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THE CORE-CENTER AND THE SHAPLEY VALUE: A COMPARATIVE STUDY

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Report 03-10

Reports in Statistics and Operations Research

The Core-Center and the Shapley Value: A comparative Study*

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Abstract

We present a study of the core-center for the class of convex games. By means of a dynamic process between coalitions, the core-center of a convex game can be obtained from the core-centers of other appealing games, namely the utopia games. Furthermore, for some subclasses of games, this formulation provides a direct connection between the core-center and the Shapley value of a game that picks up all the information of the utopia games. Our comparison with the Shapley value is also based on the properties satisfied by both solutions. To finish the paper, the airport game allows us to give some insights about the differences between the Shapley value and the core-center.
Keywords: cooperative games, TU games, convex games, Shapley value.

* **Acknowledgement**. We would like to thank Miguel Ángel Mirás Calvo for his assistance in the making of all the graphics in this paper with Matlab 6.5. The Authors also acknowledge the financial support of the Spanish Ministry for Science and Technology and FEDER through project BEC2002-04102-C02-02.

Introduction

In González-Díaz and Sánchez-Rodríguez (2003), a new framework to study solution concepts is introduced. If given a game we are able to summarize all its information through a set-valued solution then, taking the appropriate allocation within the set it should also summarize the information of the set and consequently the information of the original game too. For this purpose distributions of probability on the sets are defined in order to measure the importance of each allocation in the set. When there is no additional information on the set but the set itself, one can endow it with the uniform distribution. Furthermore, in the specific case where the set-valued solution is the core, the single-valued solution selected is named the core-center.

The advantages of an allocation method based on a game theoretic solution (nucleolus and Shapley value, among others) is that the allocation rule incorporates the information regarding all the alternatives available to the participants in the game. In our case, the core-center takes into account the information provided by all the allocations in the core, and subsequently, by the game itself.

In the present paper we focus on the analysis of the core-center for convex games. If a game is convex, players have incentives to cooperate and form the grand coalition N since the marginal contribution of any player to any coalition becomes larger (or at least does not decrease) when the coalition size increases. Convex games have nice properties: they are balanced games, i.e. their cores are non-empty; the core is the unique stable set; the Weber set, the bargaining set, and the core coincide; the kernel is the nucleolus... Furthermore, the special geometric structure of the core of a convex game allows to find the main result of this paper: the existing relation between two different "centers" of the core, the Shapley value and the core-center.

In the class of convex games, the core coincides with the convex hull of all the marginal contributions vectors, i.e., with the Weber set. The Shapley value (Shapley (1953)) for convex games is a weighted center of mass of the finite set of particles given by the extreme points of the core.

The core-center is the center of mass of the whole core considered as a continuum of particles. It is the point at which one can balance all the core allocations; in other words it is the expectation of the uniform distribution over the core. So, in some sense we can say that the core-center summarizes the behavior of the core preserving its properties and applies a principle of fairness to all the structure of the core.

The nucleolus is a solution concept for cooperative games introduced by Schmeidler (1969) to overcome the multiplicity of outcomes characteristic of its antecedent concepts, the bargaining set and the kernel. There is at least one important difference between the core-center and the nucleolus. Given a game with a non-empty core, as it is proved in Maschler et al. (1979), not all redundant constraints of the core can be suppressed in the computation of the nucleolus. The core-center behaves differently. If two games have the same core, then they also have the same core-center. The nucleolus, also called the lexicographic center of a game, defines a unique vector for each game. When the core is non empty it contains the nucleolus. More geometric properties of the nucleolus are given in Maschler et al. (1979), where the location of the nucleolus and kernel within the core is characterized in geometric terms. The core-center also has some parallelism with the nucleolus: the core-center is the "gravity center" of the core meanwhile the nucleolus is a "lexicographic center". Furthermore, the core-center sums up all the information regarding the core.

We refer to González-Díaz and Sánchez-Rodríguez (2003) for a detailed analysis and axiomatic characterizations of the core-center.

What we do in this work is analyzing in detail the core by means of a dynamic process among coalitions. Initially, we start with the imputations set. The value $v(S)$ represents the utility that a coalition S can obtain independently of $N \setminus S$. Once players accept that value, the set of good allocations for coalition $N \setminus S$ is reduced. The core is the resultant of this process when all the values of the characteristic function are considered. The core-center picks up this information through the utopia games. Basically, the utopia games measure the looses experimented by any coalition. Following with the analysis, we prove that for special classes of convex games the core-center can be expressed as the Shapley value of a specific game. The comparative study is also based on the properties satisfied by both solutions. Finally, the airport game shows some of the differences between the Shapley value and the core-center.

The outline of this paper is the following: in the first Section we introduce notions on cooperative game theory and geometric tools; in Section 2 the utopia games are introduced and the main results are stated; in Section 3, the core-center is illustrated by means of a very well known game: the airport game; in Section 4 we make a comparison between the core-center and the Shapley value studying the properties they satisfy; and finally, in section 5 we give insights to some open questions.

1 Game Theory Background

A cooperative $n-$person game with transferable utility, shortly a TU game, is a pair (N, v) where N is a finite set of players and $v : 2^N \rightarrow \mathbb{R}$ is a function assigning to each coalition $S \in 2^N$ a real number $v(S)$, where $v(\emptyset) = 0$. For each subgroup or coalition S, the value $v(S)$ indicates what the players of S obtain by cooperation among themselves. The set of all n-person games G^n forms a $(2^n - 1)$-dimensional space.

Given a coalition $S \subset N$, and $T \subset S$, $S \backslash T = \{i \in S : i \notin T\}$.

A game (N, v) is called *convex* if $v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$ for all $S \subset T \subset N \backslash \{i\}$. The amount $v(S \cup i) - v(S)$ is called the $i's$ marginal contribution to a coalition S. Convexity says that for all $i \in N$, the $i's$ marginal contribution does not decrease as the coalition becomes larger. The set of all n-person convex games will be denoted by CG^n.

Given a game (N,v), the preimputations set is defined by,

$$I^*(N,v) = \left\{ x = (x_i)_{i\in N} \in \mathbb{R}^n : \sum_{i\in N} x_i = v(N) \right\}.$$

A *solution concept* on G^n is a function φ that associates to any game (N,v) a subset $\varphi(N,v)$ of its preimputation set $I^*(N,v)$.

An *allocation rule* is a function φ which, given a game (N,v), selects one preimputation $\varphi(N,v)$ in $I^*(N,v)$.

The imputation set consists on the individually rational preimputations, i.e.,

$$I(N,v) = \{x \in \mathbb{R}^n : \sum_{i\in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

The imputation set is non-empty iff $v(N) \geq \sum\limits_{i\in N} v(\{i\})$. If $v(N) = \sum\limits_{i\in N} v(\{i\})$, the game is called degenerate and $I(N,v) = (v(\{1\}),...,v(\{n\}))$; in the case where $v(N) > \sum\limits_{i\in N} v(\{i\})$, then $I(N,v)$ is an $(n-1)$-dimensional simplex with extreme points $f^1(N,v),...,f^n(N,v)$ where $f^i_k(N,v) = v(\{k\})$ for all $k \neq i$ and $f^i_i(N,v) = v(N) - \sum\limits_{k\in N\backslash\{i\}} v(\{k\})$.

The core of a game (N,v), Gillies (1953), is defined by $C(N,v) = \{x \in I(N,v) : \sum_{i\in S} x_i \geq v(S)$ for all $S \subset N\}$. It can be said that all the allocations in the core satisfy the minimal requirements that any coalition might demand in the game. So, any allocation in the core ensures that everyone gains, or at least does not loose, from cooperation . If for all $S \subset N$, $v(S) = \sum_{i\in S} v(\{i\})$, the game is called additive and $C(N,v) = \{(v(\{1\}),...,v(\{n\})\}$. Furthermore if a game (N,v) satisfies that $v(S) = \sum_{j\in S} v(j)$ for all $S \subset N, S \neq N$ and $v(N) > \sum_{i\in N} v(\{i\})$, then $C(N,v) = I(N,v)$. Bondareva (1963) and Shapley (1967) established necessary and sufficient conditions for the non-emptyness of the core. Let $(N,v) \in G^n$, $C(N,v) \neq \varnothing$ iff $v(N) \geq \sum\limits_{S\in 2^N\backslash\{\varnothing\}} \lambda_S v(S)$ for all $\{\lambda_S\}_{S\in 2^N}$ such that $\lambda_S \geq 0$ for all $S \in 2^N\backslash\{\varnothing\}$ and $\sum\limits_{S\in 2^N\backslash\{\varnothing\}} \lambda_S e_S = e_N$, where $e_S \in \mathbb{R}^n$ and $(e_S)_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$.

Games with a non-empty core are called balanced games. The set of all n-person balanced games will be denoted by BG^n. Note that each convex game is balanced, but not every balanced game is convex.

For $S \subseteq N$, we denote by $\Pi(S)$ the set of all possible orderings of the elements in S, i.e., bijective functions from S to $\{1,...,s\}$, where $s = |S|$ is the cardinality of S. A generic order of S is denoted by $\sigma_S \in \Pi(S)$. For all $i \in N$ and $\sigma_N \in \Pi(N)$, let $P_{\sigma_N}(\{i\}) = \{j \in N : \sigma_N(j) < \sigma_N(i)\}$ the set of predecessors of i with respect to σ_N. Given $\sigma \in \Pi(N)$ and $S \in 2^N\backslash\varnothing$, let $\sigma_S \in \Pi(S)$ denote the order induced by σ in S.

Let us consider the map that associates to every game $(N,v) \in G^n$ and any order $\sigma \in \Pi(N)$ the marginal's contribution vector,:

$$\begin{array}{rccc} m: & G^n \times \Pi(N) & \to & \mathbb{R}^n \\ & ((N,v),\sigma) & \rightsquigarrow & m^\sigma(N,v) \end{array}$$

where the i-th coordinate of the marginal vector $m^\sigma(N,v)$, $\sigma \in \Pi(N)$, is defined by

$$m_i^\sigma(N,v) = v(P_\sigma(\{i\}) \cup \{i\}) - v(P_\sigma(\{i\}))$$

and consider the following relation of equivalence:

$$\sigma_1 \sim \sigma_2 \Longleftrightarrow m^{\sigma_1}(N,v) = m^{\sigma_2}(N,v).$$

The quotient set will be denoted by $\pi(N)/\sim$.

The Shapley value of a game (N,v) is defined as the average of all marginal vectors and denoted by Sh, i.e.,

$$\begin{aligned} Sh(N,v) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(N,v) \\ &= \frac{1}{n!} \sum_{[\sigma] \in \pi(N)/\sim} |[\sigma]|\, m^{[\sigma]}(N,v) \end{aligned}$$

which can be expressed alternatively as

$$Sh_i(N,v) = \sum_{S \subseteq N: i \in S} \frac{(s-1)!(n-s)!}{n!} \left(v(S) - v(S \backslash \{i\})\right), \quad \text{for all } i \in N.$$

If (N,v) is a convex game, then the marginal vectors $m^\sigma(N,v)$ are the extreme points of $C(N,v)$, i.e.,[1]

$$\begin{aligned} C(N,v) &= conv\{m^\sigma(N,v) : \sigma \in \Pi(N)\} \\ &= conv\left\{m^{[\sigma]}(N,v) : [\sigma] \in \pi(N)/\sim\right\}. \end{aligned}$$

For the subclass of convex games, the Shapley value is the center of mass of the extreme points of the core where the weight of each extreme point is the number of permutations that originate it. Usually, when working with convex games, the Shapley value is called, by an abuse of language, the barycenter of the core.

Now we define formally the core-center and a property which will be used along the paper.

Definition 1. *Let BG^n denote the class of balanced games. The Core-Center μ is the allocation rule defined as follows:*

[1] Given a set $A \subseteq \mathbb{R}^N$, we denote by $conv(A)$ its convex hull.

$$\begin{array}{rccl} \mu: & BG^n & \longrightarrow & \mathbb{R}^n \\ & (N,v) & \longmapsto & \mu((N,v)) \end{array}$$

where $\mu((N,v))$ denotes the center of gravity of the core of the game (N,v)[2].

A convex polytope P is the convex hull of a finite set $V = \{v_1, ..., v_n\}$ of points in $\mathbb{R}^n$. Clearly the core of a game is a polytope.

Definition 2. *A set $\{a^0, a^1, \ldots, a^n\}$ in $\mathbb{R}^n$ is said to be geometrically independent if for any scalars $t_i \in \mathbb{R}$, the equations*

$$\sum_{i=1}^{n} t_i = 0 \quad \textit{and} \quad \sum_{i=1}^{n} t_i a^i = 0$$

imply that $t_0 = t_1 = \cdots = t_n = 0$. Note that $\{a^0, a^1, \ldots, a^n\}$ is geometrically independent if and only if the vectors $a^1 - a^0, \ldots, a^n - a^0$ are linearly independent.

Definition 3. *Let $\{a^0, a^1, \ldots, a^n\}$ be a geometrically independent set in $\mathbb{R}^n$. The n-simplex S_n spanned by $a^0, a^1, \ldots, a^n$ is the set of all points x of $\mathbb{R}^n$ such that*

$$x = \sum_{i=1}^{n} t_i a^i \quad \textit{where} \quad \sum_{i=1}^{n} t_i = 1 \quad \textit{and} \quad t_i \geq 0, \, \forall i$$

Each a^i is a vertex of the n-simplex. The numbers t_i are the barycentric coordinates for x of S_n with respect to $a^0, a^1, \ldots, a^n$.The subscript of S_n is the dimension of the simplex. An n-simplex is regular if the distance between any two vertices is constant.

Definition 4. *The centroid or barycentre, of an n-simplex S_n spanned by the points $a^0, a^1, \ldots, a^n$ is*

$$\Theta(S_n) = \sum_{i=0}^{n} \frac{a^i}{n+1}$$

Any convex polytope can be partitioned into simplices. The volume of a polytope is usually computed decomposing the polytope into simplices for which the volume is easily computed and summed up.

Let P be a polytope in $\mathbb{R}^n$, if P is split into $P_1, ..., P_s$ such that $P = \bigcup_{i=1}^{s} P_i$ and $Vol(P_i \cap P_j) = 0$ for all $i \neq j$, then $Vol(P) = \sum_{i=1}^{s} Vol(P_i)$. The particular case where the elements of the partition are simplices, is called a triangulation of P.

[2] We consider our sets as homogeneous bodies, so the center of gravity (or center of mass) of a set can be calculated as the expectation of the uniform distribution defined over it.

2 The utopia games

In this section we introduce a new class of games; they will play an important role to make comparisons between core-center and Shapley value.

Definition 5. *Let (N, v) be a convex game, and let $T \in 2^N \setminus\{\emptyset\}$. Consider $H \in 2^T\setminus\{\emptyset\}$, then we define the game (N, v_T^H) where*

$$v_T^H(S) = \begin{cases} v((T\cap S)\cup(N\backslash T)) - v(N\backslash T) + v(S\backslash(T\cap S)) & \text{if } H \subset S \\ v((T\cap S)\cup(N\backslash T)) - v(N\backslash T) + \sum\limits_{j\in S\backslash(T\cap S)} v(j) & \text{otherwise} \end{cases}$$

It can be easily showed that for all $T \in 2^N$, $T \neq \emptyset$ and $\emptyset \neq H \subset T$,

- $v_T^H(\emptyset) = 0$
- $v_T^H(N) = v(N)$
- Whenever $T \cap S = \emptyset$, $v_T^H(S) = \sum\limits_{j\in S} v(j)$.

Furthermore we define the game $(N, v_\emptyset)$ where players have not incentives to cooperate in subcoalitions,

$$v_\emptyset(S) = \begin{cases} \sum\limits_{j\in S} v(\{j\}) & \text{if } S \neq N \\ v(N) & \text{if } S = N \end{cases}$$

Now, we describe in detail the game v_T^H. The first consideration is that for each coalition $S \neq T$, the value $v_T^H(S)$ is the sum of two quantities: on the one side the marginal contribution of the players of T that are in S to $N\backslash T$, and in the other side, the contribution of players of S that are not in T. The second consideration concerns the coalition formation; in a convex game, it is well known that players have incentives to cooperate and share the total amount given by $v(N)$.

Observe that, independently of H, what a coalition $S \subseteq T$ obtains in the game v_T^H is its marginal contribution to $N\backslash T$, i.e., $v_T^H(S) = v(S\cup(N\backslash T)) - v(N\backslash T)$, and in the case where $S = T$, $v_T^H(T) = v(N) - v(N\backslash T)$.

Take now $S \subset N$ such that $T \cap S \neq \emptyset, S$; the contribution of players of S that are not in T, depends on the head-coalition H. Fixed a head-coalition H, then the contribution of players of S that are not in T is the maximum utility that the players can guarantee by themselves whenever $H \subset S$, i.e., $v(S\backslash(T\cap S))$; otherwise, that contribution is computed by $\sum\limits_{j\in S\backslash(T\cap S)} v(j)$. Roughly speaking, players of H can be thought as the ones who have the key to allow cooperation.

The main idea underlying the games v_T^H is that players of T are the ones who have the power in the game, but always respecting the minimal rights of players in $N\backslash T$. Furthermore, the game also establishes a hierarchical structure among the members of T.

Next proposition shows that these games are convex.

Proposition 1. *Let* $(N,v) \in CG^n$, $T \in 2^N \setminus \{\emptyset\}$ *and* $H \in 2^T \setminus \{\emptyset\}$, *then* $(N, v_T^H) \in CG^n$.

Proof. We prove that $v_T^H(R \cup \{i\}) - v_T^H(R) \leq v_T^H(S \cup \{i\}) - v_T^H(S)$ for all $R \subset S \subset N \setminus \{i\}$. Let us suppose first that $i \notin T$. We distinguish three cases:

c1) $H \subseteq R \subset S$.

As $v_T^H(S \cup \{i\}) - v_T^H(S) = v((S \cup \{i\}) \setminus T) - v(S \setminus T)$ and $v_T^H(R \cup \{i\}) - v_T^H(R) = v((R \cup \{i\}) \setminus T) - v(R \setminus T)$, the convexity condition for the game (N, v_T^H) holds by the convexity of the game (N,v).

c2) $H \not\subseteq R$ and $H \subseteq S$.

In this case

$$\begin{aligned} v_T^H(S \cup \{i\}) - v_T^H(S) &= v((S \cup \{i\}) \setminus T) - v(S \setminus T) \\ v_T^H(R \cup \{i\}) - v_T^H(R) &= v(\{i\}). \end{aligned}$$

Hence, again by the convexity of the game (N,v), the result holds.

c3) $H \not\subseteq S$. Trivial, since $i \notin T$ we have $v_T^H(R \cup \{i\}) - v_T^H(R) = v_T^H(S \cup \{i\}) - v_T^H(S) = v(\{i\})$.

Let us suppose now that $i \in T$. We have again three different cases to study:

c4) $H \subseteq R \cup \{i\} \subset S \cup \{i\}$. We distinguish two subcases: $i \notin H$ and $i \in H$.

$i \notin H$. In this case,

$$v_T^H(S \cup \{i\}) - v_T^H(S) = v((T \cap S) \cup (N \setminus T) \cup \{i\}) - v((T \cap S) \cup (N \setminus T))$$

$$v_T^H(R \cup \{i\}) - v_T^H(R) = v((T \cap R) \cup (N \setminus T) \cup \{i\}) - v((T \cap R) \cup (N \setminus T))$$

As a consequence of the convexity condition the result holds.

$i \in H$. Now,

$$\begin{aligned} v_T^H(S \cup \{i\}) - v_T^H(S) = {} & v((T \cap S) \cup (N \setminus T) \cup \{i\}) - \\ & - v((T \cap S) \cup (N \setminus T)) + v(S \setminus (T \cap S)) - \sum_{j \in S \setminus (T \cap S)} v(\{j\}) \end{aligned}$$

$$\begin{aligned} v_T^H(R \cup \{i\}) - v_T^H(R) = {} & v((T \cap R) \cup (N \setminus T) \cup \{i\}) - \\ & - v((T \cap R) \cup (N \setminus T)) + v(R \setminus (T \cap R) - \sum_{j \in R \setminus (T \cap R)} v(\{j\}) \end{aligned}$$

Again by the convexity of the game (N, v),

$$v((T \cap S) \cup (N \backslash T) \cup \{i\}) - v((T \cap S) \cup (N \backslash T)) \geq$$
$$\geq v((T \cap R) \cup (N \backslash T) \cup \{i\}) - v((T \cap R) \cup (N \backslash T))$$

Besides,

$$v(S \backslash (T \cap S)) - \sum_{j \in S \backslash (T \cap S)} v(\{j\}) \geq v(R \backslash (T \cap R)) - \sum_{j \in R \backslash (T \cap R)} v(\{j\})$$

since

$$v(S \backslash (T \cap S)) - v(R \backslash (T \cap R)) = v(S \backslash (T \cap S)) - v(R \backslash (T \cap S \cap R))$$

and

$$\sum_{j \in S \backslash (T \cap S)} v(\{j\}) - \sum_{j \in R \backslash (T \cap R)} v(\{j\}) =$$
$$= \sum_{j \in S \backslash (T \cap S)} v(\{j\}) - \sum_{j \in R \backslash (T \cap S \cap R)} v(\{j\}) = \sum_{j \in S \backslash (R \cup (T \cap S \cap (N \backslash R))} v(\{j\})$$

c5) $H \not\subset R \cup \{i\}$ and $H \subseteq S \cup \{i\}$. We distinguish two subcases: $i \notin H$ and $i \in H$.
$i \notin H$ Then,

$$v_T^H(S \cup \{i\}) - v_T^H(S) = v((T \cap S) \cup (N \backslash T) \cup \{i\}) - v((T \cap S) \cup (N \backslash T))$$
$$v_T^H(R \cup \{i\}) - v_T^H(R) = v((T \cap R) \cup (N \backslash T) \cup \{i\}) - v((T \cap R) \cup (N \backslash T))$$

Therefore, by the convexity of the game (N, v), the convexity condition also holds for the game (N, v_T^H).
$i \in H$. Then,

$$v_T^H(S \cup \{i\}) - v_T^H(S) = v((T \cap S) \cup (N \backslash T) \cup \{i\}) -$$
$$- v((T \cap S) \cup (N \backslash T)) + v(S \backslash (T \cap S)) - \sum_{j \in S \backslash (T \cap S)} v(\{j\})$$

$$v_T^H(R \cup \{i\}) - v_T^H(R) = v((T \cap R) \cup (N \backslash T) \cup \{i\}) - v((T \cap R) \cup (N \backslash T))$$

And, repeating analogous reasonings, we obtain the convexity condition since $v(S \backslash (T \cap S)) - \sum_{j \in S \backslash (T \cap S)} v(\{j\}) \geq 0$.

c6) $H \not\subset S \cup \{i\}$.

$$v_T^H(S \cup \{i\}) - v_T^H(S) = v((T \cap S) \cup (N \backslash T) \cup \{i\}) - v((T \cap S) \cup (N \backslash T))$$
$$v_T^H(R \cup \{i\}) - v_T^H(R) = v((T \cap R) \cup (N \backslash T) \cup \{i\}) - v((T \cap R) \cup (N \backslash T))$$

So, the convexity condition also holds.

Now all the cases have been analyzed in detail, so the game $(N, v_T^H) \in CG^n$. $\square$

For our purpose in the present Section, we will focus in two special cases of these games, those such that $|T| \leq 2$.

Let $i \in N$, and $T = \{i\}$ (in this case, $H = T$). From now on we will identify for each $i \in N$, the game $(N, v_{\{i\}}^{\{i\}})$ with $(N, v_{\{i\}})$. The game $(N, v_{\{i\}})$ can be thought as the utopia game for player i, and it will be referred to as the $\{i\}$-utopia game. Now observe its characteristic function below, it shows that player i is the powerful one since all the coalitions to which i does not belong, are formed separately (each player gets his $v(\{j\})$ if one such coalition is formed).

For all $S \subset N$,

$$v_{\{i\}}(S) = \begin{cases} v(N) - v(N\setminus \{i\}) + v(S\setminus\{i\}) & \text{if } i \in S \\ \sum\limits_{l \in S} v(\{l\}) & \text{if } i \notin S \end{cases}$$

In this game player i can be thought as the one who has the key for cooperation, the other players by themselves can get only the sum of their individual values.

Let us describe now the games v_T^H where T is a two player coalition, $T = \{i_1, i_2\}$ with $i_1, i_2 \in N$ and $i_1 \neq i_2$. To this extent, we can define three games:

$$(N, v_{\{i_1,i_2\}}^{\{i_1\}}), \ (N, v_{\{i_1,i_2\}}^{\{i_2\}}), \text{ and } (N, v_{\{i_1,i_2\}}^{\{i_1,i_2\}})$$

that we denote to simplify notation by $(N, v_{(i_1,i_2)})$, $(N, v_{(i_2,i_1)})$, and $(N, v_{i_1 i_2})$, respectively.

$$\begin{aligned} v_{(i_1,i_2)}(S) &= \begin{cases} v((T \cap S) \cup (N\setminus T)) - v(N\setminus T) + v(S\setminus(T \cap S)) & i_1 \in S \\ v((T \cap S) \cup (N\setminus T)) - v(N\setminus T) + \sum\limits_{j \in S\setminus(T\cap S)} v(\{j\}) & i_1 \notin S \end{cases} \\ &= \begin{cases} v(N) - v(N\setminus\{i_1, i_2\}) + v(S\setminus\{i_1, i_2\}) & i_1, i_2 \in S \\ v(N\setminus\{i_2\}) - v(N\setminus\{i_1, i_2\}) + v(S\setminus\{i_1\}) & i_1 \in S, i_2 \notin S \\ v(N\setminus\{i_1\}) - v(N\setminus\{i_1, i_2\}) + \sum\limits_{l \in S\setminus\{i_2\}} v(\{l\}) & i_2 \in S, i_1 \notin S \\ \sum\limits_{l \in S} v\{l\}) & i_1 \notin S, i_2 \notin S \end{cases} \end{aligned}$$

We call this game the (i_1, i_2)-utopia game. Analogously we can define the (i_2, i_1)-utopia game interchanging the role of i_1 and i_2.

The characteristic function of the $i_1 i_2$-utopia game is the following:

$$v_{i_1 i_2}(S) = \begin{cases} v(N) - v(N\setminus\{i_1, i_2\}) + v(S\setminus\{i_1, i_2\}) & i_1, i_2 \in S \\ v(N\setminus\{i_1\}) - v(N\setminus\{i_1, i_2\}) + \sum\limits_{l \in S\setminus\{i_2\}} v(\{l\}) & i_2 \in S, i_1 \notin S \\ v(N\setminus\{i_2\}) - v(N\setminus\{i_1, i_2\}) + \sum\limits_{l \in S\setminus\{i_1\}} v(\{l\}) & i_1 \in S, i_2 \notin S \\ \sum\limits_{l \in S} v\{l\}) & i_1 \notin S, i_2 \notin S \end{cases}$$

The $i_1 i_2$-utopia game is not going to be used anymore so there will be no place for confusion with notation; a (i_1, i_2)-utopia game will denote the game which is good for both players 1 and 2 but excellent for player 1.

Remark. Expressions for marginal contributions to a coalition for the i-utopia games and for the (i_1, i_2)-utopia games can be obtained without effort. Let $i \in N$, $S \subset N$ and $j \notin S$, then

$$v_{\{i\}}(S \cup \{j\}) - v_{\{i\}}(S) = \begin{cases} v((S \cup \{j\}) \backslash \{i\}) - v(S \backslash \{i\}) & i \in S, i \neq j \\ v(N) - v(N \backslash \{i\}) + v(S) - \sum\limits_{l \in S} v(\{l\}) & i = j \\ v(\{j\}) & i \notin S, i \neq j \end{cases}$$

Similar expressions can be found for the (i_1, i_2)-utopia games. Notice that by the convexity of the game (N, v_T^H),

$$C(N, v_T^H) = conv\{m^\sigma(N, v_T^H)\}.$$

Lemma 1. *Let $(N, v) \in G^n$, $n > 2$, such that $v(N) > \sum\limits_{i \in N} v(\{i\})$. Then,*

a) $\Theta_i(I(N, v)) = v(\{i\}) + \frac{v(N) - \sum\limits_{k \in N} v(\{k\})}{n}$ *for all* $i \in N$.

b) If $v(S) = \sum\limits_{i \in S} v(\{i\})$, *for all* $S \subset N$, $S \neq N$ *then* $\mu(N, v) = \Theta(I(N, v)) = Sh(N, v)$.

c) $Vol(I(N, v)) = \frac{1}{(n-1)!} n^{1/2} \left(v(N) - \sum\limits_{j \in N} v(\{j\}) \right)^{n-1}$ *where* Vol *is the Lebesgue measure on* $\mathbb{R}^{n-1}$.

Proof. The statements in a) and b) are straightforward, and c) is a consequence of the definition of an $(n-1)$-dimensional simplex and its corresponding volume. □

Corollary 1. *Let $(N, v) \in G^2$ with $N = \{1, 2\}$, such that $v(N) > \sum\limits_{i \in N} v(\{i\})$, then,*

$$Sh(N, v) = \Theta(I(N, v)) = \mu(N, v)$$

and it corresponds to the center of mass of the segment

$$[(v(1), v(N) - v(1)), (v(N) - v(2), v(2))].$$

Proof. This immediately yields from Lemma 1. □

Next definition classifies each game attending to the relevant utility of the coalitions in relation with their members.

Definition 6. *Given a game $(N,v) \in G^n$, we say that $(N,v) \in G^n_r \Leftrightarrow \exists 1 \leq r \leq n-1$ such that*

- $v(S) = \sum\limits_{i \in S} v(\{i\})$ *for all* $|S| \leq r$ *and*
- *There exists a coalition S, $|S| = r+1$ such that* $v(S) > \sum\limits_{i \in S} v(\{i\})$

Then, for each game (N,v) there is $1 \leq r \leq n-1$ such that $(N,v) \in G^n_r$. The case where for all $S \subset N$, $v(S) = \sum_{i \in S} v(\{i\})$ has been avoided in the definition because in this case the core is indeed a single point, and then $\mu(N,v) = (v(1), \ldots, v(n))$. Observe that $I(N,v) = C(N,v)$ if and only if $r = n-1$, and then by Lemma 1 the core-center is the centroid of the imputation set. A game is in G^n_r if all the restrictions originated by m player coalitions are redundant in its core when $1 < m \leq r$, and there is at least one coalition with $r+1$ players such that its corresponding restriction is not redundant.

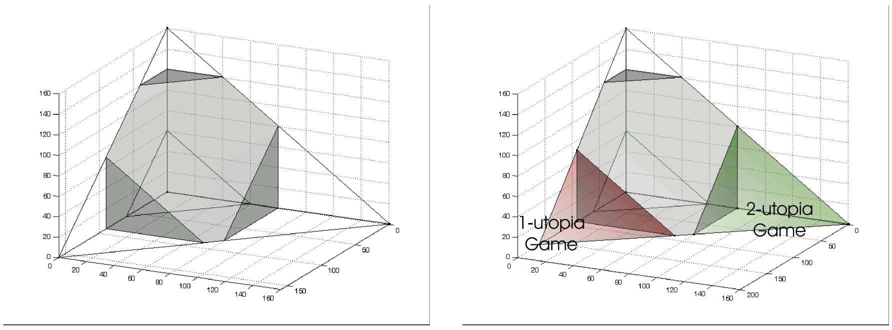

Figure 1: Core of a game in G^4_2 and the cores of a pair of its utopia games

Lemma 2. *Let $(N,v) \in CG^n \cap G^n_{n-2}$. Then,*

a) For all $i \in N$, $C(N, v_{\{i\}}) = I(N, v_{\{i\}})$ and consequently,

$$\frac{Vol(C(N,v_{\{i\}}))}{Vol(I(N,v))} = \left(\frac{v(N \backslash \{i\}) - \sum_{j \in N \backslash \{i\}} v(\{j\})}{v(N) - \sum_{j \in N} v(\{j\})} \right)^{n-1}$$

where Vol is the Lebesgue measure on $\mathbb{R}^{n-1}$.

b) Let $i, j \in N$, $i \neq j$, then $C\left(N, (v_{\{i\}})_{\{j\}}\right) = m^{\sigma}(N,v)$ where $\sigma \in \pi(N)$ is such that $\sigma(i) = n$ and $\sigma(j) = n-1$.

c) $I(N,v) = \left(\bigcup\limits_{i \in N} C(N, v_{\{i\}}) \right) \bigcup C(N,v)$.

d) For all $i, j \in N$, $C(N, v_{\{i\}}) \cap C(N, v_{\{j\}})$ and $C(N,v) \cap C(N, v_{\{i\}})$ are null measure sets.

e) $Vol(I(N,v)) = \sum_{i\in N} Vol(C(N, v_{\{i\}})) + Vol(C(N,v))$ *where* Vol *is the Lebesgue measure on* $\mathbb{R}^{n-1}$.

Proof. a) Let $i \in N$. It is easy to check that

$$v_{\{i\}}(S) = \begin{cases} v(N) - v(N\setminus \{i\}) + v(S\setminus\{i\}) & \text{if } i \in S \text{ and } |S| \geq n-1 \\ v(N) - v(N\setminus \{i\}) + \sum\limits_{j\in S\setminus\{i\}} v(j) & \text{if } i \in S \text{ and } |S| < n-1 \\ \sum\limits_{j\in S} v(j) & \text{if } i \notin S \end{cases} \tag{1}$$

Besides, let $\sigma \in \pi(N)$ such that $\sigma(i) = n$, then

$$\begin{aligned} m_i^\sigma(N, v_{\{i\}}) &= v(N) - \sum_{l\in N\setminus\{i\}} v(\{l\}) \\ m_k^\sigma(N, v_{\{i\}}) &= v(\{k\}) \text{ for all } k \neq i \end{aligned}$$

and take $j \in N$, $j \neq i$, then for all $\sigma \in \pi(N)$ such that $\sigma(j) = n$,

$$\begin{aligned} m_i^\sigma(N, v_{\{i\}}) &= v(N) - v(N\setminus\{i\}) \\ m_j^\sigma(N, v_{\{i\}}) &= v(N\setminus\{i\}) - \sum_{l\neq i,j} v(\{l\}) \\ m_k^\sigma(N, v_{\{i\}}) &= v(\{k\}) \text{ for all } k \neq i,j \end{aligned}$$

Therefore, $C(N, v_{\{i\}}) = I(N, v_{\{i\}})$ for all $i \in N$. The ratio between the cores is easily established following Lemma 1.

b) Let $i, j \in N$, $j \neq i$. By Proposition 1, $(N, v_{\{i\}}) \in CG^n$. Besides, $v_{\{i\}}(S) = \sum_{l\in S} v_{\{i\}}(\{l\})$ for all $S \subset N$ such that $|S| \leq n-2$, then by a),

$$C(N, (v_{\{i\}})_{\{j\}}) = I(N, (v_{\{i\}})_{\{j\}}).$$

We describe the characteristic function of the $\{j\}$-utopia game associated with the $\{i\}$-utopia game. Following (1),

$$(v_{\{i\}})_{\{j\}}(S)) = \begin{cases} v_{\{i\}}(N) - v_{\{i\}}(N\setminus \{j\}) + v_{\{i\}}(S\setminus\{j\}) & j \in S, |S| \geq n-1 \\ v_{\{i\}}(N) - v_{\{i\}}(N\setminus \{j\}) + \sum\limits_{l\in S\setminus\{j\}} v_{\{i\}}(\{l\}) & j \in S, |S| < n-1 \\ \sum\limits_{l\in S} v_{\{i\}}(\{l\}) & j \notin S \end{cases} \tag{2}$$

Furthermore, taking into account that

$$v_{\{i\}}(S\setminus\{j\}) = \begin{cases} v(N) - v(N\setminus \{i\}) + \sum\limits_{l\in S\setminus\{i,j\}} v(\{l\}) & \text{if } i \in S \\ \sum\limits_{l\in S\setminus\{j\}} v(\{l\}) & \text{if } i \notin S \end{cases} \tag{3}$$

where the last equality holds because $|S\backslash\{i,j\}| = n-2$. Besides,

$$v_{\{i\}}(\{l\}) = \begin{cases} v(N) - v(N\backslash\{i\}) & \text{if } l = i \\ v(\{l\}) & \text{if } l \neq i \end{cases} \tag{4}$$

Henceforth, substituting (3) and (4) in (2) one easily obtain with straight computations the value of any coalition S in the j-utopia game associated with the i-utopia game,

$$(v_{\{i\}})_{\{j\}}(S) = \begin{cases} v(N) - \sum\limits_{l\in N\backslash S} v(\{l\}) & j \in S, i \in S \\ v(N) - v(N\backslash\{i\}) + \sum\limits_{l\in S\backslash\{i\}} v(\{l\}) & j \notin S, i \in S \\ v(N\backslash\{i\}) - \sum\limits_{l\in N\backslash(S\cup\{i\})} v(\{l\}) & j \in S, i \notin S \\ \sum\limits_{l\in S} v(\{l\}) & j \notin S, i \notin S \end{cases}$$

Now, it is immediate to check that $(N, (v_{\{i\}})_{\{j\}})$ is an additive game with constants:

$$\begin{aligned} c_i &= v(N) - v(N\backslash\{i\}) \\ c_j &= v(N\backslash\{i\}) - \sum_{l\in N\backslash\{i,j\}} v(\{l\}) \\ c_k &= v(\{k\}) \text{ for all } k \in N\backslash\{i,j\}. \end{aligned}$$

So, for all $\sigma \in \pi(N)$, $C\left(N, (v_{\{i\}})_{\{j\}}\right) = m^\sigma(N,v)$ where $\sigma \in \pi(N)$ is such that $\sigma(i) = n$ and $\sigma(j) = n-1$.

c) First we show that $(\bigcup\limits_{i\in N} C(N, v_{\{i\}})) \bigcup C(N,v) \subset I(N,v)$. We only need to check that for all $i \in N$, $C(N, v_{\{i\}}) \subset I(N,v)$. Take $x \in C(N, v_{\{i\}})$, then by the supperaditivity of the game $(N, v_{\{i\}})$,

$$x_i \geq v_{\{i\}}(\{i\}) = v(N) - v(N\backslash\{i\}) \text{ and } x_i \geq v(\{i\})$$

and for all $k \in N\backslash\{i\}$,

$$x_k \geq v_{\{i\}}(\{k\}) \text{ and } x_k = v(\{k\})$$

hence $x \in I(N,v)$.

Consequently we only need to prove that for all $x \in I(N,v) \backslash C(N,v)$, there is $i \in N$ such that $x \in C(N, v_{\{i\}})$. Take $x \notin C(N,v)$. That implies that there is $S \subset N$, $|S| = n-1$ such that $\sum\limits_{i\in S} x_i < v(S)$. Then there exists $i \in N$ such that $\sum\limits_{j\in N\backslash\{i\}} x_j < v(N\backslash\{i\})$, and by the efficiency condition we deduce that $x_i > v(N) - v(N\backslash\{i\})$, and this only happens in $C(N, v_{\{i\}})$.

d) We first prove that for all $i \in N$, $C(N,v) \cap C(N, v_{\{i\}})$ is a null measure set.

Let $x \in C(N,v) \cap C(N,v_{\{i\}})$, then

$$\begin{aligned} x &\in C(N,v_{\{i\}}) \Rightarrow x_i \geq v_{\{i\}}(\{i\}) = v(N) - v(N\backslash\{i\}). \\ x &\in C(N,v) \Rightarrow v(\{i\}) \leq x_i \leq v(N) - v(N\backslash\{i\}) \end{aligned}$$

So, for all $x \in C(N,v) \cap C(N,v_{\{i\}})$, $x_i = v(N) - v(N\backslash\{i\})$, and then $C(N,v) \cap C(N,v_{\{i\}})$ is at most an $n-2$ dimensional space.

Now we prove that for all $i,j \in N$,

$$C(N,v_{\{i\}}) \cap C(N,v_{\{j\}}) \neq \emptyset \Rightarrow C(N,v_{\{i\}}) \cap C(N,v_{\{j\}}) \subset C(N,v) \cap C(N,v_{\{i\}})$$

which it will finish the proof.

We know the existence of $x \in C(N,v_{\{i\}}) \cap C(N,v_{\{j\}})$, which implies

$$\begin{aligned} x_i &\geq v(N) - v(N\backslash\{i\}) \\ x_j &\geq v(N) - v(N\backslash\{j\}) \\ x_k &\geq v(\{k\}) \text{ for all } k \in N\backslash\{i,j\} \end{aligned}$$

Let us suppose that $x \notin C(N,v)$, then there is $S \subset N$ such that $\sum\limits_{i \in S} x_i < v(S)$. So, the size of the coalition S must be $n-1$. Then or $i \in S$ or $j \in S$. Take w.l.o.g. $S = N\backslash\{j\}$, then

$$v(N\backslash\{j\}) > \sum_{i \in N\backslash\{j\}} x_i \geq v(N) - v(N\backslash\{i\}) + \sum_{l \in N\backslash\{i,j\}} v(\{l\})$$

As $(N,v) \in G^n_{n-2}$ $v(N\backslash\{i,j\}) = \sum\limits_{l \in N\backslash\{i,j\}} v(\{l\})$, we deduce that,
$v(N\backslash\{j\}) - \sum\limits_{l \in N\backslash\{i,j\}} v(\{l\}) = v(N\backslash\{j\}) - v(N\backslash\{i,j\})$ and
$v(N\backslash\{j\}) - \sum\limits_{l \in N\backslash\{i,j\}} v(\{l\}) > v(N) - v(N\backslash\{i\})$,
contradicting the convexity of the game (N,v).

e) It follows immediately from d). □

Theorem 1. *Let $(N,v) \in CG^n \cap G^n_{n-2}$ with $n > 2$. Then,*

$$\mu(N,v) = Sh(N,w)$$

where for all $S \subset N$,

$$w(S) = \left(\frac{1}{p}\left(p_0 v_\emptyset - \sum_{i \in N} p_i v_{\{i\}}\right)\right)(S)$$

being $Vol(C(N,v)) = p, Vol(C(N,v_\emptyset)) = p_0$, and $Vol(C(N,v_{\{i\}}) = p_i$ for all $i \in N$.

Proof. The core-center satisfies the property of fair additivity on the core[3], and by Lemma 2, the imputation set can be broken into $n+1$ pieces, the cores of the utopia games and the core of the original game. So,

$$\mu(N, v_{\emptyset}) = \mu(N, v)\frac{p}{p_0} + \sum_{i \in N} \mu(N, v_{\{i\}})\frac{p_i}{p_0}$$

By Lemma 1, $\mu(N, v_{\emptyset}) = Sh(N, v_{\emptyset})$, and $\mu(N, v_{\{i\}}) = Sh(N, v_{\{i\}})$ for all $i \in N$. Thus,

$$\begin{aligned} \mu(N, v) &= \frac{p_0}{p}\left(\mu(N, v_{\emptyset}) - \sum_{i \in N} \frac{p_i}{p_0}\mu(N, v_{\{i\}})\right) \\ &= \frac{p_0}{p} Sh(N, v_{\emptyset}) - \sum_{i \in N} \frac{p_i}{p} Sh(N, v_{\{i\}}) \\ &= \frac{1}{p}\left(p_0 Sh(N, v_{\emptyset}) - \sum_{i \in N} p_i Sh(N, v_{\{i\}})\right) \\ &= Sh(N, w). \end{aligned}$$

where the last equality holds by the additivity of the Shapley value. □

Remark. Note that the last part of the proof has the following nice feature; we start with the centroid of a game and in two steps, using both the "fair additivity" of the core-center and the "additivity" of the Shapley value, we end up with the Shapley value of a new game which we are going to call the "fair game".

Corollary 2. *Let $(N, v) \in CG^3$. Then,*

$$\mu(N, v) = Sh(N, w)$$

where $w(S) = \left(\frac{1}{p}\left(p_0 v_{\emptyset} - \sum_{i \in N} p_i v_{\{i\}}\right)\right)(S)$.

Proof. It is immediate from Theorem 1. □

Remark. The game (N, w) can be interpreted as the fair game that takes all the information in the core. The core of the fair game coincides with its imputation set and then it will be a simplex containing the core of the game (N, v). That fact is easily showed since the game (N, w) is an additive game for the coalitions of size less of equal than $n-1$ which is a consequence of the same additivity property in the games $(N, v_{\{i\}})$ and $(N, v_{\emptyset})$.

[3]This property is formally defined in González-Díaz and Sánchez-Rodríguez (2003), it is founded upon the following property of the center of gravity: Given a set A and two sets B_1 and B_2 such that $B_1 \cup B_2 = A$ and $Vol(B_1 \cap B_2) = 0$ then the center of gravity of A can be expressed as $\mu(A) = Vol(B_1)\mu(B_1) + Vol(B_2)\mu(B_2)$.

Furthermore with easy computations in the game (N, w) one can derive the following expression for the individual values in the game (N, w),

$$w(\{i\}) = v(\{i\}) - \frac{p_i}{p}\left(v(N) - v(N\backslash\{i\}) + v(\{i\})\right) \text{ for all } i \in N$$

and as a consequence,

$$\mu(N, v) = w(\{i\}) + \frac{v(N) - \sum_{k\in N} w(\{k\})}{n}.$$

To illustrate the utopia games and the fair game we give the following example. In addition, special emphasis is made on the ratios $\frac{p_i}{p}$.

Example 3 players. Let us take the game (N, v), where $N = \{1, 2, 3\}$ and $v(\{i\}) = 0$ for all $i \in N$, $v(1, 2) = 2, v(1, 3) = v(2, 3) = 5$, and $v(N) = 10$. Then

Coalition	$1-$ utopia game	$\{1,2\}-$ utopia game	$w(S)$
1	$v(123) - v(23)$	$v(13) - v(3)$	-2.7174
2	$v(2)$	$v(23) - v(3)$	-2.7174
3	$v(3)$	$v(3)$	-0.6957
12	$v(123) - v(23) + v(2)$	$v(123) - v(3)$	-5.4348
13	$v(123) - v(23) + v(3)$	$v(13)$	-3.4130
23	$v(2) + v(3)$	$v(23)$	-3.4130
123	$v(123)$	$v(123)$	10

and,

$$\begin{aligned} Sh(N, v) &= (2.8333, 2.8333, 4.3333) \\ \mu(N, v) &= (2.6594, 2.6594, 4.6812) \end{aligned}$$

besides, for all $i \in N$,

$$\begin{aligned} r_i &= \frac{p_i}{p_0} = \left(\frac{v(N\backslash\{i\}) - \sum_{k\in N\backslash\{i\}} v(\{k\})}{v(N) - \sum_{k\in N} v(\{k\})}\right)^2 \\ \text{and } r &= \frac{p}{p_0} \end{aligned}$$

With easy computations we obtain, $\frac{p_1}{p_0} = \frac{p_2}{p_0} = \frac{1}{4}, \frac{p_3}{p_0} = \frac{1}{25}$. Then, $\frac{p}{p_0} = 1 - \frac{1}{p_0}(p_1 + p_2 + p_3) = \frac{23}{50}$. So, players 1 and 2 are symmetric and less powerful than player 3.

Observe that $C(N, v_{\{1,2\}}) = C(N, v_{\{2,1\}}) = C(N, v_{\{12\}})$. Furthermore $x \in C(N, v_{\{1,2\}})$ if and only if

$$\begin{aligned} v(1, 3) - v(3) \leq &\; x_1 \leq v(1, 2, 3) - v(2, 3) \\ v(2, 3) - v(3) \leq &\; x_2 \leq v(1, 2, 3) - v(1, 3) \\ &\; x_3 = v(\{3\}) \end{aligned}$$

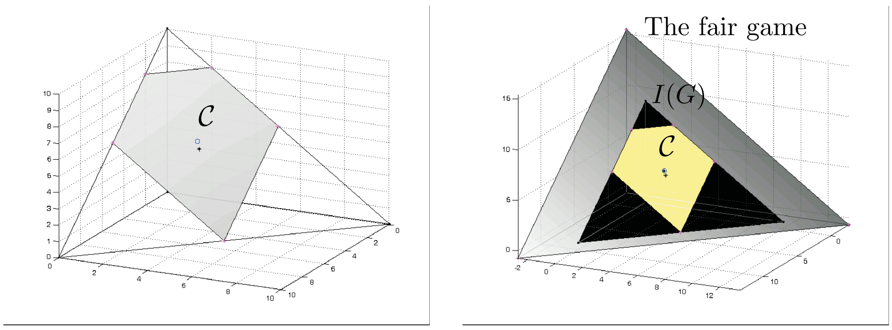

Figure 2: The core of a game and its corresponding “fair” game

In our figure the core of that game is the single point $(5, 5, 0)$.

The ratios r_i have a nice interpretation since for all $i \in N$, $0 \leq r_i \leq 1$ and $\sum r_i = 1$. So, we are really determining a probability distribution over the set of imputations. Given $i \in N$, r_i is the probability that the coalition $\{i\}$ receives a good allocation and the coalition $(N\backslash\{i\})$ is disappointed. Further r is the probability that the grand coalition gets an allocation in the core. So the greater the utopia game is for a player i, the worse his situation in the game is because his losses with regard to the original set of imputations are also greater, i.e. he has lost many good allocations. Looking at Figure 2 it can be seen that in the original game the “big” utopia games are those for players 1 and 2, so in the core of the fair game the “bad” section which has been added for player 3 with regard to the core of the original game is smaller than those for the other two players.

Let us assume that $(N, v) \in CG^n \cap G^n_{n-2}$ where $n > 2$ and $v(N) > \sum_{k \in N} v(\{k\})$.

Definition 7. *A player i is a* dummy player *if and only if*

$$v(N\backslash\{i\}) + v(\{i\}) = v(N), \ \textit{i.e.}\ r_i = 1.$$

A player i is a strong player *if and only if* $v(N\backslash\{i\}) = \sum_{k \in N\backslash\{i\}} v(\{k\})$, *i.e.* $r_i = 0$.

Let us notice that the strong players are the powerful ones. In contrast, dummy players receive only their individual values.

Let $(N, v) \in CG^n \cap G^n_{n-2}$. Next proposition shows that for these games there is at most one dummy player, and in such a case the core of the game coincides exactly with one face of the simplex (the one that gives $v(\{i\})$ to that player).

Proposition 2. *Let us assume that $v(N) > \sum_{k \in N} v(\{k\})$ and let $(N, v) \in CG^n \cap G^n_{n-2}$, then there is at most one dummy player, namely i, and in that case,*

$$C(N, v) = \{x_N : x_i = v(\{i\}) \ \textit{and}\ x_{N\backslash\{i\}} \in C(N\backslash\{i\}, v_{N\backslash\{i\}})\}.$$

Proof. Suppose there are two players $i, j \in N$ such that

$$\begin{aligned} v(N) &= v(N \backslash \{i\}) + v(\{i\}) \\ &= v(N \backslash \{j\}) + v(\{j\}) \end{aligned}$$

Henceforth,

$$\begin{aligned} v(N) - v(N \backslash \{j\}) &= v(\{j\}) \\ v(N \backslash \{i\}) - v(N \backslash \{i,j\}) &= v(N) - v(\{i\}) - v(N \backslash \{i,j\}) \\ &= v(N) - v(\{i\}) - \sum_{k \in N \backslash \{i,j\}} v(\{k\}) \end{aligned}$$

As $v(N) > \sum_{k \in N} v(\{k\})$, there is a contradiction with the convexity condition since $v(N) - v(N \backslash \{j\})$ should be greater or equal than $v(N \backslash \{i\}) - v(N \backslash \{i,j\})$.

The second part of the proof immediately yields since as $v(N) = v(N \backslash \{i\}) + v(\{i\})$, if $x \in C(N, v)$ then

$$x_i = v(\{i\}) \text{ and } \sum_{k \in N \backslash \{i\}} x_k = v(N \backslash \{i\}).$$

□

Remark. Proposition 2 tells us that we can avoid dummy players in order to compute the centroid. Let us denote by D_v the set of dummy players of the game (N, v). Then,

$$\mu_i(N, v) = \begin{cases} v(\{i\}) & \text{if } i \in D_v \\ \mu_i(N \backslash D_v, v_{N \backslash D_v}) & \text{if } i \notin D_v \end{cases}$$

Let us describe now the structure of the utopia games when we give power to the coalitions of size $n-3$.

Lemma 3. *Let $(N, v) \in CG^n \cap G^n_{n-3}$ with $n > 3$ and $i \in N$. Then,*

a) $(N, v_{\{i\}}) \in CG^n \cap G^n_{n-2}$.

b) $C(N, (v_{\{i\}})_{\{j\}}) = I(N, (v_{\{i\}})_{\{j\}})$ *for all* $i, j \in N$ *with* $i \neq j$.

c) $\mu(N, v_{\{i\}}) = Sh(N, w_{\{i\}})$, *where*

$$w_{\{i\}}(S) = \left(\frac{1}{Vol(C(N, v_{\{i\}}))} \left(p_{i_0}(v_{\{i\}})_{\emptyset} - \sum_{j \in N \backslash \{i\}} p_{i_j}(v_{\{i\}})_{\{j\}} \right) \right) (S)$$

where

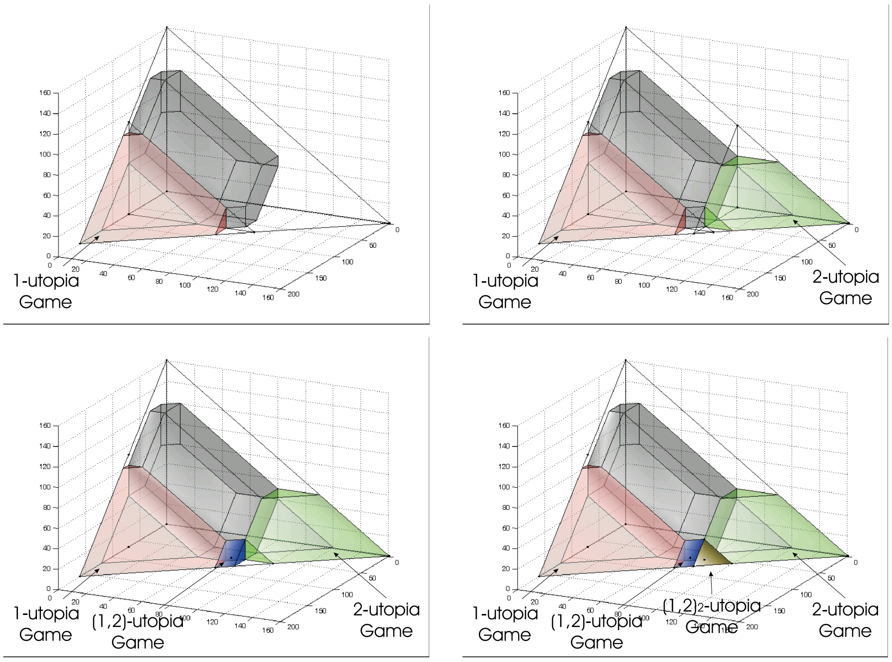

Figure 3: Example of a game in G_3^4 with some of its utopia games

$$p_{i_0} = Vol(C(N,(v_{\{i\}})_{\emptyset})) = \frac{1}{(n-1)!}n^{1/2}\left(v(N\backslash\{i\}) - \sum_{k\in N\backslash\{i\}} v(\{k\})\right)^{n-1}$$

$$p_{i_j} = Vol(C(N,v_{\{i\}})_{\{j\}}) = \frac{1}{(n-1)!}n^{1/2}\left(v(N\backslash\{i,j\}) - \sum_{k\in N\backslash\{i,j\}} v(\{k\})\right)^{n-1}$$

for all $j \in N\backslash\{i\}$.

Proof. a) We will show that $v_{\{i\}}(S) = \sum_{k\in S} v_{\{i\}}(\{k\})$ for all S such that $|S| \leq n-2$.

One can easily test that for all $S \subset N$,

$$v_{\{i\}}(S) = \begin{cases} v(N) - v(N\backslash\{i\}) + v(S\backslash\{i\}) & \text{if } i \in S \text{ and } |S| \geq n-2 \\ v(N) - v(N\backslash\{i\}) + \sum_{j\in S\backslash\{i\}} v(j) & \text{if } i \in S \text{ and } |S| < n-2 \\ \sum_{j\in S} v(j) & \text{if } i \notin S \end{cases}$$

When $|S| < n-2$, the result is immediate. The case where $|S| = n-2$ holds because $|S\backslash\{i\}| = n-3$, and then $v(S\backslash\{i\}) = \sum_{j\in S\backslash\{i\}} v(j)$, and so $v_{\{i\}}(S) = \sum_{k\in S} v_{\{i\}}(\{k\})$ for all S such that $|S| \leq n-2$.

b) and c) are immediately yield by Lemma 2 and Theorem 1. $\square$

Lemma 4. *Let $(N,v) \in CG^n \cap G^n_{n-3}$ with $n > 3$ and let $(N, v_{(i_1,i_2)})$ the (i_1,i_2)-utopia game where i_1 and $i_2 \in N$ and $i_1 \neq i_2$.*

a) $Sh(N, v_{(i_1,i_2)}) = \frac{\sum_{[\sigma]\in\pi(N)/\sim} m^\sigma(N,v)}{2n-2}$.

b) $Sh(N, v_{(i_1,i_2)}) = \mu(N, v_{(i_1,i_2)})$.

c) $Vol(C(N, v_{(i_1,i_2)})) = \frac{1}{(n-2)!} n^{\frac{1}{2}} \Big(v(N\backslash\{i_1,i_2\}) - \sum_{i_l \in N\backslash\{i_1,i_2\}} v(\{i_l\}) \Big)^{n-2} \cdot$

$$\cdot \Big(v(N) - v(N\backslash\{i_1\}) + v(N\backslash\{i_1,i_2\}) - v(N\backslash\{i_2\}) \Big)$$

Proof. a) First we obtain expressions for the marginal contributions given the characteristic function of the $(i_1,i_2)-$utopia game

$$v_{(i_1,i_2)}(S) = \begin{cases} v(N) - v(N\backslash\{i_1,i_2\}) + v(S\backslash\{i_1,i_2\}) & \text{if } i_1, i_2 \in S \\ v(N\backslash\{i_2\}) - v(N\backslash\{i_1,i_2\}) + v(S\backslash\{i_1\}) & \text{if } i_1 \in S \text{ but } i_2 \notin S \\ v(N\backslash\{i_1\}) - v(N\backslash\{i_1,i_2\}) + \sum_{l\in S\backslash\{i_2\}} v(\{l\}) & \text{if } i_2 \in S \text{ but } i_1 \notin S \\ \sum_{l\in S} v\{l\}) & \text{if } i_1 \notin S \text{ and } i_2 \notin S \end{cases}$$

Let $\pi_1(N) = \{\sigma \in \pi(N) : \sigma(i_1) = n\}$, then for all $\sigma \in \pi_1(N)$,

$$\begin{aligned} m^\sigma_{i_1}(N, v_{(i_1,i_2)}) &= v(N) - v(N\backslash\{i_1\}) + v(N\backslash\{i_1,i_2\}) - \sum_{l\in N\backslash\{i_1,i_2\}} v(\{l\}) \\ m^\sigma_{i_2}(N, v_{(i_1,i_2)}) &= v(N\backslash\{i_1\}) - v(N\backslash\{i_1,i_2\}) \\ m^\sigma_{i_k}(N, v_{(i_1,i_2)}) &= v(\{i_k\}) \text{ for all } i_k \neq i_1, i_2 \end{aligned}$$

It is easy to check that $|\pi_1(N)| = (n-1)!$

Let $\pi_2(N) = \{\sigma \in \pi(N) : \sigma(i_2) < \sigma(i_1)$ and there is $i_n \in N\backslash\{i_1,i_2\}$ such that $\sigma(i_n) = n\}$, then

$$\begin{aligned} m^\sigma_{i_1}(N, v_{(i_1,i_2)}) &= v(N) - v(N\backslash\{i_1\}) + v((P_\sigma(i_1)\backslash\{i_2\}) - \sum_{l\in P_\sigma(i_1)\backslash\{i_2\}} v(\{l\}) \\ &= v(N) - v(N\backslash\{i_1\}) \\ m^\sigma_{i_2}(N, v_{(i_1,i_2)}) &= v(N\backslash\{i_1\}) - v(N\backslash\{i_1,i_2\}) \\ m^\sigma_{i_k}(N, v_{(i_1,i_2)}) &= v(\{i_k\}) \ \forall \ i_k \text{ such that } \sigma(i_k) < \sigma(i_1) \\ m^\sigma_{i_k}(N, v_{(i_1,i_2)}) &= v((P_\sigma(i_k) \cup \{i_k\})\backslash\{i_1,i_2\}) - v((P_\sigma(i_k))\backslash\{i_1,i_2\}) \\ &\qquad \forall \ i_k \text{ such that } \sigma(i_k) > \sigma(i_1) \end{aligned}$$

Let us note that $v(P_\sigma(i_1)\backslash\{i_2\}) = \sum_{l\in P_\sigma(i_1)\backslash\{i_2\}} v(\{l\})$ since $|P_\sigma(i_1)\backslash\{i_2\}| \leq n-3$, and applying similar reasonings we obtain,

$$m^\sigma_{i_k}(N, v_{(i_1,i_2)}) = \begin{cases} v(\{i_k\}) & i_k \in N\backslash\{i_1, i_2\}\text{: } i_k \neq i_n \\ v(N\backslash\{i_1, i_2\}) - \sum_{i_l\in N\backslash\{i_1,i_2,i_n\}} v(\{i_l\}) & i_k = i_n \end{cases}$$

So, we can write,

$$\begin{aligned} m^\sigma_{i_1}(N, v_{(i_1,i_2)}) &= v(N) - v(N\backslash\{i_1\}) \\ m^\sigma_{i_2}(N, v_{(i_1,i_2)}) &= v(N\backslash\{i_1\}) - v(N\backslash\{i_1, i_2\}) \\ m^\sigma_{i_k}(N, v_{\{i\}}) &= v(\{i_k\}) \text{ for all } i_k \text{ such that } i_k \neq i_n \\ m^\sigma_{i_n}(N, v_{\{i\}}) &= v(N\backslash\{i_1, i_2\}) - \sum_{i_l\in N\backslash\{i_1,i_2,i_n\}} v(\{i_l\}) \end{aligned}$$

Observe that for each $\sigma \in \pi(N)$ such that $\sigma(i_2) < \sigma(i_1)$ and there is $i_n \in N\backslash\{i_1, i_2\}$ such that $\sigma(i_n) = n$, there are exactly $\frac{(n-1)!}{2}$ marginal vectors that give the same point. The total number of permutations in this subclass is $\frac{(n-1)!}{2}(n-2)$ and the number of equivalence classes is equal to $n-2$; i.e. $n-2$ different extreme points.

Let $\pi_3(N) = \{\sigma \in \pi(N)$ such that $\sigma(i_2) = n$ and $\sigma(i_1) = n-1\}$. Easily it can be showed that for all $\sigma \in \pi_3(N)$,

$$\begin{aligned} m^\sigma_{i_1}(N, v_{(i_1,i_2)}) &= v(N\backslash\{i_2\}) - \sum_{i_l\in N\backslash\{i_1,i_2\}} v(\{i_l\}) \\ m^\sigma_{i_2}(N, v_{(i_1,i_2)}) &= v(N) - v(N\backslash\{i_2\}) \\ m^\sigma_{i_k}(N, v_{(i_1,i_2)}) &= v(\{i_k\}) \text{ for all } i_k \text{ such that } i_k \neq i_1, i_2 \end{aligned}$$

In this case we have $(n-2)!$ marginal vectors that coincide in the same point.

Let $\pi_{4_a}(N) = \{\sigma \in \pi(N)$ such that $\sigma(i_2) < n$ and $\sigma(i_2) > \sigma(i_1)\}$. Then, take $\sigma \in \pi_{4_a}(N)$,

$$\begin{aligned} m^\sigma_{i_1}(N, v_{(i_1,i_2)}) &= v(N\backslash\{i_2\}) - v(N\backslash\{i_1, i_2\}) \\ m^\sigma_{i_2}(N, v_{(i_1,i_2)}) &= v(N) - v(N\backslash\{i_2\}) \\ m^\sigma_{i_k}(N, v_{(i_1,i_2)}) &= v(\{i_k\}) \forall\, i_k \text{ such that } \sigma(i_k) < \sigma(i_1) \\ m^\sigma_{i_k}(N, v_{(i_1,i_2)}) &= v((P_\sigma(i_k) \cup \{i_k\})\backslash\{i_1\}) - \sum_{i_l\in P_\sigma(i_k)\backslash\{i_1\}} v(\{i_l\}) \\ &\qquad \forall\, i_k \text{ such that } \sigma(i_1) < \sigma(i_k) < \sigma(i_2) \\[1em] m^\sigma_{i_k}(N, v_{(i_1,i_2)}) &= v((P_\sigma(i_k)) \cup \{i_k\})\backslash\{i_1, i_2\}) - v((P_\sigma(i_k))\backslash\{i_1, i_2\}) \\ &\qquad \forall\, i_k \text{ such that } \sigma(i_k) > \sigma(i_2) \end{aligned}$$

Equivalently,

$$\begin{aligned}
m_{i_1}^{\sigma}(N, v_{(i_1,i_2)}) &= v(N\backslash\{i_2\}) - v(N\backslash\{i_1, i_2\}) \\
m_{i_2}^{\sigma}(N, v_{(i_1,i_2)}) &= v(N) - v(N\backslash\{i_2\}) \\
m_{i_k}^{\sigma}(N, v_{(i_1,i_2)}) &= v(\{i_k\}) \text{ for all } i_k \text{ such that } \sigma(i_k) < n \\
m_{i_k}^{\sigma}(N, v_{(i_1,i_2)}) &= v(N\backslash\{i_1, i_2\}) - \sum_{i_l \in N\backslash\{i_1,i_2,i_k\}} v(\{i_l\}) \text{ if } \sigma(i_k) = n
\end{aligned}$$

It is straightforward to check that that $|\pi_{4_a}(N)| = \frac{(n-1)!}{2}(n-2)$, with $n-2$ different classes.

Let $\pi_{4_b}(N) = \{\sigma \in \pi(N)$ such that $\sigma(i_2) = n$ and $\sigma(i_1) < n-1\}$. Now it is not difficult to check that:

$$\begin{aligned}
m_{i_1}^{\sigma}(N, v_{(i_1,i_2)}) &= v(N\backslash\{i_2\}) - v(N\backslash\{i_1, i_2\}) \\
m_{i_2}^{\sigma}(N, v_{(i_1,i_2)}) &= v(N) - v(N\backslash\{i_2\}) \\
m_{i_k}^{\sigma}(N, v_{(i_1,i_2)}) &= v(\{i_k\}) \text{ for all } i_k \text{ such that } \sigma(i_k) < n-1 \\
m_{i_k}^{\sigma}(N, v_{(i_1,i_2)}) &= v(N\backslash\{i_1, i_2\}) - \sum_{i_l \in N\backslash\{i_1,i_2,i_k\}} v(\{i_l\}) \text{ if } \sigma(i_k) = n-1
\end{aligned}$$

And $|\pi_{4_b}(N)| = (n-1)! - (n-2)! = (n-2)!(n-2)$, with $n-2$ different classes. But now, these classes can be regrouped with the corresponding in $\pi_{4_b}(N)$, so we obtain $\pi_{4_a}(N)$ such that $|\pi_4(N)| = |\pi_{4_a}(N)| + |\pi_{4_b}(N)| = \frac{n^2-n-2}{2}(n-2)!$ and of course the number of equivalence classes remains equal to $n-2$.

Now we can write the following expression for the Shapley value:

$$Sh_{i_1}(N, v_{(i_1,i_2)}) = \frac{1}{n!} \sum_{\sigma \in \pi(N)} m_{i_1}^{\sigma}(N, v_{(i_1,i_2)}) = \frac{1}{n!}\Big[\sum_{l=1}^{4} \sum_{\sigma \in \pi_l(N)} m_{i_1}^{\sigma}(N, v_{(i_1,i_2)})\Big] =$$

$$= \frac{1}{n!}\Big[(n-1)!\Big[v(N) - v(N\backslash\{i_1\}) + v(N\backslash\{i_1, i_2\}) - \sum_{l \in N\backslash\{i_1,i_2\}} v(\{l\})\Big]\Big] +$$

$$+\frac{1}{n!}\Big[\frac{(n-1)!}{2}(n-2)\Big[v(N) - v(N\backslash\{i_1\})\Big]\Big] + \frac{1}{n!}\Big[(n-2)!\Big[v(N\backslash\{i_2\}) - \sum_{i_l \neq i_1,i_2} v(\{i_l\})\Big]\Big]$$

$$+ \frac{1}{n!}\Big[\frac{n^2-n-2}{2}(n-2)!\Big[v(N\backslash\{i_2\}) - v(N\backslash\{i_1, i_2\})\Big]\Big].$$

Making some computations,

$$\begin{aligned}
Sh_{i_1}(N, v_{(i_1,i_2)}) &= \frac{v(N) + v(N\backslash\{i_2\}) - v(N\backslash\{i_1\})}{2} + \\
&\quad -\frac{1}{n-1}\left(\frac{(n-3)}{2} v(N\backslash\{i_1, i_2\}) + \sum_{l \in N\backslash\{i_1,i_2\}} v(\{i_l\})\right).
\end{aligned}$$

Besides for player i_2,

$$Sh_{i_2}(N, v_{(i_1,i_2)}) = \frac{v(N) + v(N\backslash\{i_1\} - v(N\backslash\{i_2\}) - v(N\backslash\{i_1, i_2\})}{2}$$

and for all $i_k \in N\backslash\{i_1, i_2\}$,

$$Sh_{i_k}(N, v_{(i_1,i_2)}) =$$
$$\frac{1}{n!}\Big((n-1)!v(\{i_k\}) + \frac{(n-1)!}{2}\Big(v(N\backslash\{i_1, i_2\}) - \sum_{l\in N\backslash\{i_1,i_2,i_k\}} v(\{i_l\})\Big)\Big)+$$
$$+\frac{1}{n!}\Big(\frac{(n-1)!}{2}(n-3)v(\{i_k\})\Big) + \frac{1}{n!}\Big((n-2)!v(\{i_k\})\Big)+$$
$$+\frac{1}{n!}\left(\frac{n^2-n-2}{2}(n-3)!\Big(v(N\backslash\{i_1, i_2\}) - \sum_{l\in N\backslash\{i_1,i_2,i_k\}} v(\{i_l\} + (n-3)v(\{i_k\})\Big)\right).$$

Simplifying we obtain,

$$Sh_{i_k}(N, v_{(i_1,i_2)}) = \frac{n-2}{n-1}v(\{i_k\}) + \frac{1}{n-1}\Big(v(N\backslash\{i_1, i_2\}) - \sum_{l\in N\backslash\{i_1,i_2,i_k\}} v(\{i_l\})\Big).$$

Observe that the number of different vertices of the core is $2(n-1)$ that coincides with the cardinal of the quotient set $\frac{\pi(N)}{\sim}$.

$$|\pi(N)/\sim| = 1 + n - 2 + 1 + n - 2 = 2(n-1)$$

With some computations it can be checked that

$$\frac{\sum_{[\sigma]\in\pi(N)/\sim} m^\sigma(N, v)}{2n-2} = Sh(N, v_{(i_1,i_2)})$$

b) First observe that

$$m^\sigma_{i_2}(N, v_{(i_1,i_2)}) = \begin{cases} v(N\backslash\{i_1\}) - v(N\backslash\{i_1, i_2\}) & \text{for all } \sigma \in \pi_1(N) \cup \pi_2(N) \\ v(N) - v(N\backslash\{i_2\}) & \text{for all } \sigma \in \pi_3(N) \cup \pi_4(N) \end{cases}$$

and these values determine the hyperplanes defined by the restrictions imposed by the coalitions $\{i_2\}$ and $N\backslash\{i_2\}$, i.e. for all $x \in C(N, v_{(i_1,i_2)})$,

$$\begin{aligned} x_{i_2} &\geq v_{(i_1,i_2)}(\{i_2\}) = v(N\backslash\{i_1\}) - v(N\backslash\{i_1, i_2\}) \\ \sum_{i\neq i_2} x_i &\geq v_{(i_1,i_2)}(N\backslash\{i_2\}) \Leftrightarrow x_{i_2} \leq v(N) - v_{(i_1,i_2)}(N\backslash\{i_2\}) = v(N) - v(N\backslash\{i_2\}) \end{aligned}$$

Besides, by the convexity of the game we have that $v(N) - v(N\backslash\{i_2\}) \geq v(N\backslash\{i_1\}) - v(N\backslash\{i_1, i_2\})$.

Two cases must be considered:

i) $v(N) - v(N \backslash \{i_2\}) = v(N \backslash \{i_1\}) - v(N \backslash \{i_1, i_2\})$. Then the hyperplanes coincide and

$$Vol(C(N, v_{(i_1,i_2)}) = 0.$$

ii) $v(N) - v(N \backslash \{i_2\}) > v(N \backslash \{i_1\}) - v(N \backslash \{i_1, i_2\})$. Then the hyperplanes are parallel and

$$Vol(C(N, v_{(i_1,i_2)}) > 0.$$

Because of the simmetry of the core of this game we can deduce that for player i_2,

$$\mu_{i_2}(N, v_{(i_1,i_2)}) = \frac{v(N \backslash \{i_1\}) - v(N \backslash \{i_1, i_2\}) + v(N) - v(N \backslash \{i_2\})}{2}$$

and it coincides with the corresponding coordinate of the Shapley value.

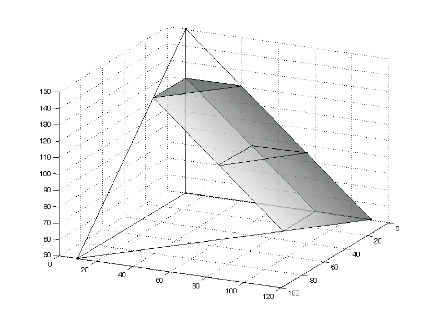

Figure 4: The core of the game $(N, v_{(i_1,i_2)})$

In the Figure 4 they are shown both the symmetries with regard the hyperplane $x_{i_2} = \mu_{i_2}(N, v_{(i_1,i_2)})$ and the simplex which is inside the hyperplane itself. Note that there are $2n - 2$ different extreme points, $n - 1$ will be in one of the two hyperplanes and the remaining $n - 1$ will be placed in the other one, besides they are going to be the symmetric with regard the hyperplane $x_{i_2} = \mu_{i_2}(N, v_{(i_1,i_2)})$.

For any $i_k \in N \backslash \{i_1, i_2\}$, we know that for all $x \in C(N, v_{(i_1,i_2)})$ (see proof of Lemma 4 a))

$$v(\{i_k\}) \leq x_k \leq v(N \backslash \{i_1, i_2\}) - \sum_{l \in N \backslash \{i_1, i_2, i_k\}} v(\{i_l\})$$

Then we can write,

$$\mu_{i_k}(N, v_{(i_1,i_2)}) = v(\{i_k\}) + t$$

where

$$0 \leq t \leq v(N \backslash \{i_1, i_2\}) - \sum_{l \in N \backslash \{i_1, i_2\}} v(\{i_l\})$$

Since $v(N \backslash \{i_1, i_2\}) - \sum_{l \in N \backslash \{i_1, i_2\}} v(\{i_l\})$ is a constant, we can say that for all $i_k \in N \backslash \{i_1, i_2\}$,

$$\mu_{i_k}(N, v_{(i_1,i_2)}) = v(\{i_k\}) + t$$

Now we have to compute the center of mass of the convex hull of the $n-1$ points on the hyperplane

$$x_{i_2} = \mu_{i_2}(N, v_{(i_1,i_2)}) = \frac{v(N\backslash\{i_1\}) - v(N\backslash\{i_1,i_2\}) + v(N) - v(N\backslash\{i_2\})}{2}$$

The coordinates of players $i_3, ..., i_n$ in the $n-1$ points $u_0, u_3, u_4, \ldots, u_n$ are the following:

$$u_0 : \qquad (u_0)_{i_k} = 0 \text{ for all } k \neq 1,2$$

$$u_l : (l = 3, \ldots, n) \quad \begin{cases} (u_l)_{i_k} = v(N\backslash\{i_1,i_2\}) - \sum\limits_{l \in N\backslash\{i_1,i_2\}} v(\{i_l\}) & l = k \\ (u_l)_{i_k} = 0 & l \neq k \end{cases}$$

Now, considering the vectors $u_l - u_0$ it is straightforward to check that these points are geometrically independent in $\mathbb{R}^{n-2}$, so we indeed have a $n-2$ simplex, and then for all $i_k \in N\backslash\{i_1,i_2\}$,

$$\mu_{i_k}(N, v_{(i_1,i_2)}) = v(\{i_k\}) + \frac{v(N\backslash\{i_1,i_2\}) - \sum\limits_{l \in N\backslash\{i_1,i_2\}} v(\{i_l\})}{n-1}$$

which coincides with the expression for the Shapley value since if we sum and rest the amount $\frac{v(\{i_k\})}{n-1}$ to the expression obtained for the Shapley value we obtain that for all $i_k \in N\backslash\{i_1,i_2\}$,

$$\begin{aligned} Sh_{i_k}(N, v_{(i_1,i_2)}) &= \frac{n-2}{n-1} v(\{i_k\}) + \frac{1}{n-1}\left(v(N\backslash\{i_1,i_2\}) - \sum_{l \in N\backslash\{i_1,i_2,i_k\}} v(\{i_l\}) \right) \\ &= v(\{i_k\}) + \frac{1}{n-1}\left(v(N\backslash\{i_1,i_2\}) - \sum_{l \in N\backslash\{i_1,i_2\}} v(\{i_l\}) \right) \\ &= \mu_{i_k}(N, v_{(i_1,i_2)}) \end{aligned}$$

Now, only remains to prove that $Sh_{i_1}(N, v_{(i_1,i_2)}) = \mu_{i_1}(N, v_{(i_1,i_2)})$, which it is a direct consequence of the efficiency property, satisfied by both Shapley value and core-center.

c) This expression for the volume is a consequence of the volume of the simplex calculated in b) and the distance between the two hyperplanes also obtained in the previous part of this Lemma. □

Lemma 5. *Let $(N, v) \in CG^n \cap G^n_{n-3}$ with $n > 3$ and let $(N, v_{(i_1,i_2)})$ the (i_1,i_2)-utopia game where i_1 and $i_2 \in N$ and $i_1 \neq i_2$.*

a) For all $S \subset N$, $(v_{(i_1,i_2)})_{\{i_2\}}(S) = (v_{\{i_2\}})_{\{i_1\}}(S)$ and $C(N, (v_{(i_1,i_2)})_{\{i_2\}}) = C(N, (v_{\{i_2\}})_{\{i_1\}})$.

b) $\mu(N,(v_{(i_1,i_2)})_{\{i_2\}}) = Sh(N,(v_{(i_1,i_2)})_{\{i_2\}})$

c) For all $i,j \in N$,

$$C(N,v_{\emptyset}) = \Big(\bigcup_{i\in N} C(N,v_{\{i\}})\Big) \cup \Big(\bigcup_{i<j} (C(N,v_{(i,j)}) \cup C(N,(v_{(i,j)})_{\{j\}}))\Big) \cup C(N,v)$$

d) All the intersections between two cores of the before division have null measure.

Proof. a) First, we recall that

$$v_{\{i_2\}}(S) = \begin{cases} v(N) - v(N\setminus\{i_2\}) + v(S\setminus\{i_2\}) & \text{if } i_2 \in S \\ \sum\limits_{l\in S} v(\{l\}) & \text{if } i_2 \notin S \end{cases}$$

$$v_{(i_1,i_2)}(S) = \begin{cases} v(N) - v(N\setminus\{i_1,i_2\}) + v(S\setminus\{i_1,i_2\}) & i_1,i_2 \in S \\ v(N\setminus\{i_2\}) - v(N\setminus\{i_1,i_2\}) + v(S\setminus\{i_1\}) & i_1 \in S, i_2 \notin S \\ v(N\setminus\{i_1\}) - v(N\setminus\{i_1,i_2\}) + \sum\limits_{l\in S\setminus\{i_2\}} v(\{l\}) & i_2 \in S, i_1 \notin S \\ \sum\limits_{l\in S} v(\{l\}) & i_1 \notin S, i_2 \notin S \end{cases}$$

Then

$$\begin{aligned} (v_{(i_1,i_2)})_{\{i_2\}}(S) &= \begin{cases} v_{\{i_1,i_2\}}(N) - v_{\{i_1,i_2\}}(N\setminus\{i_2\}) + v_{\{i_1,i_2\}}(S\setminus\{i_2\}) & i_2 \in S \\ \sum\limits_{l\in S} v_{\{i_1,i_2\}}(\{l\}) & i_2 \notin S \end{cases} \\ &= \begin{cases} v(N) - v(N\setminus\{i_1,i_2\}) + v(S\setminus\{i_1,i_2\}) & i_2 \in S, i_1 \in S \\ v(N) - v(N\setminus\{i_2\}) + \sum\limits_{l\in S\setminus\{i_2\}} v(\{l\}) & i_2 \in S, i_1 \notin S \\ v(N\setminus\{i_2\}) - v(N\setminus\{i_1,i_2\}) + \sum\limits_{l\in S\setminus\{i_2,i_1\}} v(\{l\}) & i_2 \notin S, i_1 \in S \\ \sum\limits_{l\in S} v(\{l\}) & i_1 \notin S, i_2 \notin S \end{cases} \end{aligned}$$

Besides,

$$\begin{aligned} (v_{\{i_2\}})_{\{i_1\}})(S) &= \begin{cases} v_{\{i_2\}}(N) - v_{\{i_2\}}(N\setminus\{i_1\}) + v_{\{i_2\}}(S\setminus\{i_1\}) & i_1 \in S \\ \sum\limits_{l\in S} v_{\{i_2\}}(\{l\}) & i_1 \notin S \end{cases} \\ &= \begin{cases} v(N) - v(N\setminus\{i_1,i_2\}) + v(S\setminus\{i_2,i_1\}) & i_1 \in S, i_2 \in S \\ v(N\setminus\{i_2) - v(N\setminus\{i_1,i_2\}) + \sum\limits_{l\in S\setminus\{i_1\}} v(\{l\}) & i_1 \in S, i_2 \notin S \\ v(N) - v(N\setminus\{i_2\}) + \sum\limits_{l\in S\setminus\{i_2\}} v(\{l\}) & i_1 \notin S, i_2 \in S \\ \sum\limits_{l\in S} v(\{l\}) & i_1 \notin S, i_2 \notin S \end{cases} \end{aligned}$$

b) It follows from a) and Lemma 3.

The proofs of c), and d) follow similar lines to those of Lemma 2 and we omit the details. □

Remark. Notice that the ratios r_{i_1,i_2} corresponds with the following formula:

$$\frac{\frac{1}{(n-2)!}n^{1/2}\left(v(N\backslash\{i_1,i_2\})-\sum_{l\in N\backslash\{i_1,i_2\}}v(\{i_l\})\right)^{n-2}(v(N)-v(N\backslash\{i_1\})+v(N\backslash\{i_1,i_2\})-v(N\backslash\{i_2\}))}{\frac{1}{(n-1)!}n^{1/2}\left(v(N)-\sum_{j\in N}v(\{j\})\right)^{n-1}}=$$

$$(n-1)\left(\frac{v(N\backslash\{i_1,i_2\})-\sum_{l\in N\backslash\{i_1,i_2\}}v(\{i_l\})}{v(N)-\sum_{j\in N}v(\{j\})}\right)^{n-1}\frac{v(N)-v(N\backslash\{i_1\})+v(N\backslash\{i_1,i_2\})-v(N\backslash\{i_2\})}{v(N)-\sum_{j\in N}v(\{j\})}$$

This number can be interpreted as the probability that the coalition (i_1, i_2) is really happy with the part that they receive in the division of $v(N)$, but inside the coalition itself, the happiness is bigger for player i_1. Happiness for (i_1, i_2) implies unhappines for the players of $N\backslash\{i_1, i_2\}$.

And, the ratios $r_{(i_1,i_2)_{i_2}}$ coincide with

$$r_{(i_1,i_2)_{i_2}}=\left(\frac{v(N\backslash\{i,j\})-\sum_{k\in N\backslash\{i,j\}}v(\{k\})}{v(N)-\sum_{j\in N}v(\{j\}}\right)^{n-1}$$

Observe that $r_{i_1,i_2} = r_{i_2,i_1}$ and $r_{(i_1,i_2)_{i_2}} = r_{(i_2,i_1)_{i_1}}$. With that in mind we can write the following lemma.

Lemma 6. *Let $(N, v) \in CG^n \cap G^n_{n-3}$ with $n > 3$ and let $(N, v_{(i_1,i_2)})$ the (i_1, i_2)-utopia game where $i_1$and $i_2 \in N$ and $i_1 \neq i_2$.The following equality holds:*

$$r_{i_1,i_2}\mu(N,v_{(i_1,i_2)})+r_{(i_1,i_2)_{i_2}}\mu(N,(v_{(i_1,i_2)})_{\{i_2\}})=$$
$$=r_{i_2,i_1}\mu(N,v_{(i_2,i_1)})+r_{(i_2,i_1)_{i_1}}\mu(N,(v_{(i_2,i_1)})_{i_1}).$$

Proof. It is a consequence of the following fact,

$$C(N,v_{(i_1,i_2)})\cup C(N,\left(v_{(i_1,i_2)}\right)_{\{i_2\}})=C(N,v_{(i_2,i_1)})\cup C(N,\left(v_{(i_2,i_1)}\right)_{i_1}).$$

□

The last result gives insights of symmetry between any two player coalition. Now, we are in conditions to state our main theorem that gives a direct relation between the core-center and the Shapley value of the fair game.

Theorem 2. *Let $(N, v) \in CG^n \cap G^n_{n-3}$ with $n > 3$. Then,*

$$\mu(N,v)=Sh(N,w)$$

where for all $S \subset N$,

$$w(S) = \left(\frac{1}{p}\left(p_0 v_\emptyset - \sum_{i\in N} p_i w_{\{i\}} - \sum_{i,j\in N}\frac{1}{2}\left(p_{i,j}v_{(i,j)} + p_{(i,j)_j}\left(v_{(i,j)}\right)_{\{j\}}\right)\right)\right)(S)$$

$$\begin{aligned}
Vol(C(N, v_\emptyset)) &= p_0 \\
Vol(C(N, v_{\{i\}})) &= p_i \textit{ for all } i \in N \\
Vol(C(N, v_{(i,j)})) &= p_{i,j} \textit{ for all } i, j \in N \\
Vol(C(N, \left(v_{(i,j)}\right)_{\{j\}})) &= p_{(i,j)_j} \textit{ for all } i, j \in N \\
p &= p_0 - \sum_{i\in N} p_i - \sum_{i<j}(p_{i,j} + p_{(i,j)_j}) = Vol(C(N, v))
\end{aligned}$$

Proof. Taking into account that,

$$C(N, v_\emptyset) = \Big(\bigcup_{i\in N} C(N, v_{\{i\}})\Big) \cup \Big(\bigcup_{i<j}\left(C(N, v_{(i,j)}) \cup C(N, (v_{(i,j)})_{\{j\}})\right)\Big) \cup C(N, v)$$

and we find out that $\mu(N, v_\emptyset) =$

$$\begin{aligned}
&\sum_{i\in N}\tfrac{p_i}{p_0}\mu(N, v_{\{i\}}) + \sum_{i<j}\left(\tfrac{p_{i,j}}{p_0}\mu(N, v_{\{i,j\}}) + \tfrac{p_{(i,j)_j}}{p_0}\mu(N, \left(v_{\{i,j\}}\right)_{\{j\}})\right) + \mu(N, v)\tfrac{p}{p_0} = \\
&= \sum_{i\in N}\tfrac{p_i}{p_0}\mu(N, v_{\{i\}}) + \sum_{i,j\in N}\tfrac{1}{2}\left(\tfrac{p_{i,j}}{p_0}\mu(N, v_{\{i,j\}}) + \tfrac{p_{(i,j)_j}}{p_0}\mu(N, \left(v_{\{i,j\}}\right)_{\{j\}})\right) + \mu(N, v)\tfrac{p}{p_0}
\end{aligned}$$

where the last equality hold by Lemma 6. Applying lemmas 1 and 4, and the additivity of the Shapley value we deduce,

$$\begin{aligned}
&\mu(N, v) = \frac{p_0}{p}\mu(N, v_\emptyset) - \sum_{i\in N}\frac{p_i}{p}\mu(N, v_{\{i\}}) - \\
&\sum_{i,j\in N}\frac{1}{2}\Big(\frac{p_{i,j}}{p}\mu(N, v_{(i,j)}) + \frac{p_{(i,j)_j}}{p}\mu(N, (v_{(i,j)})_{\{j\}})\Big) = \frac{p_0}{p}Sh(N, v_\emptyset) - \sum_{i\in N}\frac{p_i}{p}Sh(N, w_{\{i\}}) \\
&\qquad - \sum_{i,j\in N}\frac{1}{2}\left(\frac{p_{i,j}}{p}Sh(N, v_{(i,j)}) + \frac{p_{(i,j)_j}}{p}Sh(N, (v_{(i,j)})_{\{j\}})\right) = Sh(N, w).
\end{aligned}$$

□

Corollary 3. *Let* $(N, v) \in CG^4$. *Then,*

$$\mu(N, v) = Sh(N, w)$$

where $w(S) = \left(\frac{1}{p}\left(p_0 v_\emptyset - \sum_{i\in N} p_i w_{\{i\}} - \sum_{i,j\in N}\frac{1}{2}\left(p_{i,j}v_{(i,j)} + p_{(i,j)_j}\left(v_{(i,j)}\right)_{\{j\}}\right)\right)\right)(S)$.

Proof. It is immediate from Theorem 2. □

Remark. The fair game takes into account all the possibilities in the game. Once the players know their individual values and the value of the grand coalition, they observe that the core of the game $(N, v_{\emptyset})$ (the imputation set) contains all the possibilities to share $v(N)$ among them. Just in this point it would be fair that they obtain the center of the core. But, with all the characteristic function on the table, would it be fair? Rational players would think on the possibility of forming subcoalitions, and it would be fair to play a game that picks all that information. So, players notice that each $S \subset N$ such that $v(S) > \sum_{i\in S} v(\{i\})$ is imposing a constraint in the imputation set. What the fair game does is to take into account all that information by means of a probability measure. For the coalitions of two players we also have a probabilistic interpretation: when they form their own coalition there are two possible ways of cooperate, so we assign probability $\frac{1}{2}$ to the coalition $\{i, j\}$ and $\frac{1}{2}$ to the coalition $\{j, i\}$.

3 The airport game and the core-center

A cooperative TU cost game is a pair (N, c) where $N = \{1, \ldots, n\}$ is a finite set of agents and $c : 2^N \to \mathbb{R}$ is a map assigning to each coalition $S \in 2^N$, a real number $c(S)$ that represents the minimum costs that the agents of S can guarantee by themselves independently of the agents of $N\backslash S$, where $c(\emptyset) = 0$. The corresponding cost savings game (N, v) is defined by $v(S) = \sum\limits_{i\in S} c(\{i\}) - c(S)$ for all $S \in 2^N$.

Consider the airport game (Littlechild and Owen (1973)). Suppose, that we have three types of planes. $N = \{1, 2, 3\}$, and $c_1 < c_2 < c_3$. Then the corresponding cost game (N, c),

$$\begin{array}{ll} c(1,2,3) = c_3 & \\ c(1,2) = c_2 & c(1) = c_1 \\ c(1,3) = c_3 & c(2) = c_2 \\ c(2,3) = c_3 & c(3) = c_3 \end{array}$$

It is well known that the Shapley value of the airport game corresponds with the following formula:

$$Sh(N, c) = \left(\frac{c_1}{3}, \frac{c_1}{3} + \frac{c_2 - c_1}{2}, \frac{c_1}{3} + \frac{c_2 - c_1}{2} + c_3 - (c_2 + c_1)\right).$$

The cost to pay by each firm only depends on the planes that are less or equal than theirs and so, it is independent on the big planes. Now we provide some examples to study the centroid.

	$Sh(N,c)$	$\mu(N,c)$
$c_1 = 2, c_2 = 4, c_3 = 4$	$(0.6667, 1.6667, 1.6667)$	$(0.8889, 1.5556, 1.5556)$
$c_1 = 2, c_2 = 4, c_3 = 5$	$(0.6667, 1.6667, 2.6667)$	$(0.8889, 1.5556, 2.5556)$
$c_1 = 2, c_2 = 4, c_3 = 10$	$(0.6667, 1.6667, 7.6667)$	$(0.8889, 1.5556, 7.5556)$
$c_1 = 2, c_2 = 40, c_3 = 100$	$(0.6667, 19.6667, 79.667)$	$(0.9915, 19.504, 79.504)$
$c_1 = 2, c_2 = 40, c_3 = 1000$	$(0.6667, 19.6667, 979.67)$	$(0.9915, 19.504, 979.504)$

As we can observe the centroid for the planes different from the big one is not affected by changes on the cost of the big one. The reason is that, although the core changes, we always have the same structure for the other type of planes, i.e. we are moving the core vertically over the axis of the big plane. On the contrary, what happens if we change the cost of the other planes? The answer is that for the core-center all the planes are going to have their allocation changed. The motivation for this could be that if there are little planes and very big ones, the little planes, even when they do not need those improvements in the airport, they are also going to take some benefit from all the resources of the big companies, and so, it seems fair to pay something else.

4 A comparison by means of properties

- The Shapley value is an allocation rule that satisfies efficiency, symmetry on the characteristic function, dummy player and additivity on the characteristic function[4]. All these properties are satisfied by the core-center, except additivity on the characteristic function.

- The Shapley value satisfies the strong monotonicity property that says: given (N,v) and $(N,w) \in G^n$, if for all $S \subset N\backslash\{i\}$, $v(S\cup i)-v(S) \geq w(S\cup i)-w(S)$, then $\varphi_i(N,v) \geq \varphi_i(N,w)$. As a direct consequence if for all $S \subset N\backslash\{i\}$, $v(S\cup i) - v(S) = w(S \cup i) - w(S)$, then $\varphi_i(N,v) = \varphi_i(N,w)$. Besides, Young (1985) characterized the Shapley value using efficiency, symmetry and strong monotonicity.

  The centroid does not verify this property. Consider the following games with $N = \{1,2,3\}$ and $i = \{3\}$.

$v(1,2,3) = 10$		$w(1,2,3) = 15$	
$v(1,2) = 2$	$v(1) = 0$	$w(1,2) = 7$	$w(1) = 0$
$v(1,3) = 3$	$v(2) = 0$	$w(1,3) = 3$	$w(2) = 0$
$v(2,3) = 3$	$v(3) = 0$	$w(2,3) = 3$	$w(3) = 0$

[4] φ satisfies additivity on the characteristic function if for any two games (N,v) and (N,w), $\varphi(N,v+w) = \varphi(N,v) + \varphi(N,w)$, where $(v+w)(S) = v(S) + w(S)$ for all $S \subset N$.

r(1,2,3)=1000	
r(1,2)=992	r(1)=0
r(1,3)=3	r(2)=0
r(2,3)=3	r(3)=0

For all $S \subset N \backslash \{3\}$,

$$v(S \cup \{3\}) - v(S) = w(S \cup \{3\}) - w(S) = r(S \cup \{3\}) - r(S)$$

and,

$$\begin{aligned} Sh_3(N,v) &= Sh_3(N,w) = Sh_3(N,r) = 3.6667. \\ \mu_3(N,v) &= 3.5983,\ \mu_3(N,w) = 3.8017,\ \mu_3(N,r) = 4. \end{aligned}$$

- There are other characterizations of the Shapley value, for instance, using the property of balanced contributions. Obviously, the core-center does not satisfy that property.

5 Some conclusions

There is no doubt about the fact that it is interesting to know if Theorem 2 can be extended to the entire class of convex games. Besides, the utopia games give a lot of information concerning the game, and up to here, they have been defined for convex games. So at this point there are some open questions; can we define the utopia games for any balanced game? and, the core-center of any balanced game, can it be expressed by means of these games? Even when we have many insights on how these games behave, much more research on this topic is needed.

As the final conclusion just insist in the fact that the core-center provides a new focus to search for connections between set-valued solutions and allocation rules. Of course, many things still remain to be explored in this new field.

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