UNIVERSIDADE DE
SANTIAGO DE COMPOSTELA
DEPARTAMENTO DE
ESTATÍSTICA E INVESTIGACIÓN OPERATIVA

On the meaning, properties and computing of a coalitional Shapley value

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Report 11-04

Reports in Statistics and Operations Research

On the meaning, properties and computing of a coalitional Shapley value

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Abstract

In this paper we introduce a value for cooperative games with transferable utility and a coalitional structure. This value can be interpreted as the Shapley value of a certain modified game associated to the initial game or as a modified Shapley value. An axiomatic characterization, that makes use of five independent properties, is provided for this value. Finally, we provide a tool for the calculus involving the so-called multilinear extension.

Keywords: cooperative game, Shapley value, coalition structure, coalitional value, axiomatic characterization, multilinear extension.

MSC (2011) classification: 91A12.

1 Introduction

A value determines the payoffs allocated to each player in a cooperative game. Two of the most important values for cooperative games are the Shapley ([12]) and the Banzhaf ([4]) values. When information on relationships among players is available, coalitional values are most appropriate tools to establish these payoffs. These agreements among players are modeled by a set of a priori unions, i.e., a partition of the set of players. Two of these coalitional values were proposed by Owen in [9] and [10]. Alonso-Meijide et al. [1] present a comparative study of these two coalitional values. The coalitional value proposed in [9] divides

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among the players the Shapley value of the so-called quotient game whereas the one proposed in [10] assigns to each player the Banzhaf value of a certain modified game. In this paper, we propose a new coalitional value that assigns to each player the Shapley value of this game. A strong parallelism among the new value and the Banzhaf-Owen value can be established.

The paper is organized as follows. Some preliminaries are introduced in Section 2. In Section 3 the proposed coalitional value and its interpretations are presented. In Section 4 we characterize the value and show that the properties used in the characterization are logically independent. Finally, the multilinear extension of the new value is obtained in Section 5.

2 Preliminaries

2.1 TU games

A cooperative game with transferable utility (briefly, a *TU game*) is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is a function that assigns to each coalition of players $S \subset N$ the real number $v(S)$. Moreover, it is assumed that $v(\emptyset) = 0$. We denote by G the set of all TU games. A *simple game* is a pair (N, v) where v allocates to each coalition $S \subset N$ the value 0 or 1 in such a way that $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) = 1$ implies that $v(T) = 1$ for all $S \subset T$. We say that $S \subset N$ is a *minimal winning coalition* if $v(S) = 1$ and $v(T) = 0$ for all $T \subset S$.

Two players $i, j \in N$ are *symmetric* in a TU game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$.

A *value* is a map f that allocates to each TU game $(N, v) \in G$ a vector $f(N, v) \in \mathbb{R}^N$, where $f_i(N, v)$ is the payoff of each player $i \in N$.

Definition 1 *(Shapley [12]) The Shapley value φ assigns to each $(N, v) \in G$ and $i \in N$ the real number*

$$\varphi_i(N, v) = \sum_{S \subset N \setminus \{i\}} \frac{s!\,(n - s - 1)!}{n!} \left[v(S \cup \{i\}) - v(S) \right],$$

where $s = |S|$.

Definition 2 *(Banzhaf [4]) The Banzhaf value β assigns to each $(N, v) \in G$ and $i \in N$ the real number*

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subset N \setminus \{i\}} \left[v(S \cup \{i\}) - v(S) \right].$$

Definition 3 *(Owen [7]) Let (N, v) be a cooperative game with n players. The multilinear extension (MLE) of v is the real function of n variables*

$$f(q_1, \ldots, q_n) = \sum_{S \subset N} \prod_{j \in S} q_j \prod_{j \notin S} (1 - q_j) v(S).$$

The MLE supports a probabilistic interpretation. If X is a coalition of players formed at random, assuming that q_j is the probability that the player j belongs to X for $j = 1, \ldots, n$ and that these probabilities are independent, then for every $S \subset N$ we have:

$$prob\{X = S\} = \prod_{j \in S} q_j \prod_{j \notin S} (1 - q_j).$$

Therefore $f(q_1, \ldots, q_n) = E[v(X)]$, where E represents the mathematical expectation. The MLEs are particularly useful for the calculus of some of the values in the context of TU games.

2.2 TU games with a coalition structure

Let us consider a set of players N. A *coalition structure* $P = \{P_1, \ldots, P_m\}$ over N is a partition of N, that is, $\bigcup_{k=1}^{m} P_k = N$ and $P_k \cap P_h = \emptyset$ when $k \neq h$. Along this paper, we will use the trivial coalition structure over N where each union is a singleton, given by $P^n = \{\{1\}, \ldots, \{n\}\}$.

Given $i \in N$, $P(i)$ denotes the family of coalition structures over N where $\{i\}$ is a singleton union, that is, if $P \in P(i)$ then $\{i\} \in P$.

For each $i \in P_k \in P$, P_{-i} denotes the partition of $P(i)$ obtained from P when player i leaves the union P_k and becomes isolated, i.e., $P_{-i} = \{P_h \in P : h \neq k\} \cup \{P_k \setminus \{i\}, \{i\}\}$.

A *TU game with a coalition structure* is a triple (N, v, P) where $(N, v) \in G$ and P is a coalition structure over N. We denote by G^{cs} the set of all TU games with a coalition structure.

Given $(N, v, P) \in G^{cs}$ with $P = \{P_1, \ldots, P_m\}$, the associated *quotient game* (M, v^P) is the TU game played by the unions, where $M = \{1, \ldots, m\}$ and $v^P(R) = v\left(\bigcup_{k \in R} P_k\right)$ for all $R \subset M$.

A *coalitional value* is a map g that assigns to each TU game with a coalition structure $(N, v, P) \in G^{cs}$ a vector $g(N, v, P) \in \mathbb{R}^N$, where $g_i(N, v, P)$ is the payoff of each player $i \in N$.

Given a value f on G, a coalitional value g on G^{cs} is called a *coalitional f-value* when $g(N, v, P^n) = f(N, v)$ for all $(N, v) \in G$.

Definition 4 *(Owen [9]) The Owen value, ϕ, is the value on $G^{cs}(N)$ that is defined as follows:*

$$\phi_i(N, v, P) = \sum_{R \subset M \setminus \{k\}} \sum_{T \subset P_k \setminus \{i\}} \frac{t!(p_k - t - 1)!r!(m - r - 1)!}{p_k!m!} \Big[v(Q \cup T \cup \{i\}) - v(Q \cup T) \Big]$$

for each $i \in N$ and $(N, v, P) \in G^{cs}$, where $P_k \in P$ is the union such that $i \in P_k$, $m = |M|$, $p_k = |P_k|$ and $Q = \bigcup_{h \in R} P_h$.

3 The value Γ and its interpretations

One of the coalitional values which appears in the literature is the Banzhaf-Owen coalitional value.

Definition 5 *(Banzhaf-Owen [10]) The Banzhaf-Owen coalitional value is defined by*

$$\Psi_i(N, v, P) = \sum_{R \subset M \setminus \{k\}} \sum_{T \subset P_k \setminus \{i\}} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \Big[v(Q \cup T \cup \{i\}) - v(Q \cup T) \Big]$$

for all $i \in N$ and all $(N, v, P) \in G^{cs}$.

In Alonso-Meijide et al. [1] it is showed that this coalitional value is a coalitional Banzhaf value.

Moreover, it is straightforward to prove that, given $(N, v, P) \in G^{cs}$ and a player $i \in P_k \in P$, the Banzhaf-Owen coalitional value of i can be interpreted as the Banzhaf value of this player applied to the TU game played by the unions other than P_k and by the players in P_k.

Proposition 1 *(Laruelle and Valenciano [6]) For all $(N, v, P) \in G^{cs}$ and all $i \in P_k$, $P_k \in P$,*

$$\Psi_i(N, v, P) = \beta_i \left(M \setminus \{k\} \cup P_k, v^{M \setminus \{k\} \cup P_k} \right)$$

where

$$v^{M \setminus \{k\} \cup P_k}(R \cup T) = v \left(\bigcup_{h \in R} P_h \cup T \right)$$

for all $R \subset M \setminus \{k\}$ and all $T \subset P_k$.

In fact, Laruelle and Valenciano [6] showed that the Banzhaf-Owen value admits a triple role when we are trying to answer the question of what is the relevance of each player i if decisions are made according to a voting rule, and voters of each union of the partition P distinct of which contains the voter i act as a block (i.e., the vote is not divided into any of the other unions). Specifically, the Banzhaf-Owen value can be interpreted as the Banzhaf value of a modified game, as a modified Banzhaf value, or as a generalized Banzhaf value of an associated game. They argued that this triple role can not be applied to other values. However, the Owen value admits the second and third interpretation, substituting the Banzhaf value by the Shapley value.

In this paper we study the coalitional value that assigns to each player the Shapley value of the modified game considered by Laruelle and Valenciano. Thus it admits a similar interpretation to the first one. It also admits, as we will see, the second interpretation. We begin by defining it.

Definition 6 *The coalitional value Γ is defined by*

$$\Gamma_i(N, v, P) = \sum_{R \subset M \setminus \{k\}} \sum_{T \subset P_k \setminus \{i\}} \frac{(r+t)!(m + p_k - r - t - 2)!}{(m + p_k - 1)!} \left[v(Q \cup T \cup \{i\}) - v(Q \cup T) \right]$$

for all $i \in N$ and all $(N, v, P) \in G^{cs}$, where $P_k \in P$ is the union such that $i \in P_k$.

3.1 The Shapley value of a "modified game"

Proposition 2 *For all $(N, v, P) \in G^{cs}$ and all $i \in P_k$, $P_k \in P$,*

$$\Gamma_i(N, v, P) = \varphi_i \left(M \setminus \{k\} \cup P_k, v^{M \setminus \{k\} \cup P_k} \right).$$

Example 1 *Consider the four-player simple game (N,v), where N={1,2,3,4} and the minimal winning coalitions are {1,3} and {2,3,4}. Suppose that the coalition structure is P={{1,2},{3,4}}. First we will calculate the index* $\Gamma(N,v,P)$ *with the Definition 5.*

Note that in this example: M={{1,2},{3,4}}, so m=2.
For i=1 we have: $1 \in P_1$, $M \setminus \{1\} = \{\{3,4\}\}$ *and* $P_1 \setminus \{1\} = \{2\}$.

Q	T	$(r+t)!(m+p_k-r-t-2)!$	$v(Q \cup T \cup \{1\}) - v(Q \cup T)$
$\varnothing$	$\varnothing$	$0!(2+2-2)! = 2$	0
$\varnothing$	$\{2\}$	$(0+1)!(2+2-0-1-2)! = 1$	0
$\{3,4\}$	$\varnothing$	$(1+0)!(2+2-1-0-2)! = 1$	1
$\{3,4\}$	$\{2\}$	$(1+1)!(2+2-1-1-2)! = 2$	0

Then:

$$\Gamma_1(N,v,P) = \frac{2 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 + 2 \cdot 0}{(2+2-1)!} = \frac{1}{3!} = \frac{1}{6}.$$

Similarly, we obtain:
$\Gamma_2(N,v,P) = \frac{1}{6}$, $\Gamma_3(N,v,P) = \frac{1}{2}$ *and* $\Gamma_4(N,v,P) = 0$.

Next we will show that we obtain the same results if we apply the previous interpretation. Consider now the modified game $(M \setminus \{k\} \cup P_k, v^{M \setminus \{k\} \cup P_k})$.
For $i = 1$ *and* $i = 2$ *we have that* $i \in P_1$ *and:*

$$M \setminus \{1\} \cup P_1 = \{\{1\},\{2\},\{3,4\}\}.$$

Let us calculate the Shapley value of this modified game:

Permutations	{1}	{2}	{34}
{1}{2}{34}	*0*	*0*	*1*
{1}{34}{2}	*0*	*0*	*1*
{2}{1}{34}	*0*	*0*	*1*
{2}{34}{1}	*0*	*0*	*1*
{34}{1}{2}	*1*	*0*	*0*
{34}{2}{1}	*0*	*1*	*0*
	1	*1*	*4*

Then we obtain:

$$\varphi_1(M \setminus \{1\} \cup P_1, v^{M \setminus \{1\} \cup P_1}) = \frac{1}{6} = \Gamma_1(N,v,P) \text{ and}$$

$$\varphi_2(M \setminus \{1\} \cup P_1, v^{M \setminus \{1\} \cup P_1}) = \frac{1}{6} = \Gamma_2(N,v,P).$$

For i=3 and i=4 we have that $i \in P_2$ *and:*

$$M \setminus \{2\} \cup P_2 = \{\{1,2\},\{3\},\{4\}\}$$

Let us calculate the Shapley value of this modification of the game:

Permutations	{12}	{3}	{4}
{12}{3}{4}	*0*	*1*	*0*
{12}{4}{3}	*0*	*1*	*0*
{3}{12}{4}	*1*	*0*	*0*
{3}{4}{12}	*1*	*0*	*0*
{4}{3}{12}	*1*	*0*	*0*
{4}{12}{3}	*0*	*1*	*0*
	3	*3*	*0*

Note that:

$$\varphi_3(M \setminus \{2\} \cup P_2, v^{M \setminus \{2\} \cup P_2}) = \frac{3}{6} = \frac{1}{2} = \Gamma_3(N, v, P) \textit{ and}$$

$$\varphi_4(M \setminus \{2\} \cup P_2, v^{M \setminus \{2\} \cup P_2}) = 0 = \Gamma_4(N, v, P).$$

3.2 A "modified Shapley value" of the game

The coalitional value Γ, given by Definition 5, admits the following heuristic interpretation. Consider $i \in P_k$, let π be a permutation of the players in N where players in P_h, $h \neq k$, act together, and $\Pi^*(N)$ the set of such permutations. We denote by B^π(i)=$\{j \in N$ such that $\pi(j) < \pi(i)\}$ the set of players preceding i in the order π. We assume that:

1. Players agree to go to a certain point of negotiation.
2. All possible arrival orders where players in P_h, $h \neq k$, act together are equally probable.
3. When a player arrives, he receives a payment equal to his contribution to the coalition formed by players who came before him.

Then, if $i \in P_k$:

$$\Gamma_i(N, v, P) = \frac{1}{(m + p_k - 1)! \prod_{h=1, h \neq k}^{m} p_h!} \sum_{\pi \in \Pi^*(N)} \left[v(B^\pi(i) \cup \{i\}) - v(B^\pi(i)) \right].$$

Once union P_k is fixed, it is supossed that players in P_k are independent but unions other than P_k act together. Thus, the coalitional value Γ appears as a "modified Shapley value" of the original game. The normative point of view is kept, on condition that the unions P_h, with $h \neq k$, do not split the vote. That is, Γ_i is the Shapley value of player i when players in P_k arrived in any order (the arrival orders for players in P_k are equally probable), but players in unions other than P_k can only act together.

Example 2 *Consider the four-player simple game (N,v) at Example 1. Now, we are going to check that the previous interpretation is true. Let us calculate the "modified Shapley value"*[5]*. For players 1 and 2 (note that $P_1 = \{1, 2\}$) we have:*

[5]For $i \in P_k$, we will denote the corresponding value by $\varphi^{P_k}(N, v)$.

Permutations	1	2	3	4
1234	*0*	*0*	*1*	*0*
1243	*0*	*0*	*1*	*0*
1342	*0*	*0*	*1*	*0*
1432	*0*	*0*	*1*	*0*
2134	*0*	*0*	*1*	*0*
2143	*0*	*0*	*1*	*0*
2341	*0*	*0*	*0*	*1*
2431	*0*	*0*	*1*	*0*
3412	*1*	*0*	*0*	*0*
4312	*1*	*0*	*0*	*0*
3421	*0*	*1*	*0*	*0*
4321	*0*	*1*	*0*	*0*
	2	*2*	*7*	*1*

Then:

$$\varphi^{\{1,2\}}(N,v) = \left(\frac{2}{12}, \frac{2}{12}, \frac{7}{12}, \frac{1}{12}\right) = \left(\frac{1}{6}, \frac{1}{6}, \frac{7}{12}, \frac{1}{12}\right).$$

Note that:

$$\Gamma_1(N,v,P) = \frac{1}{6} = \varphi_1^{\{1,2\}}(N,v) \text{ and}$$

$$\Gamma_2(N,v,P) = \frac{1}{6} = \varphi_2^{\{1,2\}}(N,v).$$

For players 3 and 4 (note that $P_2 = \{3,4\}$*) we have:*

Permutations	1	2	3	4
1234	*0*	*0*	*1*	*0*
2134	*0*	*0*	*1*	*0*
1243	*0*	*0*	*1*	*0*
2143	*0*	*0*	*1*	*0*
3124	*1*	*0*	*0*	*0*
3214	*1*	*0*	*0*	*0*
3412	*1*	*0*	*0*	*0*
3421	*0*	*1*	*0*	*0*
4312	*1*	*0*	*0*	*0*
4321	*0*	*1*	*0*	*0*
4123	*0*	*0*	*1*	*0*
4213	*0*	*0*	*1*	*0*
	4	*2*	*6*	*0*

Then:

$$\varphi^{\{3,4\}}(N,v) = \left(\frac{4}{12}, \frac{2}{12}, \frac{6}{12}, 0\right) = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}, 0\right).$$

Note that:

$$\Gamma_3(N,v,P) = \frac{1}{2} = \varphi_3^{\{3,4\}}(N,v) \text{ and}$$

$$\Gamma_4(N,v,P) = 0 = \varphi_4^{\{3,4\}}(N,v).$$

4 The axiomatic characterization

Below we introduce several properties for a coalitional value g.

A1. (*Efficiency*). For all $(N,v) \in G$, $\sum_{i \in N} g_i(N,v,P^n) = v(N)$.

A2. (*Symmetry*). If $i,j \in N$ are symmetric players in $(N,v) \in G$, then $g_i(N,v,P^n) = g_j(N,v,P^n)$.

A3. (*Equal marginal contributions*). If (N,v) and (N,w) are TU games with a common player set N, and some player $i \in N$ satisfies $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subset N \backslash \{i\}$, then $g_i(N,v,P^n) = g_i(N,w,P^n)$.

A4. (*Neutrality under individual desertion*). If $(N,v,P) \in G^{cs}$, $P_k \in P$, and $i,j \in P_k$, $i \neq j$, then $g_i(N,v,P) = g_i(N,v,P_{-j})$.

A5. (*1-Quotient game property*). If $(N,v,P) \in G^{cs}$ and $P \in P(i)$ for some $i \in N$, then $g_i(N,v,P) = g_k(M,v^P,P^m)$, where $P_k = \{i\}$.

The properties of efficiency, symmetry and equal marginal contributions are standard in the literature. We use them in the context of TU games with a trivial coalition structure. For a detailed discussion about the properties A4 and A5, we refer the reader to Alonso-Meijide et al. [1].

Proposition 3 *A coalitional value g satisfies A1, A2 and A3 if, and only if, it is a coalitional Shapley value, i.e.*

$$g_i(N,v,P^n) = \varphi_i(N,v) \text{ for all } i \in N \text{ and all } (N,v) \in G.$$

Proof.
In Young [13] it is proved that the Shapley value is the unique value which satisfies efficiency, symmetry and equal marginal contributions (this last property is called independence in [13]). There is only one difference between efficiency and A1, symmetry and A2, or independence and A3: while the properties considered by Young are applied in the context of values for TU games, A1-A3 are properties to be satisfied by coalitional values. On the other hand, since A1-A3 should be proved for the trivial coalition structures where each union is a singleton, is not difficult to prove the proposition by adapting the proof of Young. □

Theorem 1 Γ *is the only coalitional value that satisfies A1-A5. Equivalently,* Γ *is the only coalitional Shapley value that satisfies A4 and A5.*

Proof.

a) *Existence.*

1. Γ satisfies A1-A3. According to Proposition 3, this is equivalent to prove that Γ is a coalitional Shapley value.

Let $(N,v) \in G$. If we consider the trivial coalition structure $P^n = \{\{1\},\ldots,\{n\}\}$, we obtain that for every $P_k \in P^n$, the TU game $\left(M \setminus \{k\} \cup P_k, v^{M\setminus\{k\}\cup P_k}\right)$ coincides with (N,v). Thus, by Proposition 2 we deduce that $\Gamma(N,v,P^n) = \varphi(N,v)$.

2. Γ satisfies the property of neutrality under individual desertion.

Let $(N,v,P) \in G^{cs}$, $P_k \in P$, and $i,j \in P_k$ be distinct players. Let us consider

$$M' = \{1,\ldots,m,m+1\}, P_{-j} = \{P'_1,\ldots,P'_{m+1}\},$$

where $P'_h = P_h$ for all $h \in M \setminus \{k\}$, $P'_k = P_k \setminus \{j\}$, $P'_{m+1} = \{j\}$, $|M'| = m'$ and $p'_k = |P'_k|$. Since $m' = m+1$ and $p'_k = p_k - 1$,

$$\begin{aligned}
\Gamma_i(N,v,P_{-j}) &= \sum_{R\subset M'\setminus\{k\}} \sum_{T\subset P'_k\setminus\{i\}} \frac{(r+t)!(m'+p'_k-r-t-2)!}{(m'+p'_k-1)!}\left[v(Q\cup T\cup\{i\}) - v(Q\cup T)\right] \\
&= \sum_{R\subset M'\setminus\{k\}} \sum_{T\subset P'_k\setminus\{i\}} \frac{(r+t)!(m+p_k-r-t-2)!}{(m+p_k-1)!}\left[v(Q\cup T\cup\{i\}) - v(Q\cup T)\right] \\
&= \sum_{R\subset M'\setminus\{k,m+1\}} \sum_{T\subset P_k\setminus\{i,j\}} \frac{(r+t)!(m+p_k-r-t-2)!}{(m+p_k-1)!}\left[v(Q\cup T\cup\{i\}) - v(Q\cup T)\right] \\
&+ \sum_{R\subset M'\setminus\{k\}, m+1\in R} \sum_{T\subset P_k\setminus\{i,j\}} \frac{(r+t)!(m+p_k-r-t-2)!}{(m+p_k-1)!}\left[v(Q\cup T\cup\{i\}) - v(Q\cup T)\right] \\
&= \sum_{R\subset M\setminus\{k\}} \sum_{T\subset P_k\setminus\{i,j\}} \left\{\frac{(r+t)!(m+p_k-r-t-2)!}{(m+p_k-1)!}\left[v(Q\cup T\cup\{i\}) - v(Q\cup T)\right]\right. \\
&+ \left.\frac{(r+t+1)!(m+p_k-r-t-3)!}{(m+p_k-1)!}\left[v(Q\cup T\cup\{j\}\cup\{i\}) - v(Q\cup T\cup\{j\})\right]\right\} \\
&= \sum_{R\subset M\setminus\{k\}} \sum_{T\subset P_k\setminus\{i,j\}} \frac{(r+t)!(m+p_k-r-t-2)!}{(m+p_k-1)!}\left[v(Q\cup T\cup\{i\}) - v(Q\cup T)\right] \\
&+ \sum_{R\subset M\setminus\{k\}} \sum_{T\subset P_k\setminus\{i\}, j\in T} \frac{(r+t)!(m+p_k-r-t-2)!}{(m+p_k-1)!}\left[v(Q\cup T\cup\{i\}) - v(Q\cup T)\right] \\
&= \Gamma_i(N,v,P).
\end{aligned}$$

3. Γ satisfies 1-quotient game property.

Let $(N,v,P) \in G^{cs}$ be such that $P \in P(i)$ for some $i \in N$. So, there exists $k \in M$ such that $P_k = \{i\}$.

Thus

$$\begin{aligned}
\Gamma_i(N,v,P) &= \sum_{R\subset M\setminus\{k\}} \sum_{T\subset P_k\setminus\{i\}} \frac{(r+t)!(m+p_k-r-t-2)!}{(m+p_k-1)!} \left[v(Q\cup T\cup\{i\}) - v(Q\cup T)\right] \\
&= \sum_{R\subset M\setminus\{k\}} \frac{r!(m-r-1)!}{m!} \left[v(Q\cup\{i\}) - v(Q)\right] \\
&= \sum_{R\subset M\setminus\{k\}} \frac{r!(m-r-1)!}{m!} \left[v^P(R\cup\{k\}) - v^P(R)\right] \\
&= \varphi_k(M,v^P) \\
&= \Gamma_k(M,v^P,P^m).
\end{aligned}$$

b) *Uniqueness.*

Let us suppose that g^1 and g^2 are two coalitional Shapley values satisfying A4 and A5. Hence, if uniqueness is not true, there exists a coalitional game (N,v,P) and a player $i\in N$ such that $g_i^1(N,v,P)\neq g_i^2(N,v,P)$. We can assume, without loss of generality, that P is the partition with the maximum number of coalitions which satisfies this condition.

Since g^1 and g^2 are two different coalitional values, it follows that $m<n$. Let us take $P_k\in P$ such that $i\in P_k$. There are two possible cases:

- $|P_k|=1$. So, $P\in P(i)$. By A5 we have that $g_i^1(N,v,P)=g_k^1(M,v^P,P^m)$ and $g_i^2(N,v,P)=g_k^2(M,v^P,P^m)$.
  Since g^1 and g^2 are coalitional Shapley values

  $$g_k^1(M,v^P,P^m)=\varphi_k(M,v^P)=g_k^2(M,v^P,P^m).$$

  Therefore, $g_i^1(N,v,P)=g_i^2(N,v,P)$, which is a contradiction.
- $|P_k|>1$. Then, there exists $j\in P_k$ such that $j\neq i$. By A4, $g_i^1(N,v,P)=g_i^1(N,v,P_{-j})$ and $g_i^2(N,v,P)=g_i^2(N,v,P_{-j})$.
  By the maximality of partition P we obtain that $g_i^1(N,v,P_{-j})=g_i^2(N,v,P_{-j})$.
  Then, we obtain that $g_i^1(N,v,P)=g_i^2(N,v,P)$, which is a contradiction. □

Remark 1 *(Independence of the properties).*

i) The Banzhaf-Owen coalitional value satisfies A2-A5, but not A1.

ii) Let i and j be two fixed players, $i\neq j$. Let g be defined as follows:

- For every $(N,v,P)\in G^{cs}$ such that $N=\{i,j\}$,

  $$g_i(N,v,P)=\min\{v(N)-v(\{j\}),v(\{i\})\} \text{ and } g_j(N,v,P)=\max\{v(N)-v(\{i\}),v(\{j\})\}.$$

- Otherwise, for $(N,v,P)\in G^{cs}$ such that $N\neq\{i,j\}$, $g_k(N,v,P)=\Gamma_k(N,v,P)$ for all $k\in N$.

Notice that g satisfies A1 and A3-A5, but not A2.

iii) Let us consider the class of coalitional games

$$\mathcal{C}=\{(N,v,P)\in G^{cs}: v=a_S\delta_S \text{ for some } S\subsetneq N, a_S\in\mathbb{R}\},$$

where δ_S is the Dirac game of coalition S, that is, $\delta_S(T)=1$ if $T=S$ and $\delta_S(T)=0$, otherwise. The coalitional value defined by

$$g_i(N,v,P)=\begin{cases}\Gamma_i(N,v,P) & \text{if} \quad (N,v,P)\notin\mathcal{C}\\ 0 & \text{if} \quad (N,v,P)\in\mathcal{C}\end{cases}$$

satisfies A1, A2, A4 and A5, but not A3.

iv) The coalitional value introduced by Amer et al. [3] and defined for all $i\in P_k$ by

$$\mu_i(N,v,P)=\sum_{R\subset M\setminus\{k\}}\sum_{T\subset P_k\setminus\{i\}}\frac{r!(m-r-1)!}{m!}\frac{1}{2^{p_k-1}}\left[v(Q\cup T\cup\{i\})-v(Q\cup T)\right]$$

satisfies the properties A1-A3 and A5, but not A4.

v) The coalitional value given by $g(N,v,P)=\varphi(N,v)$ for all $(N,v,P)\in G^{cs}$ satisfies the properties A1-A4, but not A5.

5 The multilinear extension of the Shapley coalitional value

As it is well known, both Shapley and Banzhaf values of any game v can be easily obtained from its multilinear extension. Indeed, $\varphi(N,v)$ can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $q_1=q_2=\ldots=q_n$ of the cube $[0,1]^N$ (Owen [7]), while the partial derivatives of that multilinear extension evaluated at point $(\frac{1}{2},\ldots,\frac{1}{2})$ give $\beta(N,v)$ (Owen [8]).

In the context of games with a coalition structure, the multilinear extension technique has been also applied to compute the Owen value Φ (Owen and Winter [11]), the Owen-Banzhaf value Ψ (Carreras and Magaña [5]) and the coalitional Banzhaf value π (Alonso et al. [2]).

We first introduce in this section the procedure to obtain the coalitional value by the multilinear extension. Then, before proving that this procedure indeed yields the coalitional value Γ, we calculate the coalitional value Γ using the MLE procedure in an example.

Procedure to obtain Γ by the multilinear extension

Suppose the game (N,v) and the coalition structure P=$\{P_1,\ldots,P_m\}$ are given. To compute Γ_i with $i\in P_k$ the following rules are given:

1. Obtain the MLE f(q_1,...,q_n).

2. For any $h \neq k$ and any $i \in P_h$, replace the variable q_i by p_h. This yields a new function of q_i, $i \in P_k$, and p_h, $h \neq k$.

3. In the function obtained by 2., reduce all higher exponents to 1, i.e., replace each p_l^n for n≥1 with p_l. This gives us the multilinear function

$$g((q_i)_{i \in P_k}, (p_h)_{h \neq k}).$$

4. The coalitional value results from calculating:

$$\Gamma_i(N, v, P) = \int_0^1 \frac{\partial g}{\partial q_i}(t, \ldots, t)dt.$$

Proposition 4 *Rules 1.-4. of the procedure above lead to the calculation of the coalitional value* Γ.

Proof.

Since both the coalitional value Γ and the calculation procedure described above are linear operators, we only need to show that they coincide in unanimity games, i.e., games of the form, for $T \subset N$,

$$u_T(Q) = \begin{cases} 1 & \text{if } T \subset Q \\ 0 & \text{otherwise.} \end{cases}$$

For a unanimity game u_T, the MLE has the simple form:

$$f(q_1, \ldots, q_n) = \prod_{i \in T} q_i.$$

For some $i \in T \cap P_k$ this function is revised by rules 2. and 3. to:

$$g((q_i)_{i \in P_k}, (p_h)_{h \neq k}) = \prod_{i \in T \cap P_k} q_i \prod_{h \in M, h \neq k, T \cap P_k \neq \varnothing} p_h.$$

Now,

$$g_i((q_i)_{i \in P_k}, (p_h)_{h \neq k}) = \frac{\partial g((q_i)_{i \in P_k}, (p_h)_{h \neq k})}{\partial q_i} = \prod_{j \in T \cap P_k, j \neq i} q_j \prod_{h \in M, h \neq k, T \cap P_h \neq \varnothing} p_h.$$

Moreover,

$$g_i(t, \ldots, t) = t^{|P_k \cap T| - 1} \, t^{K_T - 1} = t^{K_T + |P_k \cap T| - 2},$$

where $K_T = |\{h \in M : P_h \cap T \neq \varnothing\}|$, and

$$\int_0^1 g_i(t, \ldots, t)dt = \int_0^1 t^{K_T + |P_k \cap T| - 2} dt = \left[\frac{t^{K_T + |P_k \cap T| - 1}}{K_T + |P_k \cap T| - 1} \right]_0^1 = \frac{1}{K_T + |P_k \cap T| - 1}.$$

It is straightforward to prove that if $i \notin T$ procedure 1.-4. yields zero.

Let us consider a finite set N and the unanimity game u_T, with $T \subset N$. Given a partition P=$\{P_1, \dots, P_m\}$ we consider $T' = \{h \in M$ such that $P_h \cap T \neq \varnothing\}$ and T'_h=$T \cap P_h$. The quotient game (M, u_T^P) is defined as:

$$u_T^P(R) = \begin{cases} 1 & \text{if } T' \subset R \\ 0 & \text{if } T' \not\subset R \end{cases}, \textit{for all } R \subset M.$$

If $i \in T'_k = P_k \cap T$, for some $k \in M$, we have:
$u_T(Q \cup S \cup \{i\})$-$u_T(Q \cup S)$= 1 if and only if $Q = \cup_{r \in R} P_r$ where $T' \setminus \{k\} \subset R \subset M \setminus \{k\}$ and $T'_k \setminus \{i\} \subset S \subset P_k \setminus \{i\}$.

Thus if $i \in T'_k$:

$$\Gamma_i(N, u_T, P) = \sum_{T' \setminus \{k\} \subset R \subset M \setminus \{k\}} \; \sum_{T'_k \setminus \{i\} \subset S \subset P_k \setminus \{i\}} \frac{(r+s)!(m+p_k-r-s-2)!}{(m+p_k-1)!} = \frac{1}{K_T + |P_k \cap T| - 1}.$$

If $i \notin T$ it is easy to prove that $\Gamma_i(N, u_T, P) = 0$.

Moreover, we can give an alternative proof.

We know, by the second interpretation, that:

$$\Gamma_i(N, u_T, P) = \varphi_i(M \setminus \{k\} \cup P_k, u_T^{M \setminus \{k\} \cup P_k}).$$

Then, for all $R \subset M \setminus \{k\}$ and $S \subset P_k$:

$$u_T^{M \setminus \{k\} \cup P_k}(R \cup S) = u_T^{M \setminus \{k\} \cup P_k}(\bigcup_{h \in R} P_h \cup S) = \begin{cases} 1 & \text{if } T \subset \bigcup_{h \in R} P_h \cup S \\ 0 & \text{if } T \not\subset \bigcup_{h \in R} P_h \cup S \end{cases}$$

$$= \begin{cases} 1 & \text{if } \{h \in M, h \neq k : \, P_h \cap T \neq \varnothing\} \cup (P_k \cap T) \subset \bigcup_{h \in R} P_h \cup S \\ 0 & \text{if } \{h \in M, h \neq k : \, P_h \cap T \neq \varnothing\} \cup (P_k \cap T) \not\subset \bigcup_{h \in R} P_h \cup S \end{cases}$$

$$= u^{M \setminus \{k\} \cup P_k}_{(P_k \cap T) \cup \{h \in M, h \neq k \, : \, P_h \cap T \neq \varnothing\}}(D), \text{ for all } D = Q \cup S = \bigcup_{h \in R} P_h \cup S \subset M \setminus \{k\} \cup P_k.$$

We know that:

$$\varphi_i(N, u_T) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{if } i \notin T \end{cases}.$$

Then,

$$\Gamma_i(N, u_T, P) = \varphi_i\left(M \setminus \{k\} \cup P_k, u_T^{M \setminus \{k\} \cup P_k}\right) = \varphi_i\left(M \setminus \{k\} \cup P_k, u^{M \setminus \{k\} \cup P_k}_{(P_k \cap T) \cup \{h \in M, h \neq k \, : \, P_h \cap T \neq \varnothing\}}\right)$$

$$= \begin{cases} \frac{1}{|(P_k \cap T) \cup \{h \in M, h \neq k : \, P_h \cap T \neq \varnothing\}|} & \text{if } i \in T \cap P_k \\ 0 & \text{if } i \notin T \cap P_k \end{cases}$$

$$= \begin{cases} \dfrac{1}{|P_k \cap T| + K_T - 1} & \text{if } i \in T \cap P_k \\ 0 & \text{if } i \notin T \cap P_k \end{cases}.$$

Note that we obtain the same result. □

Below, we illustrate the procedure applied to the initial example.

Example 3 *Consider the four-player simple game (N,v), where N={1,2,3,4} and the minimal winning coalitions are {1,3} and {2,3,4}. Take the coalition structure P={{1,2},{3,4}}.*

The MLE of the game is given by:

$$f(q_1, q_2, q_3, q_4) = q_1 \cdot q_3 + q_2 \cdot q_3 \cdot q_4 - q_1 \cdot q_2 \cdot q_3 \cdot q_4. \tag{1}$$

To calculate Γ_1 and Γ_2 for the coalition structure P={{1,2},{3,4}}, we replace both q_3 and q_4 by p_2, obtaining:

$$f(q_1, q_2, p_2) = q_1 \cdot p_2 + q_2 \cdot p_2 \cdot p_2 - q_1 \cdot q_2 \cdot p_2 \cdot p_2 = q_1 \cdot p_2 + q_2 \cdot p_2^2 - q_1 \cdot q_2 \cdot p_2^2.$$

Reducing the higher exponents we have:

$$g(q_1, q_2, p_2) = q_1 \cdot p_2 + q_2 \cdot p_2 - q_1 \cdot q_2 \cdot p_2, \text{ and so:}$$

$$\frac{\partial g}{\partial q_1}(q_1, q_2, p_2) = p_2 - q_2 \cdot p_2 \text{ and } \frac{\partial g}{\partial q_2}(q_1, q_2, p_2) = p_2 - q_1 \cdot p_2.$$

Hence:

$$\Gamma_1(N, v, P) = \int_0^1 \frac{\partial g}{\partial q_1}(t, t, t)dt = \int_0^1 (t - t^2)dt = \frac{1}{6} \text{ and}$$

$$\Gamma_2(N, v, P) = \int_0^1 \frac{\partial g}{\partial q_2}(t, t, t)dt = \int_0^1 (t - t^2)dt = \frac{1}{6}.$$

To compute Γ_3 and Γ_4 we replace both q_1 and q_2 by p_1 in (1) and obtain:

$$f(p_1, q_3, q_4) = p_1 \cdot q_3 + p_1 \cdot q_3 \cdot q_4 - p_1 \cdot p_1 \cdot q_3 \cdot q_4 = p_1 \cdot q_3 + p_1 \cdot q_3 \cdot q_4 - p_1^2 \cdot q_3 \cdot q_4.$$

Reducing the higher exponents we have:

$$g(p_1, q_3, q_4) = p_1 \cdot q_3 + p_1 \cdot q_3 \cdot q_4 - p_1 \cdot q_3 \cdot q_4 = p_1 \cdot q_3,$$

and so:

$$\frac{\partial g}{\partial q_3}(p_1, q_3, q_4) = p_1 \text{ and } \frac{\partial g}{\partial q_4}(p_1, q_3, q_4) = 0.$$

Hence:

$$\Gamma_3(N,v,P) = \int_0^1 \frac{\partial g}{\partial q_3}(t,t,t)dt = \int_0^1 tdt = \frac{1}{2} \textit{ and}$$

$$\Gamma_4(N,v,P) = \int_0^1 \frac{\partial g}{\partial q_4}(t,t,t)dt = \int_0^1 0dt = 0.$$

Note that we get the same results as in Example 1.

Acknowledgements

Financial support from *Ministerio de Ciencia y Tecnología* and FEDER through grants ECO2008-03484-C02-01, ECO2008-03484-C02-02, ECO2011-23460, and MTM2011-27731-C03-02 and from *Xunta de Galicia* through grant INCITE09-207-064/PR is gratefully acknowledged.

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