FINITELY REPEATED GAMES: A GENERALIZED NASH FOLK THEOREM

Julio González Díaz

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Abstract: This paper characterizes the feasible, individually rational payoffs of finitely repeated games that can be approximated arbitrarily closely by Nash equilibria.

1 Introduction

Over the past twenty years, necessary and sufficient conditions have been published for numerous “folk theorems”, assuring that the individually rational feasible payoffs of finitely or infinitely repeated games with complete information can be achieved by Nash or subgame perfect equilibria. Results of this kind concerning Nash equilibrium have been obtained by Fudenberg and Maskin (1986) for infinite games and by Benoit and Krishna (1987) for finite games, and results concerning subgame perfect equilibrium in infinite games by Abreu (1988), Abreu et al. (1994) and Wen (1994), who used an “effective minimax” payoff concept. Smith (1995) obtained a necessary and sufficient condition for the arbitrarily close approximation of strictly rational feasible payoffs by subgame perfect equilibria in finite games: that the game have “recursively distinct Nash payoffs”, a premise...
that relaxes the assumption in Benoit and Krishna (1985) that each player have multiple Nash payoffs in the stage game.

Smith claimed that this condition was also necessary for approximation of the individually rational feasible payoffs of finitely repeated games by Nash equilibria. In this paper I show that this is not so by establishing a similar but distinct sufficient condition that is weaker than both Smith’s condition and the assumptions made by Benoit and Krishna (1987), and which really is necessary. Essentially, the difference between the subgame perfect and Nash cases hinges on the weakness of the Nash solution concept: in the Nash case it is not necessary for threats of punitive action against players who deviate from the equilibrium not to involve substantial loss to the punishing players themselves, i.e. threats need not be credible. All that is required is for the action sequence $\rho_i$ played by each player $i$ in the equilibrium approximating the desired payoff to finish with a series $S_i$ of rounds in which player $i$ cannot unilaterally improve his stage payoff by deviation from $\rho_i$, and for this terminal phase to start with a series $S_i^0$ of rounds in which the other players, regardless of the cost to themselves, can punish him effectively for any prior deviation by imposing a loss that wipes out any gains he may have made in deviating.

All the results mentioned above, except for that in Wen (1994), concern the approximability of the entire set of individually rational feasible payoffs. The theorem proved in this paper is more general in that, for any game, it characterizes the set of feasible payoffs that are approximable. Notation and concepts are introduced in Section 2 below, and in Section 3 the theorem is stated and proved.


2 Basic Notation, Definitions and an Example

2.1 The Stage Game

A game in strategic form is a triplet \( G = \langle N, A, \varphi \rangle \), where:

- \( N = \{1, \ldots, n\} \) is the set of players,
- \( A = \prod_{i \in N} A_i \) and \( A_i \) denotes the set of player \( i \)'s strategies,
- \( \varphi = (\varphi_1, \ldots, \varphi_n) \) and \( \varphi_i : A \to \mathbb{R} \) is the utility function of player \( i \).

Let \( a_{-i} \) denote a strategy profile for players in \( N \setminus \{i\} \), and with \( A_{-i} \) the set of such profiles; it is assumed that \( \mu(a_{-i}) = \max_{a_i \in A_i} \{\varphi_i(a_{-i}, a_i)\} \) exists for each \( i \in N \) and \( a_{-i} \in A_{-i} \), and that \( v_i = \min_{a_{-i} \in A_{-i}} \{\mu(a_{-i})\} \) exists for each \( i \in N \). The vector \( v = \{v_1, \ldots, v_n\} \) is the minimax payoff vector. The set \( F \) of all feasible and individually rational payoffs is the convex hull of the set \( \{\varphi(a) : a \in A, \varphi(a) \geq v\} \).

To avoid confusion with the strategies of the repeated game, in what follows the strategies \( a_i \in A_i \) and strategy profiles \( a \in A \) of the stage game will be called actions and action profiles, respectively.

2.2 The Repeated Game

Let \( G(\delta, T) \) denote the game consisting in the \( T \)-fold repetition of \( G \) with payoff discount parameter \( \delta \in (0, 1] \). In this repeated game each player can decide his action in the current round in the light of all actions taken by all players in all previous rounds. Let \( \alpha_i = \{\alpha_i^1, \ldots, \alpha_i^T\} \) denote the action sequence of player \( i \), and \( \varphi_i^t(\alpha) \) the stage payoff of player \( i \) at stage \( t \) when all agents play in accordance with \( \alpha \); then player \( i \)'s payoff \( \psi_i(\alpha) \) in \( G(\delta, T) \) when \( \alpha \) is played is defined to be his average discounted stage payoff: \( \psi_i(\alpha) = (1/\delta^T) \sum_{t=1}^T \delta^{t-1} \varphi_i^t(\alpha) \).

\(^2\)Or, \( \psi_i(\alpha) = (1/T) \sum_{t=1}^T \varphi_i^t(\alpha) \) if there are no discounts (\( \delta = 1 \)).
2.3 Minimax-Bettering Ladders

For any \( m \)-player subset \( M \) of \( N \), let \( A_M = \prod_{i \in M} A_i \) and let \( G(a_M) \) be the game induced for the \( n - m \) players in \( N \setminus M \) when the actions of the members of \( M \) constitute the fixed profile \( a_M \in A_M \). By abuse of language, if \( i \in N \setminus M \), \( a_M \in A_M \) and \( \sigma \in A_{N \setminus M} \) we write \( \varphi_i(\sigma) \) for \( i \)'s payoff at \( \sigma \) in \( G(a_M) \). A minimax-bettering ladder belonging to a game \( G \) is defined to be a triplet \( \{ N, A, \Sigma \} \) where

- \( N \) is a strictly increasing sequence \( \{ \emptyset = N_0 \subset N_1 \subset \cdots \subset N_h \} \) of \( h+1 \) subsets of \( N \) \( (h \geq 1) \),
- \( A \) is a sequence of action profiles \( \{ a_{N_1} \in A_{N_1}, \ldots, a_{N_h-1} \in A_{N_h-1} \} \) and
- \( \Sigma \) is a sequence \( \{ \sigma^1, \ldots, \sigma^h \} \) of Nash equilibria of \( G = G(a_{N_0}), G(a_{N_1}), \ldots, G(a_{N_{h-1}}) \), respectively, such that at \( \sigma^g \) the players of \( G(a_{N_{g-1}}) \) receiving payoffs strictly greater than their minimax payoff are exactly the members of \( N_g \setminus N_{g-1} \):

\[
\varphi_i(\sigma^g) > v_i \quad \forall i \in N_g \setminus N_{g-1} \quad \varphi_i(\sigma^g) \leq v_i \quad \forall i \in N \setminus N_g.
\]

In algorithmic terms, if the first \( g-1 \) rungs of the ladder have been constructed, then for the \( g \)-th rung to exist the current game \( G(a_{N_{g-1}}) \) must have an equilibrium \( \sigma^g \) such that there exist members \( i \) of \( N \setminus N_{g-1} \) for whom \( \varphi_i(\sigma^g) > v_i \); \( N_g \setminus N_{g-1} \) is defined as precisely this subset of players of \( G(a_{N_{g-1}}) \); and the game played in the next step is defined by some action profile \( a_{N_g} \). The set \( N_h \) will be called the top rung of the ladder, a ladder with top rung \( N_h \) is said to be maximal if there is no ladder with top rung \( N_{h'} \) such that \( N_h \) is a proper subset of \( N_{h'} \), and a game \( G \) is said to be decomposable as a complete minimax-bettering ladder if it has a minimax-bettering ladder with \( N \) as its top rung. It is shown below that being decomposable as a complete minimax-bettering ladder is a necessary and sufficient condition for it to be possible to approximate all payoff vectors in \( F \) by Nash equilibria of \( G(\delta, T) \) for some \( \delta \) and \( T \). Clearly, being decomposable as complete a minimax-bettering ladder is a weaker property than the requirement in Smith (1995), that at each step \( g-1 \) of a similar kind of ladder there be action profiles \( a_{N_{g-1}}, b_{N_{g-1}} \) such that the games \( G(a_{N_{g-1}}) \) and \( G(b_{N_{g-1}}) \) have Nash equilibria \( \sigma^g_a \) and \( \sigma^g_b \) with \( \varphi_i(\sigma^g_a) \neq \varphi_i(\sigma^g_b) \) for a non-empty
set of players (the members of $N_g \setminus N_{g-1}$).

2.4 An example

In the equilibrium strategy profile constructed in Theorem 1 below, the action profile sequence in the terminal phase $S_i$ referred to in the Introduction, consists of repetitions of $(a_{N_i-1}, \sigma^g), (a_{N_i-2}, \sigma^{g-1}), \ldots, (a_{N_2}, \sigma^2)$ and $\sigma$, where $g_i$ is the unique integer such that $i \in N_{g_i} \setminus N_{g_i-1}$; and the $\sigma^j$ are Nash equilibria of the corresponding games $G(a_{N_j-1})$. Since player $i$ is a player in all these games, he can indeed gain nothing by unilateral deviation during this phase. In the potentially punishing series of rounds $S_i^0$, the action profile sequence consists of repetitions of $(a_{N_i-1}, \sigma^g)$, in which $i$ obtains more than his minimax payoff, with the accompanying threat of punishing prior unilateral deviation by $i$ (or other members of $N_{g_i} \setminus N_{g_i-1}$) by minimaxing him instead.

As an illustration of the above ideas, consider the three player game $G$ shown in Figure 1. Its minimax payoff vector is $(0,0,0)$, and its only Nash equilibrium is the action profile $\sigma^1 = (T,l,L)$, with payoff vector $(0,0,3)$. Thus $N_1 = \{3\}$; player 3 can be punished by 1 and 2 by playing to one of his minimax profiles instead of playing $(T,l,\cdot)$. If player 3 now plays $R$ ($a_{N_1} = R$), the resulting game $G(a_{N_1}) = G(R)$ has an equilibrium $\sigma^2 = (T,l)$ with payoff vector $(0,3)$. Thus $N_2 = \{2,3\}$ and player 2 can be punished by 1 and 3 by playing to one of his
minimax profiles instead of playing \((T, \cdot, R)\). Finally if players 2 and 3 now play \(r\) and \(R\) \((a_{N_2} = (r, R))\), the resulting game \(G(a_{N_2}) = G(r, R)\) has the trivial equilibrium \(\sigma^3 = (T)\) with payoff 1 for player 1, who can therefore be punished by 2 and 3 if they play to one of his minimax profiles instead of playing \((\cdot, r, R)\).

2.5 Further Preliminaries

**Proposition 1.** Given a game \(G\), all maximal ladders have the same top rung.

**Proof.** Suppose there are maximal ladders \(L = \{N, A, \Sigma\}, L' = \{N', A', \Sigma'\}\) with \(N = \{N_0 \subset N_1 \subset \cdots \subset N_h\}\) and \(N' = \{N'_0 \subset N'_1 \subset \cdots \subset N'_k\}\) such that, \(N_h \neq N'_k\) and suppose without loss of generality that \(N'_k \setminus N_h\) is nonempty. For each \(j \in N'_k\) let \(g_j\) be the unique integer such that \(j \in N'_{g_j} \setminus N'_{g_j-1}\), and consider \(i \in N'_k \setminus N_h\) such that \(g_i = \min_{j \in N'_k \setminus N_h} g_j\). Then \(N'_k \setminus N_h\), and we can define an action profile \(a_{N_h}\) by:

\[
(a_{N_h})_j = \begin{cases} 
(a'_{N'_{g_j-1}})_j & j \in N'_{g_j-1} \\
(\sigma'^{g_j})_j & j \in N_h \setminus N'_{g_j-1}
\end{cases}
\]

where \(\sigma'^{g_j} \in \Sigma'\) is an equilibrium of the game \(G(a'_{N'_{g_j-1}})\) defined by the action profile \(a'_{N'_{g_j-1}} \in A'\).

Now the restriction of \(\sigma'^{g_j}\) to \(N_k \setminus N_h\), which we denote by \(\sigma^{h+1}\), is an equilibrium of \(G(a_{N_h})\) (since \(\sigma'^{g_j}\) is an equilibrium of \(G(a'_{N'_{g_j-1}})\), and \(N_k \setminus N_h \subset N(k' \setminus N'_{g_j-1})\), and the subset of members \(j\) of \(N_k \setminus N_h\) for whom \(\varphi_j(\sigma^{h+1}) > v_j\) is precisely the set \(N'_k \setminus N_h\). Since this set, which we denote by \(N_{h+1}\), is nonempty (it contains \(i\)), the triplet \(L'' = \{N'', A'', \Sigma''\}\) defined by

- \(N'' = \{N_0 \subset N_1 \subset \cdots \subset N_h \subset N_{h+1}\}\),
- \(A'' = \{a_{N_1}, \ldots, a_{N_{h-1}}, a_{N_h}\}\),
- \(\Sigma'' = \{\sigma^1, \ldots, \sigma^h, \sigma^{h+1}\}\)
is a ladder, the top rung of which properly contains that of $L$. Thus $L$ is not maximal, which proves the Proposition.

Consider a game $G$ with maximal ladders with top rung $N_{\text{max}}$. Given $\hat{a} \in A_{N_{\text{max}}}$, let $\Lambda(\hat{a}) = \{\lambda = (\hat{a}, \sigma) \in A \mid \sigma \text{ Nash Equilibrium of } G(\hat{a})\}$, and let $\Lambda = \bigcup_{\hat{a} \in A_{N_{\text{max}}}} \Lambda(\hat{a})$. The set $S$ of all $N_{\text{max}}$-feasible payoffs of $G$ is defined to be the intersection of $F$ with the convex hull of the set $\{\varphi(\lambda) \mid \lambda \in \Lambda\}$. If $u \in S$, then by the definition of $N_{\text{max}}$, $u_i = v_i$ for all $i \in N \setminus N_{\text{max}}$. Besides, when $N_{\text{max}} = N$ we have $\Lambda = A$ and $S = F$.

The promised result concerning the approximability of all payoffs in $F$ by Nash equilibrium payoffs is obtained below as an immediate corollary of a more general theorem concerning the approximability of all payoffs in $S$. In this more general case, the collaboration of the members of $N_{\text{max}}$ is secured by a strategy analogous to that sketched in the Example of Section 2.4, while the collaboration of the members of $N \setminus N_{\text{max}}$ is also ensured because none of them is able to obtain any advantage by unilateral deviation from any action profile in $\Lambda$.

### 3 The Theorem

In the theorem that follow, the set of action profiles $A$ may consist either of pure or mixed action profiles; in the latter case, perfect monitoring is assumed, i.e. all players are cognizant not only of the pure actions actually put into effect at each stage, but also of the mixed actions of which they are realizations. Also, public randomization is assumed: at each stage of the repeated game, players can let their actions depend on the realization of an exogenous continuous random variable.\(^3\) Public randomization is not crucial, but facilitates the proofs.

\(^3\)The assumption of public randomization is almost without loss of generality, Fudenberg and Maskin (1991) having shown that, any correlated mixed action can be approximated by alternating pure actions with the appropriate frequencies.
Theorem 1. Given a game \( G \) with a maximal minimax-bettering ladder with top rung \( N_{\text{max}} \), and given a feasible payoff \( u \in F \), then a necessary and sufficient condition for there to exist for each \( \varepsilon > 0 \), an integer \( T_0 < \infty \) and a positive real number \( \delta_0 < 1 \) such that for all \( T \geq T_0 \) and \( \delta \in [\delta_0, 1] \), \( G(\delta, T) \) has a Nash equilibrium payoff \( w \) such that \( \|w - u\| < \varepsilon \) is that \( u \) be \( N_{\text{max}} \)-feasible (i.e. \( u \in S \)).

Proof. \( \iff \) Let \( a \in \Lambda \) be an action profile of \( G \) such that \( \varphi(a) = u \), and let \( L = \{N, A, \Sigma\} \) be a maximal minimax-bettering ladder of \( G \). As noted above, the members of \( N \setminus N_{\text{max}} \) have no incentive for unilateral deviation from \( a \) (by the definition of \( \Lambda \)). In order to calculate how many repetitions of \( G(a_{N_{h-1}}) \) will be necessary for the members of \( N \setminus \{i\} \) to be able to punish a player \( i \in N_{\text{max}} \) for prior deviation, let us define \( \bar{\mu}(a) = \mu(a_{-i}) - \varphi_i(a) \) (the maximum "illicit" profit that player \( i \) can obtain by unilateral deviation from \( a \)), \( \bar{\mu}_i = \max \{\bar{\mu}(a_{-i}), \bar{\mu}((a_{N_{h-1}}, \sigma^h_{-i}), \ldots, \bar{\mu}(\sigma^1_{-i})\} \) and \( r_i := \min \{r \in \mathbb{N} | r(\varphi_i(\sigma^h) - v_i) > \bar{\mu}_i\} \). Clearly, there exists \( \delta_i \in (0, 1) \) such that \( \bar{\mu}_i - \sum_{k=1}^{r_i} \delta_i^k(\varphi_i(\sigma^h) - v_i) < 0 \), so if the discount parameter \( \delta \) is at least \( \delta_i \), \( r_i \) repetitions of \( G(a_{N_{h-1}}) \) suffice to allow player \( i \) to be punished. Further, if \( \delta \geq \delta_0 = \max_{i \in \mathbb{N}} \delta_i \) and, for all \( g \in \{1, \ldots, h\} \), \( q_g := \max_{i \in \mathbb{N}} \{q_g \} \), then \( q_g \) repetitions of \( G(a_{N_{h-1}}) \) suffice to allow any player in \( N_g \setminus N_{g-1} \) to be punished. Given \( \varepsilon > 0 \), we therefore define the action profile sequence

\[
\rho := \{a, \ldots, a, \lambda^h, \ldots, \lambda^h, \lambda^{h-1}, \ldots, \lambda^{h-1}, \ldots, \lambda^1, \ldots, \lambda^1\}
\]

where \( \lambda^k = (a_{N_{k-1}}, \sigma^k) \) (with \( a_{N_{k-1}} \in A \) and \( \sigma^k \in \Sigma \)), and \( q_0 \) is the smallest integer such that:

\[
\left\| q_0 \varphi(a) + q_h \varphi(\lambda^h) + \cdots + q_1 \varphi(\lambda^1) \right\| \frac{q_0 + q_h + \cdots + q_1}{q_0 + q_h + \cdots + q_1} - \varphi(a) < \varepsilon
\]

and \( T_0 = q_0 + q_1 + \cdots + q_h \); and for \( T \geq T_0 \) and \( \delta \in [\delta_0, 1] \) we prescribe for
G(δ, T) the strategy profile in which all players play ρ unless and until there is a unilateral deviation, in which case the deviating player is minimaxed by all the others. It is straightforward to check that this profile is a Nash equilibrium of G(δ, T), and since its payoff vector w differs from u by less than T₀ε/T if δ = 1 (by inequality 1), the same is certainly true if δ < 1, in which case the earlier stage payoffs (ϕ(a)) receive greater weight than those of the endgame.

Corollary 1. If the game G is decomposable as a complete minimax-bettering ladder, then for all u ∈ F and for all ε > 0, there exist T₀ < ∞ and δ₀ < 1 such that for all T ≥ T₀ and δ ∈ [δ₀, 1] there exists a Nash Equilibrium payoff w of G(δ, T), with ∥w − u∥ < ε.

Proof. Since N = N_max ⇒ F = S, this result is an immediate consequence of Theorem 1.

Corollary 2. If the game G is not decomposable as a complete minimax-bettering ladder, then for all T < ∞ and δ ∈ (0, 1] the players i of G(δ, T) in N \ N_max receive their minimax stage payoffs v_i at all Nash equilibria of G(δ, T).

Proof. For all u ∈ S, u_i = v_i for all i ∈ N \ N_max, so this result follows by an
argument paralleling the proof of necessity in Theorem 1.

Remarks

• **Remark 1.** Theorem 1 requires no use of the concept of effective minimax payoff\(^4\), because non-equivalent utilities\(^4\) are irrelevant to the approximation of \(N_{max}\) feasible payoffs by Nash equilibria, in which there is no need for threats to be credible.

• **Remark 2.** Theorem 1 assumes neither that the set of action profiles is finite, nor that it is compact and that the \(\varphi_i\) are continuous. It requires only that the minimax payoff \(v\) exist.\(^5\)

• **Remark 3.** Corollary 1 holds for a wider class of games than the result obtained by Benoit and Krishna (1987).

• **Remark 4.** Theorem 1 raises the question of whether a similarly general result on the approximability of payoffs by equilibria also holds for subgame perfect equilibria. The main problem is to determine the subgame perfect equilibrium payoffs of players with “recursively distinct Nash payoffs” (Smith (1995)) when the game is not completely decomposable.

• **Remark 5.** The results of this paper can be easily extended to the case in which each player has a different discount \(\delta\).

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\(^4\)See Abreu et al. (1994) and Wen (1994) for details of these concepts.

\(^5\)And if it does not exist, but the \(\text{infsup}\) payoff vector does, then a similar theorem holds for the latter. I thank Vijay Krishna for pointing this out to me.
References


