstrated. In the second one, the Victoria proposal, the multilinear extension of the corresponding game is obtained, taking into account that it is a composition of several games. The example allows to apply the procedures presented in the previous section and obtain the Deegan-Packel index, the Public Good Index, and the Johnston index of the game.

5.1 Example 1. The Basque Country Parliament

The Basque Country is one of the seventeen Spanish autonomous communities. The Basque Country Parliament is made up of 75 members. After the elections of April, 2005, the Parliament was composed of 29 members of the moderate regional party PNV (party 1), 18 members of the socialist party PSE-EE (party 2), 15 members of the conservative party PP (party 3), 3 members of the communist party IU-EB (party 5), and 9 and 1 members, respectively, of the left-wing regionalist parties PCTV (party 4), and ARALAR (party 6). The normalized vector of weights is $\left(\frac{29}{75}, \frac{18}{75}, \frac{15}{75}, \frac{9}{75}, \frac{3}{75}, \frac{1}{75}\right)$ and the quota is $d = 38$.

This parliament can be analyzed as a simple game. The winning coalitions are obtained taking into account the number of members of the different parties in the parliament.

It is easy to show that this simple game can be written in terms of unanimity games (defined in the subsection 2.2), taking $S_1 = \{1\}$ and $S_2 = \{2, 3, 4\}$:

$$v = \sum_{\substack{R \subseteq S_2 \\ |R|=1}} u_{S_1 \cup R} - \sum_{\substack{R \subseteq S_2 \\ |R|=2}} u_{S_1 \cup R} + u_{S_2}.$$

Applying Lemma 4, the multilinear extension of this game is:

$$h(x_1, x_2, x_3, x_4, x_5, x_6) = x_1x_2 + x_1x_3 + x_1x_4 - x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4 + x_2x_3x_4.$$

Eliminating those monomials with coefficients different from 1, the function l is obtained:

$$l(x_1, x_2, x_3, x_4, x_5, x_6) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3x_4.$$

- *The Deegan-Packel index.*

Once the function l is obtained, by step 3 those monomials that can be divided by any other monomial of l are eliminated, obtaining the function g

$$g(x_1, x_2, x_3, x_4, x_5, x_6) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3x_4.$$

Finally, to compute Deegan-Packel index, apply step 4, to calculate:

$$g(1,1,1,1,1,1) = 4, \qquad \int_0^1 \frac{\partial g}{\partial x_1}(t, \ldots, t)\, dt = \int_0^1 3t dt = \frac{3}{2},$$

$$\int_0^1 \frac{\partial g}{\partial x_i}(t, \ldots, t)\, dt = \frac{5}{6}, \text{ for all } i \in \{2, 3, 4\}, \quad \text{and}$$

$$\int_0^1 \frac{\partial g}{\partial x_i}(t, \ldots, t)\, dt = 0, \text{ for all } i \in \{5, 6\}.$$

The Deegan-Packel index is $\rho(N, v) = \left(\frac{9}{24}, \frac{5}{24}, \frac{5}{24}, \frac{5}{24}, 0, 0\right)$.

- *The Public Good Index.*

As step 3 is similar for Deegan-Packel index and Public Good Index, taking into account the function g obtained previously and, due to the step 4, get:

$$g_1(x_1) = 3x_1 + 1, \quad g_2(x_2) = 2x_2 + 2, \quad g_3(x_3) = 2x_3 + 2, \quad g_4(x_4) = 2x_4 + 2,$$

$$g_5(x_5) = 4, \text{ and } g_6(x_6) = 4.$$

And finally, by step 5, the Public Good Index is:

$$\delta(N, v) = \left(\frac{3}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, 0, 0\right).$$

- *The Johnston index.*

Consider the function l obtained in step 2, when the Deegan-Packel power index was computed. Step 3 has several stages:

First stage. Consider the monomials with degree 2 of the function l

$$x_1x_2 + x_1x_3 + x_1x_4.$$

Second stage. Using the monomials, add all the possible monomials of degree 3 and 3 factors such that only a strict subset of its divisors with 2 factors and degree 2 are in l. Then, the next function is obtained:

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3x_4 + x_1x_2x_3 + x_1x_2x_4 +$$

$$x_1x_2x_5 + x_1x_2x_6 + x_1x_3x_4 + x_1x_3x_5 + x_1x_3x_6 + x_1x_4x_5 + x_1x_4x_6$$

and, replacing the monomials of degree 3 and 3 factors by the highest common factor of the set of its divisors in l whose degree is 2, the next result is obtained:

$$3x_1 + 3x_1x_2 + 3x_1x_3 + 3x_1x_4 + x_2x_3x_4.$$

Third stage. By the same procedure for the monomials of degree 4, get

$$9x_1 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 3x_2x_3x_4.$$

Fourth stage. Taking into account the monomials of degree 5, the next result is obtained:

$$12x_1 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 4x_2x_3x_4.$$

Stop, obtaining that

$$g(x_1, x_2, x_3, x_4, x_5, x_6) = 12x_1 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 4x_2x_3x_4.$$

Therefore, by step 4,

$$g(1, 1, 1, 1, 1, 1) = 28,$$

$$\int_0^1 \frac{\partial g}{\partial x_1}(t, \ldots, t)dt = \int_0^1 (12 + 12t)dt = 18,$$

$$\int_0^1 \frac{\partial g}{\partial x_i}(t, \ldots, t)dt = \int_0^1 (4t + 4t^2)dt = \frac{10}{3} \text{ for all } i \in \{2, 3, 4\},$$

and

$$\int_0^1 \frac{\partial g}{\partial x_i}(t, \ldots, t)dt = 0 \text{ for all } i \in \{5, 6\}.$$

Then, the Johnston index is

$$\gamma(N, v) = \left(\frac{27}{42}, \frac{5}{42}, \frac{5}{42}, \frac{5}{42}, 0, 0 \right).$$

The three indices distinguish three types of players. The Johnston index gives the greatest power to the powerful player PNV because many quasi-minimal winning coalitions S containing PNV have this player as the sole critical player of S. The differences between PNV and the other non-null players are smaller for the Public Good Index than for the Deegan-Packel index. The reason is that the Public Good Index does not take the cardinality of the set of minimal winning coalitions into account as a comparison of the corresponding mergeability properties demonstrates.

5.2 Example 2. The Victoria Proposal

To ratify the amendments to the Canadian Constitution, in accordance with the suggestions made by the Victoria Proposal, it is necessary that they are approved by all of the following:

1. Ontario and Quebec,
2. Two of the four Maritime Provinces,
3. Either British Columbia, and one of the Prairie Provinces, or all three of the Prairie Provinces.

This situation can be analyzed as a simple game v. Moreover, there exists a natural partition of the Provinces into three subsets P_1, P_2, and P_3, where $P_1 = \{1, 2\} = \{\text{Ontario}, \text{Quebec}\}$, $P_2 = \{3, 4, 5, 6\}$

$$= \{\text{New Brunswick},\ \text{Nova Scotia},\ \text{Newfoundland},\ \text{Prince Edward Island}\}$$

are the Maritime Provinces, and $P_3 = \{7, 8, 9, 10\}$

$$= \{\text{British Columbia},\ \text{Alberta},\ \text{Saskatchewan},\ \text{Manitoba}\}$$

contains the Prairie Provinces.

The game v can be expressed as a composition of games:

$$v = u[v_1, v_2, v_3],$$

where v_1 is a two-player game in which $\{1, 2\}$ is the only winning coalition, v_2 is a four-player game in which any two-player coalition (or larger) wins, v_3 is a four-player game in which a coalition S wins if (a) S has two players, and $1 \in S$ or (b) S has three or four players. Finally, u is a three-person simple game in which only the three-person coalition wins. For more details about this game and composition of games, see Owen (1995).

Taking into account Lemma 4, the multilinear extension of u, is $f(y_1, y_2, y_3) = y_1 y_2 y_3$. For v_1, v_2 and v_3, the corresponding multilinear extensions are $g_1(x_1, x_2) = x_1 x_2$, $g_2(x_3, x_4, x_5, x_6) = x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6 - 2x_3x_4x_5 - 2x_3x_4x_6 - 2x_3x_5x_6 - 2x_4x_5x_6 + 3x_3x_4x_5x_6$, and $g_3(x_7, x_8, x_9, x_{10}) = x_7x_8 + x_7x_9 + x_7x_{10} + x_8x_9x_{10} - x_7x_8x_9 - x_7x_8x_{10} - x_7x_9x_{10}$, respectively. As h, the multilinear extension of v is:

$$h(x_1, \ldots, x_{10}) = f(g_1(x_1, x_2), g_2(x_3, x_4, x_5, x_6), g_3(x_7, x_8, x_9, x_{10})) =$$

$$g_1(x_1, x_2) \times g_2(x_3, x_4, x_5, x_6) \times g_3(x_7, x_8, x_9, x_{10}).$$

Finally, applying Theorems 5, 6, and 7 as in Example 1 above, the Deegan-Packel (D-P), Public Good Index (PGI) and Johnston (J) indices are obtained. They are showed, jointly with Shapley-Shubik (S-S) and Banzhaf (B) indices, in Table 2. Taking into account that all these indices are symmetric, only the results for representatives of the four types of players are presented.

It is interesting to note that the Banzhaf, Deegan-Packel, and Johnston indices assign greater power to the Maritime Provinces than to the Prairie Provinces. By contrast, the Shapley-Shubik index favored the Prairie Provinces more than the Maritimes. The Public Good Index assigns the same power to the Prairie Provinces and to the Maritime Provinces, as the number of minimal winning coalitions containing a Praire Province or a Maritime Province is the same.

Table 2. Power indices for the Victoria Proposal

Provinces	**S-S**	**B**	**D-P**	**PGI**	**J**
Ontario and Quebec	0.3155	0.1718	0.1607	0.1600	0.2410
Maritime Provinces	0.0298	0.0469	0.0803	0.0800	0.0509
British Columbia	0.1250	0.1289	0.1250	0.1200	0.1744
Prairie Provinces	0.0417	0.0430	0.0773	0.0800	0.0466

6 Conclusions

The paper provides (Section 3.4) a new axiomatic characterization of the Public Good Index. It refers to a specific property of monotonicity that is similar to the monotonicity property proposed to characterize the Deegan-Packel index in Lorenzo-Freire *et al.* (2007). The differences between the two properties reflect the differences of the alternative axioms of mergeability which the two indices satisfy.

A monotonicity property for the Johnston index is defined. It would be of interest, to find a characterization for this measure which is identical, or at least similar, to the characterization of the Public Good Index or the Deegan-Packel index. The corresponding axiomatization of the Johnston index remains to be worked out.

In Section 4, algorithms to compute the three power indices further analyzed in the paper using the multilinear extension of the game are given. This tool has been widely used in the literature. In this paper, however, it has been applied to power indices which are not probabilistic values. The computations presented in Section 5, i.e., applications to the Basque Country Parliament and the Victoria Proposal, illustrate further implications of the theoretical results expressed in Theorems 5, 6, and 7 and their capacity to support empirical research.

7 Acknowledgements

The authors wish to thank the interesting suggestions and comments made by I. García-Jurado, M. G. Fiestras-Janeiro, and V. Kumar. Authors acknowledge the financial support of Spanish Ministry for Science and Technology and FEDER, through projects SEJ2005-07637-C02-01/ECON, and SEJ2005-07637-C02-02/ECON and *Xunta de Galicia* through project PGIDIT03PXIC20701PN.

8 References

Banzhaf, J.F. (1965): Weighted voting doesn't work: A mathematical analysis. Rutgers Law Review 19, 317-343.

Braham, M. (2005): Causation and the measurement of power. In: Power Measures III, Homo Oeconomicus 22 (G. Gambarelli and M.J. Holler, eds.), Accedo-Verlag, Munich, 645–653.

Deegan, J. and Packel, E.W. (1979): A new index of power for simple n-person games. International Journal of Game Theory 7, 113-123.

Dubey, P. (1975): On the uniqueness of the Shapley value. International Journal of Game Theory 4, 131-139.

Dubey, P. and Shapley, L.S. (1979): Mathematical properties of the Banzhaf power index. Mathematics of Operations Research 4, 99-131.

Feltkamp, V. (1995): Alternative axiomatic characterization of the Shapley and Banzhaf values. International Journal of Game Theory 24, 179-186.

Holler, M.J. (1982): Forming coalitions and measuring voting power. Political Studies 30, 262-271.

Holler, M.J. and Packel, E.W. (1983): Power, luck and the right index. Journal of Economics 43, 21-29.

Johnston, R.J. (1978): On the measurement of power: some reactions to Laver. Environment and Planning A 10, 907-914.

Lorenzo-Freire, S., Alonso-Meijide, J.M., Casas-Méndez, B., and Fiestras-Janeiro, M.G. (2007): Characterizations of the Deegan-Packel and Johnston power indices. European Journal of Operational Research 177, 431-434.

Owen, G. (1972): Multilinear extensions of games. Management Science 18, 64-79.

Owen, G. (1975): Multilinear extensions and the Banzhaf value. Naval Research Logistic Quarterly 22, 741-750.

Owen, G. (1995): Game Theory (third edition). Academic Press, 266-284.

Shapley, L.S. (1953): A value for n-person games. In: Contributions to the Theory of Games (A.W. Tucker and H.W. Kuhn, eds.), Princeton University Press, 307–317.

Shapley, L.S. and Shubik, M. (1954): A method for evaluating the distribution of power in a committee system. American Political Science Review 48, 787-792.

Weber, R.J. (1988): Probabilistic values for games. In: the Shapley Value (A. Roth, ed.), Cambridge University Press, 101-119.

Young, H.P. (1985): Monotonic solutions of cooperative games. International Journal of Game Theory 14, 65-72.

Seit Mai 2000 sind erschienen:

NR. 113: M. J. HOLLER, Telecommunications Regulation and Economic Development, May 2000.

NR. 114: M. J. HOLLER AND V. KANNIAINEN, Compatibility Effects of Set-up Costs in the Organization of Service Systems, May 2000.

NR. 115: E. SCHOLING, Messung der Bevölkerungsgesundheit in Entwicklungsländern auf der Grundlage eines MIMIC-Modells, May 2000.

NR. 116: C. MENZEL, Collusion in Auctions: A Rotating-Bid Mechanism, July 2000.

NR. 117: C. HILLINGER, Consistent Aggregation and Chaining of Price and Quantity Measures, August 2000.

NR. 118: M. J. HOLLER AND S. NAPEL, On Interpersonnel Comparison of Value, October 2000.

NR. 119: M. BRAHAM AND F. STEFFEN, Voting Power in Games with Abstentions, July 2001.

NR. 120: M. BRAHAM AND F. STEFFEN, Notes on Power in Hierarchical Structures, July 2001.

NR. 121: I. LINDNER, Probabilistic Characterization of Power Indices with Abstentions, July 2001.

NR. 122: M. BRAHAM AND F. STEFFEN, Voting Rules in Insolvency Law: A Simple Game Theoretic Approach, December 2001.

NR. 123: M.J. HOLLER, Classical, Modern, and New Game Theory, December 2001.

NR. 124: I. LINDNER AND M. MACHOVER, L.S. Penrose's Limit Theorem: Proof of a Special Case, April 2002.

NR. 125: I. LINDNER AND H. STRULIK, Distributive Politics and Economic Growth: The Markovian Stackelberg Solution, April 2002.

NR. 126: M.J. HOLLER AND S. NAPEL, A Pragmatic Interpretation of Ken Binmore's Theory of Justice, October 2002.

NR. 127: M.J. HOLLER AND G. OWEN, Present and future of Power Measures, December 2002.

NR. 128: M.J. HOLLER AND P. SKOTT, The Importance of Setting the Agenda, June 2003.

NR. 129: M.J. HOLLER AND I.LINDNER, Mediation as signal, June 2003.

NR. 130: M.BRAHAM AND M.J. HOLLER, The Impossibility of a Preference-based Power Index, July 2003.

NR. 131: F. BOLLE AND M. BRAHAM, A Difficulty with Oats: On trust, trustworthiness, and signalling, July 2003.

NR. 132: M.J. HOLLER, The Artist as a Secret Agent: Liberalism Against Populism, August 2003.

NR. 133: W.GÜTH AND S.NAPEL, Inequality Aversion in a Variety of Games: An Indirect Evolutionary Analysis, November 2003.

NR. 134: E. SCHOLING, Zur ökonomischen Analyse der Kinderarbeit, April 2004.

NR. 135: M. BRAHAM, Freedom, power, and success: A game-theoretic perspective, May 2004.

NR. 136: M.J. HOLLER AND H. NURMI, Power, Outcomes and Preferences, September 2004.

NR. 137: M.J. HOLLER, How to Sell Power Indices, September 2004.

NR. 138: M. BRAHAM AND F. STEFFEN, The Chairman's Paradox Revisited, October 2004.

NR. 139: M.J. HOLLER, George Orwell and His Cold Wars: Truth and Politics, January 2005.

NR. 140: R. HARADAU AND S. NAPEL, Holler-Packel Value and Index – A New Characterization, January 2005.

NR. 141: D. MOOKHERJEE AND S. NAPEL, Intergenerational Mobility and Macroeconomic History Dependence, April 2005.

NR. 142: M.J. HOLLER, Freedom of Choice, Power, and the Responsibility of Decision Makers, September 2005.

NR. 143: M.J. HOLLER AND S. NAPEL, Democratic Decision Procedures and Stability with Reference to the European Union, March 2006.

NR. 144: M.J. HOLLER AND H. NURMI, Measurement of Power, Probabilities, and Alternative Models of Man, March 2006.

NR. 145: J.M. ALONSO-MEIJIDE, C. BOWLES, M.J. HOLLER AND S. NAPEL, Monotonicity of power in games with a priori unions, May 2006.

NR. 146: N. MAASER AND S. NAPEL, Equal Representation in Two-tier Voting Systems, May 2006.

NR. 147: S.K. BERNINGHAUS, C. KORTH AND S. NAPEL, Reciprocity — An Indirect Evolutionary Analysis, October 2006.

NR. 148: J. GRÖNDAHL, Power in Standardisation — the Case of CEN, December 2006.

NR. 149: M.J. HOLLER, Niccolò Machiavelli on Power, January 2007.

NR. 150: M.J. HOLLER, Adam Smith's Model of Man and Some of Its Consequences, January 2007.

NR. 151: M.J. HOLLER AND T. NGUYEN, Regulating Balance Sheet Audit: A Game Theoretical Analysis, March 2007.

NR. 152: O. LESSMANN, Effective Freedom and Combined Capabilities: Two Different Conceptions of Capability, June 2007.

NR. 153: J.M. ALONSO-MEIJIDE, B. CASAS-MÉNDEZ, M.J. HOLLER AND S. LORENZO-FREIRE, Computing Power Indices: Multilinear extensions and new characterizations, June 2007.