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AN AXIOMATIZATION OF THE BANZHAF-OWEN COALITIONAL VALUE

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An axiomatization of the Banzhaf -Owen coalitional value.

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Abstract

In this paper we propose an axiomatization of the Banzhaf-Owen value on the class of TU games which can be considered to be an alternative to the one presented in Amer et al (2002). And, with the appropiate changes, we also achieve a characterization of the Banzhaf-Owen coalitional index of power whose first axiomatization appeared in Albizuri (2001).

Keywords: TU games; Coalitional value; Banzhaf value

JEL classification: C71.

1 Introduction

The Banzhaf value was introduced by Owen (1975) as an extension of the Banzhaf-Coleman power index (Banzhaf, 1965; Coleman, 1971) for the class of all finite transferable utility cooperative games (in short TU games). This value appears as an alternative to the Shapley value (Shapley, 1953). Feltkamp (1995) presented characterizations of both values which emphasized the relevant differences between them. Only one property distinguishes these values: total power, in case of the Banzhaf value, and efficiency, for the Shapley value.

In the context of TU games endowed with a coalition structure, Owen (1977) defined and characterized the Shapley value, or the so-called Owen coalitional value. Later on, other characterizations appear (for instance, Winter, 1992; Amer and Carreras, 1995; Vázquez-Brage et al 1997). Owen (1981) formulated an extension of the Banzhaf value for this context without a characterization. We will refer to this value as the Banzhaf-Owen coalitional value. The first characterization of this value appeared in Albizuri (2001) on the class of monotonic simple games. Amer et al (2002) characterized this value in the class of

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TU games. Alonso-Meijide and Fiestras-Janeiro (2002) introduced and characterized another coalitional value: the symmetric coalitional Banzhaf value.

It is worth noting that these three coalitional values can be seen as a two-level bargaining process. First, the unions split the total amount according to the Shapley value in the case of the Owen coalitional value, and using the Banzhaf value in the two remaining coalitional values. Then, each union allots its total reward among its members taking into account the possibilities of their joining other unions using the Shapley value for the Owen and the symmetric coalitional values, and the Banzhaf value in the case of the Banzhaf-Owen coalitional value.

As was mentioned in Alonso-Meijide and Fiestras-Janeiro (2002), only one property distinguishes the Owen coalitional value from the symmetric coalitional Banzhaf value: efficiency in the former value and total power in the latter. This relationship cannot be achieved between the Owen coalitional value and the Banzhaf-Owen coalitional value since the properties of quotient game and symmetry for the unions are not satisfied by the latter (see Amer et al for details).

Vázquez-Brage et al (1997) characterized the Owen coalitional value as the unique coalitional Shapley value satisfying balanced contributions in the unions and quotient game properties. In this paper, we extend this characterization to the Banzhaf-Owen coalitional value. Section 2 recalls some basic definitions. Section 3 presents our characterization of the Banzhaf-Owen coalitional value.

2 Preliminaries

A finite transferable utility cooperative game is defined by a finite set of players N, where $|N| = n$, and a real valued function v defined on the subsets of N such that $v(\Phi) = 0$. We will denote by $\mathcal{G}$ the family of all TU games with a finite player set. Let N be a finite set and a coalition $S \subseteq N$. The S-unanimity game denoted by (N, u_S) is given by $u_S(T) = 1$ if $S \subseteq T$ and otherwise 0.

The Banzhaf value β assigns to each TU game (N, v) a vector in $\mathbb{R}^n$ where each coordinate is defined as follows:[1]

$$\beta_i(N,v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N} v(S \cup i) - v(S), \text{ for any } i \in N.$$

Let us consider a finite set $N = \{1, \ldots, n\}$. We will denote by $P(N)$ the set of all partitions of N. An element $P \in P(N)$ is called a coalition structure or a system of unions of the set N. The partition where each union is a singleton will be represented by P^n, i.e., $P^n = \{\{1\}, \ldots, \{n\}\}$. Given $i \in N$, by $P(i)$ we denote the subfamily of partitions $P \in P(N)$ such that $\{i\} \in P$. A TU game with a coalition structure is a triplet (N, v, P), where $(N, v) \in \mathcal{G}$ and $P \in P(N)$. The family of all TU games with a coalition structure will be denoted by $\mathcal{U}$.

[1] Let S be a finite set. In short notation, given $i \in S$ we will write $S \setminus i$ instead of $S \setminus \{i\}$, and given $i \notin S$ we will write $S \bigcup i$ instead of $S \bigcup \{i\}$.

Let us take $(N, v, P) \in \mathcal{U}$ with $M = \{1, \ldots, m\}$ and $P = \{P_1, \ldots, P_m\}$. We defined a TU game, (M, v^P), by $v^P(R) = v\left(\bigcup_{k \in R} P_k\right)$ for any $R \subseteq M$. This game is called the quotient game of (N, v, P). Notice that the game (M, v^P) is exactly the game (N, v) whenever $P = P^n$.

The Banzhaf-Owen coalitional value assigns to each $(N, v, P) \in \mathcal{U}$ a vector whose components are given by

$$\Psi_i(N, v, P) =$$
$$\sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \left(v(Q \cup T \cup i) - v(Q \cup T)\right),$$

where $M = \{1, \ldots, m\}$, $P = \{P_1, \ldots, P_m\}$, $Q = \underset{r \in R}{\cup} P_r$, and $P_k \in P$ is the union such that $i \in P_k$.

3 An axiomatic approach

Given an allocation rule f which assigns to every $(N, v, P) \in \mathcal{U}$ the element $f(N, v, P) \in \mathbb{R}^n$, we consider the following properties:

A1. (Additivity) For all $(N, v, P^n), (N, w, P^n) \in \mathcal{U}$ and $i \in N$, $f_i(N, v + w, P^n) = f_i(N, v, P^n) + f_i(N, v, P^n)$.

A2. (Null player) If $i \in N$ is a null player in $(N, v, P^n) \in \mathcal{U}$, ($v(S \bigcup i) = v(S)$ for any $S \subseteq N \diagdown i$), then $f_i(N, v, P^n) = 0$.

A3. (Symmetry) If $i, j \in N$ are symmetric players in $(N, v, P^n) \in \mathcal{U}$, ($v(S \bigcup i) = v(S \bigcup j)$ for any $S \subseteq N \diagdown \{i, j\}$), then $f_i(N, v, P^n) = f_j(N, v, P^n)$.

A4. (Total power) For any $(N, v, P^n) \in \mathcal{U}$, it holds that

$$\sum_{i \in N} f_i(N, v, P^n) = \frac{1}{2^{n-1}} \sum_{i \in N} \sum_{S \subseteq N \diagdown i} \left(v\left(S \bigcup i\right) - v(S)\right).$$

A5. (Indifference in the unions) Given a game $(N, v, P) \in \mathcal{U}$, $P_k \in P$, and $i, j \in P_k$, it holds that

$$f_i(N, v, P) = f_i(N, v, P_{-j}),$$

where

$$P = \{P_1, \ldots, P_m\},\ M = \{1, \ldots, m\},\ \text{and}$$
$$P_{-j} = \{P_l : l \in M \diagdown k\} \bigcup \{P_k \diagdown j, \{j\}\}.$$

A6. (Quotient game for one player unions)[2] Given a game $(N, v, P) \in \mathcal{U}$ with $P \in P(i)$ for some $i \in N$, then

$$f_i(N, v, P) = f_k(M, v, P^m)$$

where $P = \{P_1, \ldots, P_m\}$, $M = \{1, \ldots, m\}$, and $i \in P_k$.

[2]It is clear that the relevance of this property appears when $|M| < |N|$.

Next we will establish the relationship between a solution defined on $\mathcal{U}$ fulfilling A1-A4 and the Banzhaf value that is defined on $\mathcal{G}$.

Lemma 1 *Let us take any solution φ defined on $\mathcal{U}$ satisfying A1-A4. Then, it holds that*

$$\varphi_i(N, v, P^n) = \beta_i(N, v) \text{ for every } (N, v, P^n) \in \mathcal{U} \text{ and every } i \in N. \tag{1}$$

Furthermore, given a solution φ defined on $\mathcal{U}$ holding (1)*, it also satisfies A1-A4.*

Proof.

Let us consider a solution φ defined on $\mathcal{U}$ satisfying A1-A4. Using the property A1, it suffices to prove the assertion of the lemma for any game $(N, u_S, P^n) \in \mathcal{U}$ where $S \subseteq N$ and u_S is the S-unanimity game.

Since φ satisfies A2, $\varphi_i(N, u_S, P^n) = 0 = \beta_i(N, u_S)$ for every $i \in N \setminus S$. Moreover, for any $i, j \in S$, by A3, $\varphi_i(N, u_S, P^n) = \varphi_j(N, u_S, P^n)$. Then,

$$\sum_{i \in N} \varphi_i(N, u_S, P^n) = \sum_{i \in S} \varphi_i(N, u_S, P^n) = |S| \varphi_i(N, u_S, P^n), \text{ for some } i \in S.$$

Taking into account that the solution φ satisfies A4, it holds that

$$\begin{aligned} |S| \varphi_i(N, u_S, P^n) = & \frac{1}{2^{n-1}} \sum_{i \in S} \sum_{T \subseteq N \setminus i} \left(u_S\left(T \bigcup i\right) - u_S(T)\right) \\ = & \frac{|S|}{2^{n-1}} \sum_{S \setminus i \subseteq T \subseteq N \setminus i} \left(u_S\left(T \bigcup i\right) - u_S(T)\right) \\ = & \frac{|S|}{2^{n-1}} 2^{n-s}. \end{aligned}$$

Then, $\varphi_i(N, u_S, P^n) = \frac{2^{n-s}}{2^{n-1}} = \frac{1}{2^{s-1}} = \beta_i(N, u_S)$ for every $i \in S$.

¿From the properties that characterize the Banzhaf value on the family of TU games $\mathcal{G}$, it is clear that any solution defined on $\mathcal{U}$ satisfying (1) fulfils properties A1-A4.□

As a consequence of this result any solution defined on $\mathcal{U}$ satisfying A1-A4 can be interpreted as an extension of the Banzhaf value and we will call it a coalitional Banzhaf value.

Property A5 means that the allocation any player receives does not change when any coalition mate quits the union which both agents belong to. This feature is stronger than the property of balanced contributions for the unions (for any game $(N, v, P) \in \mathcal{U}$, all $P_k \in P$ and all $i, j \in P_k$, it holds that $f_i(N, v, P) - f_i(N, v, P_{-j}) = f_j(N, v, P) - f_j(N, v, P_{-i})$).

Finally, A6 says that a player who forms an unitary union gets the same amount that his union receives in the game whose participants are the unions without any association among them. Clearly, this property is a weaker version of the quotient game property (for all $(N, v, P) \in \mathcal{U}$ and all $P_k \in P$, it holds that $\sum_{i \in P_k} f_i(N, v, P) = f_k\left(M, v^P, P^m\right)$). The Banzhaf-Owen coalitional value does not satisfy this property.

Lemma 2 *The Banzhaf-Owen coalitional value is a coalitional Banzhaf value.*

Proof.
We will prove that the Banzhaf-Owen coalitional value satisfies (1). Given $(N, v, P^n) \in \mathcal{U}$ and $i \in N$, let us take $k \in \{1, 2, \ldots, n\}$ such that $P_k = \{i\}$. Then,

$$\Psi_i(N, v, P^n) = \sum_{R \subseteq N \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \left(v\left(Q \cup T \cup i\right) - v\left(Q \cup T\right)\right) =$$

$$\sum_{S \subseteq N \setminus i} \frac{1}{2^{n-1}} \left(v\left(S \cup i\right) - v\left(S\right)\right) = \beta_i\left(N, v\right),$$

where $Q = \bigcup_{r \in R} P_r$.□

Lemma 3 *The Banzhaf-Owen coalitional value satisfies the indifference in the unions property.*

Proof.
Let us take $(N, v, P) \in \mathcal{U}$ with $M = \{1, \ldots, m\}$ and $P = \{P_1, P_2, \ldots, P_m\}$.

Given $P_k \in P$ and $i, j \in P_k$, we consider

$$P_{-j} = \left\{P'_1, P'_2, \ldots, P'_{m+1}\right\},$$

where $P'_l = P_l$, for every $l \in M \diagdown k$, $P'_k = P_k \setminus j$, $P'_{m+1} = \{j\}$. Taking $M' = \{1, 2, \ldots, m+1\}$, we can write

$$\Psi_i(N, v, P_{-j}) = \sum_{R \subseteq M' \setminus k} \sum_{T \subseteq P'_k \setminus i} \frac{1}{2^{m'-1}} \frac{1}{2^{p'_k - 1}} \left(v\left(Q \cup T \cup i\right) - v\left(Q \cup T\right)\right) =$$

$$\sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \left(v\left(Q \cup T \cup i\right) - v\left(Q \cup T\right)\right) = \Psi_i\left(N, v, P\right),$$

where $Q = \bigcup_{r \in R} P_r$.□

Lemma 4 *The Banzhaf-Owen coalitional value satisfies the quotient game property for one player unions.*

Proof.
Let us take $(N, v, P) \in \mathcal{U}$ where $P \in P(i)$, i.e., there exists some $k \in M = \{1, 2, \ldots, m\}$ such that $P_k = \{i\}$. Then, we can write

$$\Psi_i(N, v, P) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \left(v\left(Q \cup T \cup i\right) - v\left(Q \cup T\right)\right) =$$

$$\sum_{R\subseteq M\backslash k}\frac{1}{2^{m-1}}\left(v\left(Q\cup i\right)-v\left(Q\right)\right)=\sum_{R\subseteq M\backslash k}\frac{1}{2^{m-1}}\left(v^{P}\left(R\cup k\right)-v^{P}\left(R\right)\right)=$$

$$\beta_{k}\left(M,v^{P}\right)=\Psi_{i}\left(M,v^{P},P^{m}\right),$$

where $Q=\bigcup_{r\in R}P_{r}$.□

In the next result, we characterize the Banzhaf-Owen coalitional value as the unique posible extension of the Banzhaf value to the context of TU games with a system of unions that satisfies the two previous properties.

Theorem 5 *The Banzhaf-Owen value is the unique coalitional Banzhaf value satisfying the properties of indifference in the unions and quotient game for one player unions.*

Proof.

Existence:

It follows from previous lemmas that the Banzhaf-Owen value satisfies the mentioned properties.

Uniqueness:

Let us assume that there exist two coalitional Banzhaf values f^{1} and f^{2} satisfying indifference in the unions and the quotient game property for one player unions.

Then, we can find a game $(N,v)\in\mathcal{G}$ and a partition $P=\{P_{1},P_{2},\ldots,P_{m}\}\in P(N)$, with the maximum number of *a priori* unions such that $f_{i}^{1}(N,v,P)\neq f_{i}^{2}(N,v,P)$ for some player $i\in N$.

As f^{1} and f^{2} are coalitional Banzhaf values, necessarily $m<n$. Let us take $P_{k}\in P$ such that $i\in P_{k}$. Two possible cases appear:

- $|P_{k}|=1$. Thus, $P_{k}=\{i\}$.

By the property of quotient game for one player unions, we have

$$f_{i}^{1}(N,v,P)=f_{k}^{1}\left(M,v^{P},P^{m}\right)\text{ and }f_{i}^{2}(N,v,P)=f_{k}^{2}\left(M,v^{P},P^{m}\right).$$

Since f^{1} and f^{2} are coalitional Banzhaf values

$$f_{k}^{1}\left(M,v^{P},P^{m}\right)=\beta_{k}\left(M,v^{P}\right)=f_{k}^{2}\left(M,v^{P},P^{m}\right).$$

Then, $f_{i}^{1}(N,v,P)=f_{i}^{2}(N,v,P)$. This is a contradiction.

- $|P_{k}|>1$. There exists $j\in P_{k}$ such that $j\neq i$. By the indifference in the unions property

$$f_{i}^{1}(N,v,P)=f_{i}^{1}(N,v,P_{-j})\text{ and }f_{i}^{2}(N,v,P)=f_{i}^{2}(N,v,P_{-j}).$$

By the maximality of the partition P, we have

$$f_i^1(N, v, P_{-j}) = f_i^2(N, v, P_{-j}).$$

This implies that $f_i^1(N, v, P) = f_i^2(N, v, P)$ and then, the proof is finished.□

Remark 6 *Being a coalitional Banzhaf value, the property of indifference in the unions, and the quotient game property for one player unions are independent. For instance,*

(i) $f(N, v, P) = \beta(N, v)$ *for any* $(N, v, P) \in \mathcal{U}$ *is a coalitional Banzhaf value which satisfies A5, but not A6.*

(ii) $f(N, v, P) = \Pi(N, v, P)$[3] *for any* $(N, v, P) \in \mathcal{U}$ *is a coalitional Banzhaf value which satisfies A6, but not A5.*

(iii) Given $a \in \mathbb{R}$, $f(N, v, P) = a$ *for any* $(N, v, P) \in \mathcal{U}$ *satisfies A5-A6, but it is not a coalitional Banzhaf value.*

Remark 7 *It should be noticed that properties A1-A4 can be replaced by any other system of properties which characterizes the Banzhaf value.*

Remark 8 *Let us consider the class of monotonic simple games with a system of unions,* $\mathcal{S}$*, and*

A1'. (Transfer) For all $(N, v, P^n), (N, w, P^n) \in \mathcal{S}$ *and* $i \in N$, $f_i(N, v+w, P^n) = f_i(N, v \wedge w, P^n) + f_i(N, v \vee w, P^n)$, *where* $(v \wedge w)(S) = \max\{v(S), w(S)\}$ *and* $(v \vee w)(S) = \min\{v(S), w(S)\}$. *The system of properties A1' and A2-A6 characterizes the coalitional Banzhaf-Owen index on* $\mathcal{S}$.

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[3] Π is the symmetric coalitional Banzhaf value defined as $\Pi_i(N, v, P) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{2^{m-1}} \frac{t!(p_k - t - 1)!}{p_k!} (v(Q \cup T \cup i) - v(Q \cup T))$ where $P_k \in P$ *is the union such that* $i \in P_k$ *and* $Q = \bigcup_{r \in R} P_r$.

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