METMA V International Workshop on Spatio-Temporal Modelling
Santiago de Compostela
30 Jun - 2 Jul 2010

Measuring dependence of a space-time ARMAX storage model

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Abstract. *We introduce a space-time ARMAX storage model, analogous to the solar thermal energy model considered in Haslett [3] to describe the temperature level in a tank used for the storage of solar energy. For this model we analyze stationarity, max-stability and compute some spatial dependence coefficients.*

Keywords. *Spatial extreme events; Spatial Dependence Coefficients; Space-time ARMAX model.*

1 Introduction

In multivariate and spatial problems attention has often focused on obtaining dependence measures that capture the main characteristics of the dependence structure. For a max-stable stationary random field $\mathbf{X} = \{X_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{Z}^2}$, with marginal distributions F, the extremal coefficient, $\varepsilon(\mathbf{i},\mathbf{j})$, defined as

$$P\left(\max\left(X_{\mathbf{i}},X_{\mathbf{j}}\right)\leq x\right)=F^{\varepsilon(\mathbf{i},\mathbf{j})}(x), x\in\mathbb{R},$$

provides information about pairwise extremal dependence of $\mathbf{X}$ (see Schlather and Tawn [4]). This coefficient is related to the upper tail dependence parameter, defined in Sibuya [5] as

$$\lambda\left(\mathbf{i},\mathbf{j}\right)=\lim_{x\to x^F}P\left(X_{\mathbf{i}}>x\mid X_{\mathbf{j}}>x\right),$$

where x_F denotes the upper endpoint of F, through the relation $\lambda\left(\mathbf{i},\mathbf{j}\right)=2-\varepsilon\left(\mathbf{i},\mathbf{j}\right)$.

Unlike a Gaussian process, the dependence structure of a max-stable process is not completely characterized by its pairwise dependence structure. To overcome this problem Schlather and Tawn [4] extend the definition of the extremal coefficient to a multivariate setting of any dimension, as follows

$$P\left(\bigvee_{\mathbf{i}\in\mathbf{A}} X_{\mathbf{i}} \leq x\right) = F^{\varepsilon(\mathbf{A})}(x), \quad x\in\mathbb{R},\ \mathbf{A}\subseteq\mathbb{Z}^2.$$

This coefficient measures the extremal dependence between the variables indexed by set **A** and its simple interpretation as the effective number of independent variables, in the set **A**, from which the maximum is drawn has led to its use as a dependence measure in a wide range of practical applications.

When the spatial process **X** is isotropic the pairwise extremal dependence measures depend only on the distance $\|\mathbf{i}-\mathbf{j}\|$ between the locations **i** and **j** considered. Nevertheless, in general, we don't have isotropy and thus need to evaluate the spatial dependence in the several directions of $\mathbb{Z}^2$. To attain this we propose a matrix of bivariate tail coefficients defined as

$$\Lambda(T_s^m(\mathbf{i}),\mathbf{i}) = \begin{bmatrix} \lambda(s_4^m(\mathbf{i}),\mathbf{i} & \lambda(s_3^m(\mathbf{i}),\mathbf{i} & \lambda(s_2^m(\mathbf{i}),\mathbf{i}) \\ \lambda(s_5^m(\mathbf{i}),\mathbf{i} & \lambda(s_0^m(\mathbf{i}),\mathbf{i} & \lambda(s_1^k(\mathbf{i}),\mathbf{i}) \\ \lambda(s_6^m(\mathbf{i}),\mathbf{i} & \lambda(s_7^m(\mathbf{i}),\mathbf{i} & \lambda(s_8^m(\mathbf{i}),\mathbf{i}) \end{bmatrix},$$

where for each $\mathbf{i}=(i_1,i_2)\in\mathbb{Z}^2$, $s_j(\mathbf{i})$, $j=1,2,\ldots,8$, denote the the neighbors of **i** as follows

$$s_1(\mathbf{i})=(i_1+1,i_2),\quad s_2(\mathbf{i})=\mathbf{i}+\mathbf{1},\quad s_3(\mathbf{i})=(i_1,i_2+1),\quad s_4(\mathbf{i})=(i_1-1,i_2+1),$$

$$s_5(\mathbf{i})=(i_1-1,i_2),\quad s_6(\mathbf{i})=\mathbf{i}-\mathbf{1},\quad s_7(\mathbf{i})=(i_1,i_2-1),\quad s_8(\mathbf{i})=(i_1+1,i_2-1),$$

$s_j^m(\mathbf{i})=(s_j\circ\ldots\circ s_j)(\mathbf{i})$, k times with $m\geq 1$, $s_j^0(\mathbf{i})=\mathbf{i}$, $j=1,2,\ldots,8$, and $T_s^m(\mathbf{i})=\{s_j^m(\mathbf{i}):\ j=1,\ldots,8\}$.

Note that, for each $\mathbf{i}\in\mathbb{Z}^2$, $s_0^m(\mathbf{i})=\mathbf{i}$, $\lambda(\mathbf{i},\mathbf{i})=1$ and, for $m>1$, we have

$$\lambda(s_j^m(\mathbf{i}),\mathbf{i})=\lambda(s_t^{m-1}(\mathbf{i}),\mathbf{i})+\varepsilon(s_t^{m-1}(\mathbf{i}),\mathbf{i})-\varepsilon(s_j^m(\mathbf{i}),\mathbf{i}),\ t,j\in\{1,2,\ldots,8\}.$$

In the next section we introduce a space-time ARMAX storage model for which we analyze stationarity, max-stablility and compute some spatial dependence coefficients.

2 A space-time ARMAX storage model

In Haslett [3] the solar thermal energy model

$$X_j=\beta X_{j-1}\vee(\alpha\beta X_{j-1}+Y_j),\quad j\geq 1,\quad 0\leq\alpha\leq 1,\quad 0<\beta<1,$$

was introduced to describe the temperature level in a tank used for the storage of solar energy. This model was further investigated by Daley and Haslett [2], among others. Alpuim [1] studied its extremal behavior for the particular case $\alpha=0$. We will next present a study of an analogous space-time storage model.

Let $\mathbf{X}^{(0)}=\{X_{(i,0)}\}_{i\geq 1}$ and $\mathbf{Y}^{(j)}=\{Y_{(i,j)}\}_{i\geq 1},\ j\in\mathbb{N}$, denote independent and stationary random sequences, with, respectively, common univariate marginal distributions H and G, and consider for each subsets $\{i_1,\ldots,i_p\}\in\mathbb{N}$ and $\{j_1,\ldots,j_p\}\in\mathbb{N}$,

$$H_{(i_1,0),\ldots,(i_p,0)}(x_1,\ldots,x_p)=P(X_{(i_1,0)}\leq x_1,\ldots,X_{(i_p,0)}\leq x_p),\quad (x_1,\ldots,x_p)\in\mathbb{R}^p,$$

and

$$G_{(i_1,j_1),\ldots,(i_p,j_p)}(x_1,\ldots,x_p) = P(Y_{(i_1,j_1)} \le x_1,\ldots,Y_{(i_p,j_p)} \le x_p), \quad (x_1,\ldots,x_p) \in \mathbb{R}^p.$$

We will assume that for each $j \in \mathbb{N}$ the random sequences $\mathbf{Y}^{(j)},\ j \in \mathbb{N}$, are identically distributed.

Considering the stationary random sequence $\mathbf{X}^{(0)}$ and the stationary random field $\mathbf{Y} = \{Y_{(i,j)}\}_{(i,j)\in\mathbb{N}^2}$ we can now define a max-autoregressive random field through the relation

$$X_{(i,j)} = k\left(X_{(i,j-1)} \vee Y_{(i,j)}\right) = k^j X_{(i,0)} \vee \bigvee_{t=1}^{j} k^{j-t+1} Y_{(i,t)}, \quad (i,j) \in \mathbb{N}^2, \quad 0 < k < 1.$$

For any locations $\mathbf{r}_1 = (i_1,j_1),\ldots,\mathbf{r}_p = (i_p,j_p)$ on $\mathbb{N}^2$, and $(x_1,\ldots,x_p) \in \mathbb{R}^p$ we have

$$H_{\mathbf{r}_1,\ldots,\mathbf{r}_p}(x_1,\ldots,x_p) = H_{\mathbf{r}_1+(0,-1),\ldots,\mathbf{r}_p+(0,-1)}\left(\frac{x_1}{k},\ldots,\frac{x_p}{k}\right) \times G_{\mathbf{r}_1,\ldots,\mathbf{r}_p}\left(\frac{x_1}{k},\ldots,\frac{x_p}{k}\right).$$

If we consider $i_1 = \ldots = i_p = i \ge 1$ fixed, we find the well know Markovian sequence studied in Alpuim [1], for which was shown that for $0 = j_0 < j_1 < \ldots < j_p$

$$H_{(i,j_1),\ldots,(i,j_p)}(x_1,\ldots,x_p) = H\left(\min_{1\le s\le p} \frac{x_s}{k^{j_s}}\right) \prod_{t=1}^{p} \prod_{s=j_{(t-1)}}^{j_t-1} G\left(\min_{t\le m\le p} \frac{x_m}{k^{j_m-s}}\right). \tag{1}$$

In what follows we shall consider locations $\mathbf{r}_1 = (i_1,j_1),\ldots,\mathbf{r}_p = (i_p,j_p)$, on $\mathbb{N}^2$, such that $i_{m_1} \neq i_{m_2}$, $m_1, m_2 \in \{1,\ldots,p\}$.

The next results give necessary and sufficient conditions for $\mathbf{X}$ to be a stationary max-stable random field.

Proposition 2.1 $\mathbf{X}$ *is a stationary random field if and only if, for any locations* $\mathbf{r}_1,\ldots,\mathbf{r}_p \in \mathbb{N}^2$ *and* $(x_1,\ldots,x_p) \in \mathbb{R}^p$,

$$H_{\mathbf{r}_1,\ldots,\mathbf{r}_p}(x_1,\ldots,x_p) = H_{\mathbf{r}_1,\ldots,\mathbf{r}_p}\left(\frac{x_1}{k},\ldots,\frac{x_p}{k}\right) \times G_{\mathbf{r}_1,\ldots,\mathbf{r}_p}\left(\frac{x_1}{k},\ldots,\frac{x_p}{k}\right).$$

If the finite dimension distributions of the sequences $\mathbf{Y}^{(j)},\ j \ge 1$ associated to the the random field of innovations $\mathbf{Y}$ are multivariate extreme value distributions then $\mathbf{Y}$ is a max-stable random field.

Proposition 2.2 *The stationary random field* $\mathbf{X}$ *is max-stable if and only if* $\mathbf{Y}$ *is a max-stable random field.*

The extremal coefficients of the finite dimension distributions of $\mathbf{X}$ and $\mathbf{Y}$ coincide as shown in the next result.

Proposition 2.3 *If both* $\mathbf{X}$ *and* $\mathbf{Y}$ *are stationary max-stable random fields then the extremal coefficients of their finite dimension distributions coincide.*

From this result we know that $\varepsilon^{\mathbf{X}}(\{\mathbf{r}_1,\ldots,\mathbf{r}_p\}) = \varepsilon^{\mathbf{Y}}(\{\mathbf{r}_1,\ldots,\mathbf{r}_p\})$ for any locations $\mathbf{r}_1 = (i_1, j_1),\ldots,$ $\mathbf{r}_p = (i_p, j_p)$, on $\mathbb{N}^2$, such that $i_{m_1} \neq i_{m_2}$, $m_1, m_2 \in \{1,\ldots,p\}$. On the other hand, from (1) we obtain $\varepsilon(\{(i, j_1),\ldots,(i, j_p)\})$ as follows.

Proposition 2.4 *For any choice of* $i \geq 1$ *and* $0 = j_0 < j_1 < \ldots < j_p$,

$$\varepsilon(\{(i, j_1),\ldots,(i, j_p)\}) = k^{j_1} + \sum_{t=1}^{p}(1 - k^{j_t - j_{(t-1)}}).$$

We can then conclude that for any point $\mathbf{i} = (i, j) \in \mathbb{N}^2$ it holds

$$\varepsilon(\mathbf{i}, s_3^m(\mathbf{i})) = \varepsilon(\{(i, j),(i, j+m)\}) = 2 - k^m,\ m \geq 1,$$

and consequently $\lambda(s_3(\mathbf{i}), \mathbf{i}) = k^m$.

Lets now consider $\mathbf{X}^{(0)}$ a stationary Markov chain in discrete time with continuous state space, with distribution function such that

$$H_{(1,0),(2,0)}(x_1, x_2) = \exp(-((-\ln H(x_1))^{\delta} + (-\ln H(x_2))^{\delta})^{1/\delta}),\ (x_1, x_2) \in \mathbb{R}^2,$$

where $\delta \in [1, +\infty[$ and $H_{(1,0)}(x) = H(x) = \exp(-\exp(-x)),\ x \in \mathbb{R}$.

In this case we obtain

$$\varepsilon(\{(1,0),(2,0)\}) = \frac{\ln H_{(1,0),(2,0)}(x,x)}{\ln H(x)} = \frac{-2^{1/\delta}\exp(-x)}{-\exp(-x)} = 2^{1/\delta}, \quad \delta \geq 1,$$

and $\lambda((2,0),(1,0)) = 2 - 2^{1/\delta}$, where independence is achieved for $\delta = 1$. The measure matrix of dependence, for $m = 1$, is then given by

$$\Lambda(T_s^1(\mathbf{i}), \mathbf{i}) = \begin{bmatrix} 0 & k & 0 \\ 2 - 2^{1/\delta} & 1 & 2 - 2^{1/\delta} \\ 0 & k & 0 \end{bmatrix}.$$

The computation of the other matrices $\Lambda(T_s^m(\mathbf{i}), \mathbf{i}),\ m \geq 2$, only depends on the computation of $\lambda(T_{s_1}^m(\mathbf{1}), \mathbf{1})$ since we have already shown that, for each $m \geq 1$, $\lambda(T_{s_3}^m(\mathbf{1}), \mathbf{1}) = k^m$. As before we can first obtain the related coefficient $\varepsilon(\mathbf{1}, T_{s_1}^m(\mathbf{1})),\ m \geq 2$, which can be computed from the dependence function of $(X_{(1,0)}, X_{(m+1,0)})$.

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