UNIVERSIDADE DE
SANTIAGO DE COMPOSTELA
DEPARTAMENTO DE
ESTATÍSTICA E INVESTIGACIÓN OPERATIVA

On the estimation of α-convex sets

B. Pateiro-López, A. Rodríguez-Casal

Report 08-02

Reports in Statistics and Operations Research

On the estimation of α-convex sets

Beatriz Pateiro-López* and Alberto Rodríguez-Casal†
Departamento de Estatística e Investigación Operativa
Universidade de Santiago de Compostela

December 23, 2008

Abstract

This paper focuses on the estimation of α-convex sets. Formally, let $S \subset \mathbb{R}^d$ be a nonempty α-convex compact set with $\alpha > 0$. The goal is to estimate S based on a sample from a random variable with support S. In this setting, the α-convex hull of the sample turns out to be the natural estimator. A sufficient and necessary condition for the consistency of the estimator and its convergence rate are given. Some useful results relating α-convexity to other geometric restrictions such as the free rolling condition are also obtained.
Keywords: convexity, α-convexity, set estimation, free rolling condition

1 Introduction

The support estimation problem is formally established as the problem of estimating the support of an absolutely continuous probability measure P_X from independent observations drawn from it. Korostelëv and Tsybakov (1993) refers to Geffroy (1964), Rényi and Sulanke (1963), and Rényi and Sulanke (1964) as the first works on support estimation. Rényi and Sulanke (1963) and Rényi and Sulanke (1964) studied the case when S is a convex support in the bidimensional euclidean space and proposed a natural estimator, the convex hull of the sample $\mathcal{X}_n$. However, if S is not convex, the convex hull of the sample is not an appropriate estimator. In a more flexible framework, Chevalier (1976) and Devroye and Wise (1980) proposed to estimate the support (without any shape restriction) of an unknown probability measure by means of a smoothed version of the sample $\mathcal{X}_n$. The problem of support estimation was introduced by Devroye and Wise (1980) in connection with a practical application, the detection of abnormal behaviour of a system, plant or machine. Results on the performance of the estimator were obtained, among others, by Chevalier (1976), Devroye and Wise (1980), and Korostelëv and Tsybakov (1993). Of course, there are situations in between the two described above, that is, we can assume that the set S satisfies some shape restriction, more flexible than convexity. In Rodríguez-Casal (2007), the estimation of an α-convex support is considered. In this work we also focus on the problem of support estimation under the assumption

*Email: beatriz.pateiro@usc.es
†Email: alberto.rodriguez.casal@usc.es

of α-convexity. The α-convexity, defined in Section 2, is a condition that affects the shape of the set of interest but which is less restrictive than convexity and therefore, it allows a wider range of applications.

The paper is organized as follows. In Section 2 we introduce some notation and describe the estimator under study, the α-convex hull of a random sample of points taken in the set of interest. The main results on the behaviour of the estimator, regarding its consistency and optimal convergence rate, are stated in Section 3. In order to obtain the asymptotic properties of the estimator it will be useful to construct unavoidable families of sets. The precise definition and some general results on the construction of such families in $\mathbb{R}^d$ are stated in Section 4. All proofs are deferred to Section 5. We also include an Appendix with some useful results relating the α-convexity with other geometric restrictions.

2 Notation and preliminaries

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space, equipped with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. We denote by $B(x,r)$ and $\mathring{B}(x,r)$ the closed and open ball with centre x and radius r, respectively. In order to simplify the notation B and $\mathring{B}$ will stand for $B(0,1)$ and $\mathring{B}(0,1)$. Given $A\subset\mathbb{R}^d$, A^c, $\mathrm{int}(A)$, $\overline{A}$ and ∂A will denote the complement, interior, closure and boundary of A, respectively. The distance from a point $x\in\mathbb{R}^d$ to the set A is defined by $d(x,A)=\inf\{\|x-a\|:a\in A\}$. A crucial concept to be used in this paper is the following notion of α-convexity.

Definition 2.1. *A set $A\subset\mathbb{R}^d$ is said to be α-convex, for $\alpha>0$, if $A=C_\alpha(A)$, where*

$$C_\alpha(A)=\bigcap_{\{\mathring{B}(x,\alpha):\ \mathring{B}(x,\alpha)\cap A=\emptyset\}}\left(\mathring{B}(x,\alpha)\right)^c$$

is called the α-convex hull of A.

The α-convex hull of a set is intimately related to the dilation and erosion morphological operators through the closing of the set. The idea behind the closing is to define an operator that tends to recover the original shape of a set that has been previously dilated. This is achieved by eroding the dilated set. It is easy to see that

$$C_\alpha(A)=(A\oplus\alpha\mathring{B})\ominus\alpha\mathring{B},$$

where the symbols $\oplus$, $\ominus$ denote the Minkowski sum and difference, defined respectively by $A\oplus C=\{a+c:a\in A,c\in C\}$, $A\ominus C=\{x:\ \{x\}\oplus C\subset A\}$. Therefore, the α-convexity of a set can be defined in terms of its closing with respect to $\mathring{B}(0,\alpha)$. The set A is said to be α-convex if $A=(A\oplus\alpha\mathring{B})\ominus\alpha\mathring{B}$.

We will assume throughout that S is a nonempty compact subset of $\mathbb{R}^d$ and that S is α-convex for some $\alpha>0$. Assume that we are given a random sample $\mathcal{X}_n=\{X_1,\ldots,X_n\}$ from X, where X denotes a random variable in $\mathbb{R}^d$ with distribution P_X and support S. Then, $S=C_\alpha(S)$ and the α-convex hull of the sample

$$C_\alpha(\mathcal{X}_n)=(\mathcal{X}_n\oplus\alpha\mathring{B})\ominus\alpha\mathring{B} \qquad (1)$$

turns out to be a natural estimator for the set S. The α-convex hull of the sample as defined in (1) has the drawback of depending on the unknown parameter α. This difficulty can be overcome by taking a sequence of positive numbers $\{r_n\}$ converging to zero as n tends to infinity. This ensures that $r_n \leq \alpha$ for large enough n. For the sake of simplicity we assume that $r_n \leq \alpha$ for all n and define the estimator

$$S_n = C_{r_n}(\mathcal{X}_n) = (\mathcal{X}_n \oplus r_n \mathring{B}) \ominus r_n \mathring{B}. \tag{2}$$

Remark 2.1. *The definition of S_n given in (2) arises naturally in connection with the definition of the α-convex hull. It is not difficult to prove that, with probability one, $(\mathcal{X}_n \oplus r_n \mathring{B}) \ominus r_n \mathring{B}$ coincides with $(\mathcal{X}_n \oplus r_n B) \ominus r_n B$ and, therefore, we could have also defined S_n as the closing of $\mathcal{X}_n$ with respect to $B(0, r_n)$. Both definitions will be used indiscriminately in the proofs.*

In order to evaluate the performance of the estimator S_n, we will consider the distance in measure, a usual metric to quantify the similarity in content of two sets. The distance in measure between two Borel sets A and C is defined by $d_\mu(A, C) = \mu(A \Delta C)$, where μ denotes the Lebesgue measure and $A \Delta C = (A \setminus C) \cup (C \setminus A)$. Since with probability one $\mathcal{X}_n \subset S$, we obtain by the properties of the α-convex hull operator that $S_n \subset S$ and $d_\mu(S, S_n) = \mu(S \setminus S_n)$.

As in Rodríguez-Casal (2007), we require an additional condition on S which, in particular, implies the α-convexity. This assumption is related to the following definition.

Definition 2.2. *Let $A \subset \mathbb{R}^d$ be a closed set. The ball αB is said to roll freely in A if for each boundary point $a \in \partial A$ there exists some $x \in A$ such that $a \in B(x, \alpha) \subset A$.*

We assume that a ball of radius $\alpha > 0$ rolls freely in S and in $\overline{S^c}$. This free rolling type condition plays a major role in the proofs and it deserves some comments. First, it excludes the presence of sharp peaks in the set. Note that, by merely assuming α-convexity, we cannot ensure that the boundary of the set is smooth. On the other hand, assuming that a ball of radius $\alpha > 0$ rolls freely in S rules sets with isolated points out, for example. Roughly speaking, the free rolling condition in S forces the boundary points to be in direct contact with the interior of the set. Some results relating the rolling condition to the α-convexity are given in the Appendix.

3 Main results

The aim of this section is to present the results on the consistency (Theorem 3.1) and convergence rate of the estimator S_n defined in (2) (Theorem 3.2). We also include a result (Theorem 3.3) that proves that the convergence rate obtained in Theorem 3.2 cannot be improved. The concept of unavoidable family, briefly discussed in Section 4, will play a major role in the proofs of these results which are postponed to Section 5.

Theorem 3.1. *Let $S \subset \mathbb{R}^d$ be a nonempty α-convex compact set with $\alpha > 0$. Let X be a random variable with probability distribution P_X and density f whose support is S. Let $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ be a random sample from X and let $\{r_n\}$ be a sequence of positive terms which do not depend on the sample such that $r_n \leq \alpha$. Then,*

$$\lim_{n \to \infty} \mathbb{E}(d_\mu(S, S_n)) = 0$$

if and only if $\lim_{n \to \infty} n r_n^d = \infty$.

Remark 3.1. *By definition, $d_\mu(S, S_n) = \mu(S \setminus S_n) + \mu(S_n \setminus S)$. The α-convexity assumption of Theorem 3.1 ensures that $S_n \subset S$ and, therefore, $\mu(S_n \setminus S) = 0$. Anyway, if the set S is not assumed to be α-convex, a similar consistency result can be stated under an extra condition on the parameter r_n. It can be proved that, if $\{r_n\}$ is a sequence of positive terms such that $\lim_{n\to\infty} r_n = 0$ and $\lim_{n\to\infty} nr_n^d = \infty$, then $\lim_{n\to\infty} \mathbb{E}(d_\mu(S, S_n)) = 0$. Without going into details, the proof follows easily from*

$$\mathbb{E}(d_\mu(S, S_n)) = \mathbb{E}(\mu(S \setminus S_n)) + \mathbb{E}(\mu(S_n \setminus S)). \tag{3}$$

The first term in the right-hand side of (3) is studied in Theorem 3.1 and the α-convexity assumption is not needed to guarantee that $\lim_{n\to\infty} \mathbb{E}(\mu(S \setminus S_n)) = 0$ for a compact set S. For the second term in the right-hand side of (3) we have $\mathbb{E}(\mu(S_n \setminus S)) \leq \mu(S \oplus r_n B) - \mu(S)$ since, with probability one, $S_n \subset (S \oplus r_n B)$. The Lebesgue dominated convergence theorem ensures that $\lim_{n\to\infty} \mu(S \oplus r_n B) = \mu(S)$ if $\lim_{n\to\infty} r_n = 0$.

Having obtained the consistency of the estimator, we now focus on the convergence rate of $\mathbb{E}(d_\mu(S, S_n))$. Rodríguez-Casal (2007) obtains, under similar conditions on S, the almost sure convergence rate of $d_\mu(S, S_n)$. A more detail comparison of these results is given in Remark 3.2, after the statement Theorem 3.2, below.

Theorem 3.2. *Let S be a nonempty compact subset of $\mathbb{R}^d$ such that a ball of radius $\alpha > 0$ rolls freely in S and in $\overline{S^c}$. Let X be a random variable with probability distribution P_X and support S. We assume that the probability distribution P_X satisfies that there exists $\delta > 0$ such that $P_X(C) \geq \delta\mu(C \cap S)$ for all Borel subset $C \subset \mathbb{R}^d$. Let $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ be a random sample from X and let $\{r_n\}$ be a sequence of positive numbers which do not depend on the sample such that $r_n \leq \alpha$. If the sequence $\{r_n\}$ satisfies*

$$\lim_{n\to\infty} \frac{nr_n^d}{\log n} \to \infty, \tag{4}$$

then

$$\mathbb{E}(d_\mu(S, S_n)) = O\left(r_n^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}}\right). \tag{5}$$

Remark 3.2. *Rodríguez-Casal (2007) proves that, if S is under the conditions of Theorem 1 of Walther (1999) and $\{r_n\}$ is a sequence of positive numbers satisfying (4), then $d_\mu(S, S_n) = O(r_n^{-1}(\log n/n)^{2/(d+1)})$, almost surely. The convergence rate of $\mathbb{E}(d_\mu(S, S_n))$ obtained in Theorem 3.2 is, therefore, faster than the almost sure convergence rate of $d_\mu(S, S_n)$. Note that the logarithmic term vanishes in (5). Moreover, the penalty factor $r_n^{-(d-1)/(d+1)}$ is asymptotically smaller than r_n^{-1}.*

Theorem 3.3. *Under the conditions of Theorem 3.2, there exist sets S for which*

$$\liminf_{n\to\infty} r_n^{\frac{d-1}{d+1}} n^{\frac{2}{d+1}} \mathbb{E}(d_\mu(S, S_n)) > 0.$$

Remark 3.3. *We conjecture that*

$$\liminf_{n\to\infty} r_n^{\frac{d-1}{d+1}} n^{\frac{2}{d+1}} \mathbb{E}(d_\mu(S, S_n)) > 0$$

for any set S under the conditions of Theorem 3.2. The proof relies on the following "local convexity" property, which we think S fulfills. We say that S is "locally *convex" in $B(s,\tau)\cap\partial S$ for $s\in\partial S$ and $\tau>0$ if there exists $\varepsilon>0$ such that for all $t\in B(s,\tau)\cap\partial S$, the set $B(t,\varepsilon)\cap S$ is contained in the halfspace $\{x\in\mathbb{R}^d:\langle x-t,\eta(t)\rangle\leq 0\}$, being $\eta(t)$ the outward pointing unit normal vector at t.*

4 Unavoidable families in $\mathbb{R}^d$

According to the definition of the estimator S_n given in (2) and by Remark 2.1, we have

$$\begin{aligned}\mathbb{E}(d_\mu(S,S_n)) &= \mathbb{E}(\mu(S\setminus S_n))=\mathbb{E}(\mu\{x\in S: x\notin S_n\})\\ &= \mathbb{E}\int_S \mathbb{I}_{\{x\notin S_n\}}\mu(dx)=\int_S P(x\notin S_n)\mu(dx)\\ &= \int_S P(\exists y\in B(x,r_n): B(y,r_n)\cap\mathcal{X}_n=\emptyset)\mu(dx). \qquad (6)\end{aligned}$$

where $\mathbb{I}_A$ denotes the indicator function on $A\subset\mathbb{R}^d$. In order to bound (6), we make use of the concept of unavoidable family of sets, defined below.

Definition 4.1. *Let $x\in\mathbb{R}^d$, $r>0$ and $\mathcal{E}_{x,r}=\{B(y,r): y\in B(x,r)\}$. The family of sets $\mathcal{U}_{x,r}$ is said to be unavoidable for $\mathcal{E}_{x,r}$ if, for all $B(y,r)\in\mathcal{E}_{x,r}$, there exists $U\in\mathcal{U}_{x,r}$ such that $U\subset B(y,r)$.*

As a consequence of Definition 4.1, if $\mathcal{U}_{x,r_n}$ is an unavoidable family of sets for $\mathcal{E}_{x,r_n}$, then $\{\exists y\in B(x,r_n): B(y,r_n)\cap\mathcal{X}_n=\emptyset\}\subset\{\exists U\in\mathcal{U}_{x,r_n}: U\cap\mathcal{X}_n=\emptyset\}$ and

$$P(\exists y\in B(x,r_n): B(y,r_n)\cap\mathcal{X}_n=\emptyset)\leq P(\exists U\in\mathcal{U}_{x,r_n}: U\cap\mathcal{X}_n=\emptyset).$$

Moreover, if $\mathcal{U}_{x,r_n}$ is a finite family,

$$\begin{aligned}P(\exists y\in B(x,r_n): B(y,r_n)\cap\mathcal{X}_n=\emptyset) &\leq P(\exists U\in\mathcal{U}_{x,r_n}: U\cap\mathcal{X}_n=\emptyset)\\ &\leq \sum_{U\in\mathcal{U}_{x,r_n}} P(U\cap\mathcal{X}_n=\emptyset)\\ &= \sum_{U\in\mathcal{U}_{x,r_n}} P(\forall X_j, j=1,\ldots,n, X_j\notin U)\\ &= \sum_{U\in\mathcal{U}_{x,r_n}} (1-P_X(U))^n. \qquad (7)\end{aligned}$$

To sum up, if we define for each $x\in S$ a family $\mathcal{U}_{x,r_n}$ unavoidable and finite for $\mathcal{E}_{x,r_n}$ then, from (6) and (7), it follows that

$$\begin{aligned}\mathbb{E}(d_\mu(S,S_n)) &= \int_S P(\exists y\in B(x,r_n): B(y,r_n)\cap\mathcal{X}_n=\emptyset)\mu(dx)\\ &\leq \int_S \sum_{U\in\mathcal{U}_{x,r_n}}(1-P_X(U))^n\mu(dx). \qquad (8)\end{aligned}$$

From (8) it is apparent that the problem of finding an upper bound for $\mathbb{E}(d_\mu(S,S_n))$ reduces to the problem of finding a lower bound for $P_X(U)$, for all $U\in\mathcal{U}_{x,r_n}$. In view of (8) it would be desirable that both the lower bound and the number of elements of the family $\mathcal{U}_{x,r_n}$ depend in the simplest possible way on the point x.

Given a point $x \in S$, there is not just one possible unavoidable family $\mathcal{U}_{x,r_n}$ and the sets $U \subset \mathcal{U}_{x,r_n}$ can substantially change from one family to another. It is important to note that the shape of U determines the value of $P_X(U)$. Therefore, the choice of $\mathcal{U}_{x,r_n}$ is a crucial point in the resolution of (8). Proposition 4.1 defines a finite family of unavoidable sets for $\mathcal{E}_{x,r_n}$ when $x \in S$ and $d(x, \partial S) > r_n/2$. The result also gives a lower bound for the probability of such sets, which is independent of x. In the same manner, Proposition 4.2 defines a finite family of unavoidable sets for $\mathcal{E}_{x,r_n}$ and gives a lower bound for the probability of such sets when $x \in S$ and $d(x, \partial S) \leq r_n/2$. In that case the probability depends on the distance from x to the boundary of the set. Moreover, the number of sets that form the unavoidable families is independent of x in both situations. For the proofs of these results we refer to Pateiro-López (2008).

Proposition 4.1. *Let S be a nonempty compact subset of $\mathbb{R}^d$ such that a ball of radius $\alpha > 0$ rolls freely in S and in $\overline{S^c}$. Let X be a random variable with probability distribution P_X and support S. We assume that the probability distribution P_X satisfies that there exists $\delta > 0$ such that*

$$P_X(C) \geq \delta\mu(C \cap S)$$

for all Borel subset $C \subset \mathbb{R}^d$.

Then, for all $x \in S$ such that $d(x, \partial S) > r_n/2$, there exists a finite family $\mathcal{U}_{x,r_n}$ with m_1 elements, unavoidable for $\mathcal{E}_{x,r_n}$ and that satisfies

$$P_X(U) \geq L_1 r_n^d, \quad U \in \mathcal{U}_{x,r_n},$$

where the constant $L_1 > 0$ is independent of x.

Proposition 4.2. *Let S be a nonempty compact subset of $\mathbb{R}^d$ such that a ball of radius $\alpha > 0$ rolls freely in S and in $\overline{S^c}$. Let X be a random variable with probability distribution P_X and support S. We assume that the probability distribution P_X satisfies that there exists $\delta > 0$ such that*

$$P_X(C) \geq \delta\mu(C \cap S)$$

for all Borel subset $C \subset \mathbb{R}^d$.

Then, for all $x \in S$ such that $d(x, \partial S) \leq r_n/2$, there exists a finite family $\mathcal{U}_{x,r_n}$ with m_2 elements, unavoidable for $\mathcal{E}_{x,r_n}$ and that satisfies

$$P_X(U) \geq L_2 r_n^{\frac{d-1}{2}} d(x, \partial S)^{\frac{d+1}{2}}, \quad U \in \mathcal{U}_{x,r_n},$$

where the constant $L_2 > 0$ is independent of x.

5 Proofs

Proof of Theorem 3.1. Recall that, according to the definition of the estimator S_n and by (6),

$$\mathbb{E}(d_\mu(S, S_n)) = \mathbb{E}(\mu(S \setminus S_n)) = \int_S P(\exists y \in B(x, r_n) : B(y, r_n) \cap \mathcal{X}_n = \emptyset)\mu(dx).$$

Let us first assume that $\lim_{n\to\infty} nr_n^d = \infty$. We shall see that, for almost all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} P(\exists y \in B(x, r_n) : B(y, r_n) \cap \mathcal{X}_n = \emptyset) = 0. \tag{9}$$

Note that if (9) holds, then by the dominated convergence theorem

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}(d_\mu(S, S_n)) &= \lim_{n\to\infty} \int_S P(\exists y \in B(x, r_n) : B(y, r_n) \cap \mathcal{X}_n = \emptyset)\mu(dx) \\ &= \int_S \lim_{n\to\infty} P(\exists y \in B(x, r_n) : B(y, r_n) \cap \mathcal{X}_n = \emptyset)\mu(dx) \\ &= 0. \end{aligned} \tag{10}$$

Let $\mathbb{S}_d = \{u \in \mathbb{R}^d : \|u\| = 1\}$ be the unit sphere in $\mathbb{R}^d$. For $u \in \mathbb{S}_d$, we define the sets $C_u = \{x \in \mathbb{R}^d : \langle x, u\rangle \geq \|x\| \cos \pi/6\}$ and the generalized circular sectors $C_{u,r} = C_u \cap B(0, r)$, see Figure 1. For each $x \in S$ let us consider the family

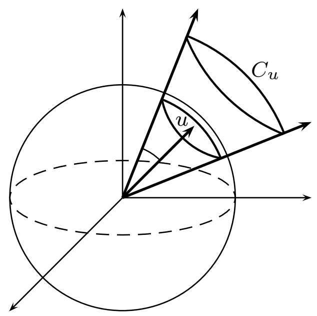

Figure 1: C_u *in* $\mathbb{R}^3$.

$\mathcal{U}_{x,r_n} = \{U^u_{x,r_n}, u \in \mathcal{W}\}$, where $\mathcal{W}$ is a finite family of unit vectors such that

$$B(0, r) = \bigcup_{u\in\mathcal{W}} C_{u,r},$$

for all $r > 0$ (it can be proved that $\mathcal{W}$ exists and is well defined) and, for each $u \in \mathcal{W}$, $U^u_{x,r_n} = \{x\} \oplus C_{u,r_n}$ is the translation of the set C_{u,r_n} by x. It follows easily that $\mathcal{U}_{x,r_n}$ is a finite unavoidable family for $\mathcal{E}_{x,r_n}$. Denote by m the number of sets of $\mathcal{U}_{x,r_n}$, which coincides with the number of unit vectors of $\mathcal{W}$. Then, using the same argument as in (8) we have that

$$P(\exists y \in B(x, r_n) : B(y, r_n) \cap \mathcal{X}_n = \emptyset) \leq \sum_{u\in\mathcal{W}} (1 - P_X(U^u_{x,r_n}))^n. \tag{11}$$

In order to give a lower bound for $P_X(U^u_{x,r_n})$ in (11) it will be useful following general version of the Lebesgue density theorem. See Devroye (1983) for the proof of the lemma.

Lemma 5.1 (Lebesgue density theorem, Devroye (1983)). *If f is a density in $\mathbb{R}^d$ and A is a compact set of $\mathbb{R}^d$ with $\mu(A) > 0$, then*

$$\lim_{h\to 0} \frac{1}{\mu(hA)} \int_{\{x\}\oplus hA} f(y)dy = f(x), \text{ almost all } x.$$

Lemma 5.1 gives us the key to bounding the probability of small compact sets in a neighbourhood of the point x, from the value of the density in x and the Lebesgue measure of the set. Thus, let us consider the compact set $C_{u,1}$ and $h > 0$. We have that

$$\{x\} \oplus hC_{u,1} = \{x\} \oplus C_{u,h} = U^u_{x,h}.$$

It follows from Lemma 5.1 that for almost all x, there exists h_x such that for all $h \le h_x$ we have

$$P_X(U^u_{x,h}) = \int_{U^u_{x,h}} f(y)dy \ge \frac{f(x)}{2}\mu(C_{u,h}). \tag{12}$$

For each $n \in \mathbb{N}$ let $h_n \equiv h_{n,x} = \min(r_n, h_x)$. Then $U^u_{x,h_n} \subset U^u_{x,r_n}$ and we can apply (12) to conclude that

$$P_X(U^u_{x,r_n}) \ge P_X(U^u_{x,h_n}) \ge \frac{f(x)}{2}\mu(C_{u,h_n}) \ge \frac{f(x)}{2}\frac{\mu(B(0,h_n))}{m} = \frac{f(x)}{2}\frac{w_d h_n^d}{m} = L_x h_n^d,$$

where, if $x \in S$, $L_x = \frac{f(x)w_d}{2m} > 0$. Returning to (11) we have

$$P(\exists y \in B(x,r_n) : B(y,r_n) \cap \mathcal{X}_n = \emptyset) \le m(1 - L_x h_n^d)^n \le me^{-nL_x h_n^d}.$$

The last inequality follows from the fact that $(1 - z)^n \le \mathrm{e}^{-nz}$, for $z \in [0, 1]$. Note that we can guarantee that $L_x h_n^d \le 1$ since $L_x h_n^d \le P_X(U^u_{x,r_n})$. Then

$$\lim_{n\to\infty} P(\exists y \in B(x,r_n) : B(y,r_n) \cap \mathcal{X}_n = \emptyset) \le \lim_{n\to\infty} me^{-nL_x h_n^d}.$$

Finally, the definition of h_n and the assumption $\lim_{n\to\infty} nr_n^d = \infty$ yield $\lim_{n\to\infty} nL_x h_n^d = \infty$. As a consequence,

$$\lim_{n\to\infty} P(\exists y \in B(x,r_n) : B(y,r_n) \cap \mathcal{X}_n = \emptyset) = 0, \text{ for almost all } x \in S,$$

which yields (10).

We now prove the converse assertion. Let us assume that $\lim_{n\to\infty} \mathbb{E}(d_\mu(S, S_n)) = 0$. Note that

$$\begin{aligned} P(\exists y \in B(x,r_n) : B(y,r_n) \cap \mathcal{X}_n = \emptyset) &\ge P(B(x,r_n) \cap \mathcal{X}_n = \emptyset) \\ &= (1 - P_X(B(x,r_n)))^n. \end{aligned} \tag{13}$$

If the sequence $\{nr_n^d\}$ does not converge to infinity as $n \to \infty$, then we may find a bounded subsequence $\{n_k r_{n_k}^d\}$. Therefore, there exists $M > 0$ such that $n_k r_{n_k}^d \le M$ for all n_k and as an immediate consequence $\lim_{k\to\infty} r_{n_k}^d = 0$. In this case Lemma 5.1 ensures that, for almost all x, for large enough k,

$$P_X(B(x,r_{n_k})) = \int_{B(x,r_{n_k})} f(y)dy \le 2f(x)\mu(B(0,r_{n_k})) = 2f(x)w_d r_{n_k}^d = L_x r_{n_k}^d, \tag{14}$$

where now $L_x = 2f(x)w_d$. In order to simplify the notation let

$$\Psi_n(x) = P(\exists y \in B(x, r_n) : B(y, r_n) \cap \mathcal{X}_n = \emptyset)$$

and consider the subsequence $\{\Psi_{n_k}(x)\}$. We now combine (13) and (14) to get

$$\begin{aligned}
\liminf_{k\to\infty} \Psi_{n_k}(x) &\geq \liminf_{k\to\infty}(1 - P_X(B(x, r_{n_k})))^{n_k} \\
&\geq \liminf_{k\to\infty}(1 - L_x r_{n_k}^d)^{n_k} \\
&\geq \liminf_{k\to\infty} \exp\left(\frac{-n_k L_x r_{n_k}^d}{1 - L_x r_{n_k}^d}\right) \\
&\geq e^{-L_x M}. \qquad (15)
\end{aligned}$$

We have used that $(1-z)^n \geq \exp(-nz/(1-z))$ for $z \in [0, 1)$. The case when $z = 0$ is straightforward and for $z \in (0, 1)$ write $(1-z)^n = \exp(n \log(1-z))$ and use the fact that $\log(1-z) > -z/(1-z)$. The last inequality holds since $\lim_{k\to\infty} r_{n_k}^d = 0$ and $\{n_k r_{n_k}^d\}$ is bounded by M. By the Fatou's Lemma and (15) we obtain

$$\lim_{k\to\infty} \mathbb{E}(d_\mu(S, S_{n_k})) = \lim_{k\to\infty} \int_S \Psi_{n_k}(x)\mu(dx)$$
$$= \liminf_{k\to\infty} \int_S \Psi_{n_k}(x)\mu(dx) \geq \int_S \liminf_{k\to\infty} \Psi_{n_k}(x)\mu(dx) > 0,$$

which is a contradiction since we are assuming that $\lim_{n\to\infty} \mathbb{E}(d_\mu(S, S_n)) = 0$ and hence every subsequence of $\mathbb{E}(d_\mu(S, S_n))$ must also converge to zero. So, the sequence $\{nr_n^d\}$ must converge to infinity and this concludes the proof of the theorem. □

Proof of Theorem 3.2. Recall that, if we define for each $x \in S$ a family $\mathcal{U}_{x,r_n}$ unavoidable and finite for $\mathcal{E}_{x,r_n}$, then

$$\begin{aligned}
\mathbb{E}(d_\mu(S, S_n)) &\leq \int_S \sum_{U\in\mathcal{U}_{x,r_n}} (1 - P_X(U))^n \mu(dx) \\
&\leq \int_S \sum_{U\in\mathcal{U}_{x,r_n}} \exp(-nP_X(U))\mu(dx).
\end{aligned}$$

The last inequality follows by applying that $(1-z)^n \leq \mathrm{e}^{-nz}$, for $z \in [0, 1]$. We divide S into two subsets

$$S = \left\{x \in S : \quad d(x, \partial S) > \frac{r_n}{2}\right\} \cup \left\{x \in S : \quad d(x, \partial S) \leq \frac{r_n}{2}\right\}$$

and then

$$\begin{aligned}
\mathbb{E}(d_\mu(S, S_n)) \leq& \int_{\left\{x\in S:\ d(x,\partial S) > \frac{r_n}{2}\right\}} \sum_{U\in\mathcal{U}_{x,r_n}} \exp(-nP_X(U))\mu(dx) \\
&+ \int_{\left\{x\in S:\ d(x,\partial S) \leq \frac{r_n}{2}\right\}} \sum_{U\in\mathcal{U}_{x,r_n}} \exp(-nP_X(U))\mu(dx). \qquad (16)
\end{aligned}$$

For those $x \in S$ such that $d(x, \partial S) > r_n/2$ we make use of the families $\mathcal{U}_{x,r_n}$ given in Proposition 4.1. Recall that Proposition 4.1 ensures the existence of suitable finite families $\mathcal{U}_{x,r_n}$ and provides a lower bound on the probability of the sets U, independent of x. Thus,

$$\begin{aligned}
&\int_{\left\{x\in S:\ \ d(x,\partial S)>\frac{r_n}{2}\right\}} \sum_{U\in\mathcal{U}_{x,r_n}} \exp(-nP_X(U))\mu(dx) \\
&\quad\leq \int_{\left\{x\in S:\ \ d(x,\partial S)>\frac{r_n}{2}\right\}} m_1 \exp(-nL_1 r_n^d)\mu(dx) \\
&\qquad = O\left(\mathrm{e}^{-L_1 n r_n^d}\right),
\end{aligned} \tag{17}$$

where m_1 denotes the finite number of elements of $\mathcal{U}_{x,r_n}$. Note that m_1 is also independent of x. Now, for those $x \in S$ such that $d(x, \partial S) \leq r_n/2$, we may consider the unavoidable families $\mathcal{U}_{x,r_n}$ given in Proposition 4.2. Let m_2 be the number of elements of $\mathcal{U}_{x,r_n}$. We have that

$$\begin{aligned}
&\int_{\left\{x\in S:\ \ d(x,\partial S)\leq\frac{r_n}{2}\right\}} \sum_{U\in\mathcal{U}_{x,r_n}} \exp(-nP_X(U))\mu(dx) \\
&\leq \int_{\left\{x\in S:\ \ d(x,\partial S)\leq\frac{r_n}{2}\right\}} m_2 \exp\left(-L_2 n r_n^{\frac{d-1}{2}} d(x,\partial S)^{\frac{d+1}{2}}\right)\mu(dx) \\
&\qquad = \int_{\mathcal{T}^{-1}([0,r_n/2])} g(\mathcal{T}(x))\mu(dx),
\end{aligned}$$

where $\mathcal{T} : S \to \mathbb{R}$ is defined as $\mathcal{T}(x) = d(x, \partial S)$ and $g(z) = m_2 \exp(-L_2 n r_n^{\frac{d-1}{2}} z^{\frac{d+1}{2}})$. It follows from the change of variables formula (see Theorem 16.12 of Billingsley (1995)) that

$$\int_{\mathcal{T}^{-1}([0,r_n/2])} g(\mathcal{T}(x))\mu(dx) = \int_{[0,r_n/2]} g(\rho)\mu\mathcal{T}^{-1}(d\rho)$$

where $\rho = \mathcal{T}(x)$ and $\mu\mathcal{T}^{-1}$ is the measure on $\mathbb{R}$ defined by $\mu\mathcal{T}^{-1}(A) = \mu(\mathcal{T}^{-1}(A))$, for $A \subset \mathbb{R}$. The measure $\mu\mathcal{T}^{-1}$ is characterized by $F(z) = \mu\{x \in S :\ d(x, \partial S) \leq z\}$. Under the stated conditions it can be proved that, for $0 \leq z < \alpha$, $F(z)$ is a polynomial of degree at most d in z, see Federer (1959). Therefore, it is a differentiable function and $F'(z)$ is bounded on compact sets. In short, we obtain

$$\begin{aligned}
&\int_{[0,r_n/2]} g(\rho)\mu\mathcal{T}^{-1}(d\rho) \\
&= \int_{[0,r_n/2]} m_2 \exp\left(-L_2 n r_n^{\frac{d-1}{2}} \rho^{\frac{d+1}{2}}\right) F'(\rho)d\rho \\
&\leq K \int_0^{\frac{r_n}{2}} m_2 \exp\left(-L_2 n r_n^{\frac{d-1}{2}} \rho^{\frac{d+1}{2}}\right) d\rho \\
&= K \int_0^{\frac{L_2 n}{2^{(d+1)/2}} r_n^d} m_2 \frac{1}{\frac{d+1}{2} L_2^{2/(d+1)}} r_n^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}} \mathrm{e}^{-v} v^{\frac{1-d}{d+1}} dv \\
&= O\left(r_n^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}}\right),
\end{aligned} \tag{18}$$

where we have used the change of variables formula $v = L_2 n r_n^{\frac{d-1}{2}} \rho^{\frac{d+1}{2}}$ and also the fact that $\int_0^\infty \mathrm{e}^{-v} v^{\frac{1-d}{d+1}} dv < \infty$. Turning to the computation of $\mathbb{E}(d_\mu(S, S_n))$ in (16), it follows from (17) and (18) that

$$\mathbb{E}(d_\mu(S, S_n)) = O\left(\mathrm{e}^{-L_1 n r_n^d} + r_n^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}}\right). \tag{19}$$

Now if (4) holds, then for all $M > 0$ there exists $N \in \mathbb{N}$ such that $n r_n^d \geq M \log n$, for all $n \geq N$ and hence

$$\mathrm{e}^{-L_1 n r_n^d} \leq \mathrm{e}^{-L_1 M \log n} = n^{-L_1 M}.$$

As a consequence

$$\limsup_{n\to\infty} \frac{\mathrm{e}^{-L_1 n r_n^d}}{r_n^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}}} \leq \limsup_{n\to\infty} \frac{n^{-L_1 M}}{r_n^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}}} = \limsup_{n\to\infty} r_n^{\frac{d-1}{d+1}} n^{\left(\frac{2}{d+1} - L_1 M\right)} = 0, \tag{20}$$

for large enough M. Remember that r_n is bounded ($r_n \leq \alpha$ by assumption). We now combine (19) and (20) to obtain

$$\mathbb{E}(d_\mu(S, S_n)) = O\left(\mathrm{e}^{-L_1 n r_n^d} + r_n^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}}\right) = O\left(r_n^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}}\right),$$

which completes the proof. □

Proof of Theorem 3.3. Let $S = B(0, \alpha)$ and assume that the distribution P_X is uniform on S. Our aim is to find a lower bound for $\mathbb{E}(d_\mu(S, S_n))$. Thus,

$$\begin{aligned} \mathbb{E}(d_\mu(S, S_n)) &= \int_S P(\exists y \in B(x, r_n) : B(y, r_n) \cap \mathcal{X}_n = \emptyset) \mu(dx) \\ &\geq \int_{\left\{x \in S:\ d(x, \partial S) \leq \frac{r_n}{2}\right\}} P(\exists y \in B(x, r_n) : B(y, r_n) \cap \mathcal{X}_n = \emptyset) \mu(dx). \end{aligned}$$

For each $x \in S$ such that $d(x, \partial S) \leq r_n/2$ let $\eta = x / \|x\|$ and

$$\tilde{x} = (\alpha + r_n - d(x, \partial S))\eta = (\|x\| + r_n)\eta. \tag{21}$$

Note that $\tilde{x} \in B(x, r_n)$ and hence

$$P(\exists y \in B(x, r_n) : B(y, r_n) \cap \mathcal{X}_n = \emptyset) \geq P(B(\tilde{x}, r_n) \cap \mathcal{X}_n = \emptyset) = (1 - P_X(B(\tilde{x}, r_n)))^n.$$

In short,

$$\mathbb{E}(d_\mu(S, S_n)) \geq \int_{\left\{x \in S:\ d(x, \partial S) \leq \frac{r_n}{2}\right\}} (1 - P_X(B(\tilde{x}, r_n)))^n \mu(dx), \tag{22}$$

where $\tilde{x}$ is given by (21). First we shall see that $P_X(B(\tilde{x}, r_n)) \leq 1/2$. Under the assumption of the uniform distribution on S, we have

$$P_X(B(\tilde{x}, r_n)) = \frac{\mu(B(\tilde{x}, r_n) \cap S)}{\mu(S)}. \tag{23}$$

Let us consider an orthogonal transformation $\mathcal{O} : \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathcal{O}(\eta) = -e_d$, where $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$. Then

$$\mathcal{O}(B(\tilde{x}, r_n) \cap S) = B(-(\alpha + r_n - d(x, \partial S))e_d, r_n) \cap B(0, \alpha).$$

It is easy to see that

$$B(-(\alpha + r_n - d(x, \partial S))e_d, r_n) \subset \{z \in \mathbb{R}^d : \ \langle z, e_d \rangle \leq 0\}$$

and, since the Lebesgue measure is invariant under orthogonal transformations, we have

$$\begin{aligned} \mu(B(\tilde{x}, r_n) \cap S) &= \mu(B(-(\alpha + r_n - d(x, \partial S))e_d, r_n) \cap B(0, \alpha)) \\ &\leq \mu(\{z \in \mathbb{R}^d : \ \langle z, e_d \rangle \leq 0\} \cap B(0, \alpha)) \\ &= \frac{1}{2}\mu(B(0, \alpha)). \end{aligned} \tag{24}$$

Combine (23) and (24) to get

$$P_X(B(\tilde{x}, r_n)) \leq \frac{1}{2}. \tag{25}$$

We return to (22) to obtain that

$$\begin{aligned} \mathbb{E}(d_\mu(S, S_n)) &\geq \int_{\left\{x \in S: \ \ d(x, \partial S) \leq \frac{r_n}{2}\right\}} (1 - P_X(B(\tilde{x}, r_n)))^n \mu(dx) \\ &\geq \int_{\{x \in S: d(x, \partial S) \leq r_n/2\}} \exp\left(\frac{-nP_X(B(\tilde{x}, r_n))}{1 - P_X(B(\tilde{x}, r_n))}\right) \mu(dx) \\ &\geq \int_{\{x \in S: d(x, \partial S) \leq r_n/2\}} \exp\left(-2nP_X(B(\tilde{x}, r_n))\right) \mu(dx). \end{aligned} \tag{26}$$

We have used again the fact that $(1 - z)^n \geq \exp(-nz/(1 - z))$ for $z \in [0, 1)$ together with (25). In view of (26) we need again an upper bound for $P_X(B(\tilde{x}, r_n))$. The bound in (25) will be now too rough for our purposes and so we shall see that it can be sharpened. Let us now consider the composed function formed by first applying the previous orthogonal transformation $\mathcal{O} : \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathcal{O}(\eta) = -e_d$ and then applying the translation by the vector $(\alpha - d(x, \partial S))e_d$, see Figure 2. Using again that the Lebesgue measure is invariant under orthogonal transformations and translations we have that

$$\mu(B(B(\tilde{x}, r_n) \cap S)) = \mu(B(-r_n e_d, r_n) \cap B((\alpha - d(x, \partial S))e_d, \alpha)).$$

The set $B(-r_n e_d, r_n) \cap B((\alpha - d(x, \partial S))e_d, \alpha)$ is the intersection of two balls with radius r_n and α such that the distance between their centres is equal to $\alpha + r_n - d(x, \partial S)$. Then,

$$B(-r_n e_d, r_n) \cap B((\alpha - d(x, \partial S))e_d, \alpha) = \mathcal{C}(h_1) \cup \mathcal{A}(h_2),$$

where

$$\mathcal{C}(h_1) = \{x \in \mathbb{R}^d : \ -h_1 \leq \langle x, e_d \rangle \leq 0\} \cap B(-r_n e_d, r_n),$$

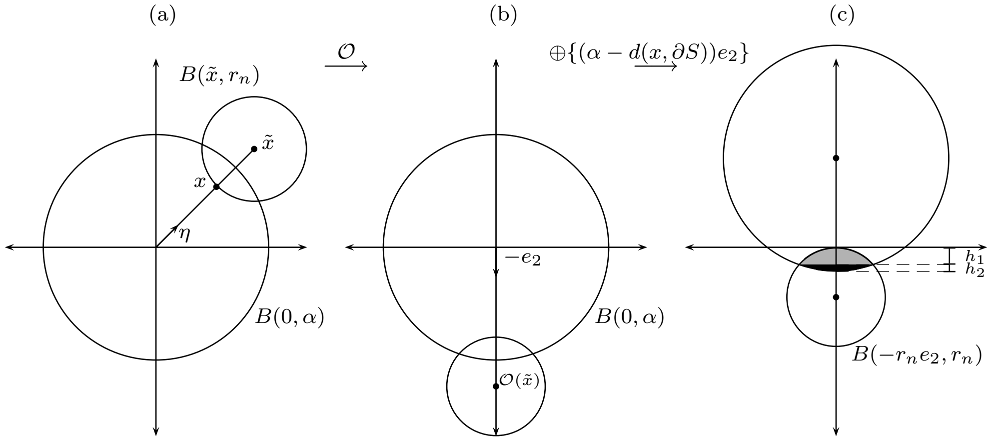

Figure 2: *(a) $B(\tilde{x}, r_n) \cap S$. (b) Result of applying an orthogonal transformation $\mathcal{O} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathcal{O}(\eta) = -e_2$. (c) Translation by the vector $(\alpha - d(x, \partial S))e_2$. In black $\mathcal{A}(h_2)$ and in gray $\mathcal{C}(h_1)$.*

and

$$\mathcal{A}(h_2) = \{z \in \mathbb{R}^d : -(h_1 + h_2) \leq \langle z, e_d \rangle \leq -h_1\} \cap B((\alpha - d(x, \partial S))e_d, \alpha).$$

The values of h_1 and h_2 are easily deduced from the Pythagorean theorem by solving the system

$$\begin{cases} (r_n - h_1)^2 + \lambda^2 = r_n^2, \\ (\alpha - h_2)^2 + \lambda^2 = \alpha^2, \\ h_1 + h_2 = d(x, \partial S). \end{cases}$$

Thus,

$$h_1 = \frac{d(x, \partial S)(2\alpha - d(x, \partial S))}{2(\alpha + r_n - d(x, \partial S))}, \quad h_2 = \frac{d(x, \partial S)(2r_n - d(x, \partial S))}{2(\alpha + r_n - d(x, \partial S))}.$$

Since $\mathcal{C}(h_1)$ and $\mathcal{A}(h_2)$ are disjoint, up to a zero measure set, we have

$$\mu(B(-r_n e_d, r_n) \cap B((\alpha - d(x, \partial S))e_d, \alpha)) = \mu(\mathcal{C}(h_1)) + \mu(\mathcal{A}(h_2)). \tag{27}$$

First, in order to find an upper bound in (27), we shall see that $\mu(\mathcal{A}(h_2)) \leq \mu(\mathcal{C}(h_1))$. It can be easily proved that $\mu(\mathcal{A}(h_2)) = \mu(\mathcal{A}_0(h_2))$, where

$$\mathcal{A}_0(h_2) = \{z = (z_1, \ldots, z_d) \in \mathbb{R}^d : \ 0 \leq \langle z, e_d \rangle \leq h_2\} \cap B(-(\alpha - h_2)e_d, \alpha).$$

Note that $\mathcal{A}_0(h_2)$ is obtained after applying an orthogonal transformation and a translation to $\mathcal{A}(h_2)$. Let $0 \leq l \leq h_2$ and define the set

$$\mathcal{A}_0(h_2, l) = \{z = (z_1, \ldots, l) \in \mathbb{R}^d : \ z \in \mathcal{A}_0(h_2)\}.$$

Then

$$\mu(\mathcal{A}_0(h_2)) = \int_0^{h_2} \mu_{d-1}(\mathcal{A}_0(h_2, l))dl$$

where μ_{d-1} denotes the $(d-1)$-dimensional Lebesgue measure and $\mathcal{A}_0(h_2, l)$ refers to the $(d-1)$-dimensional sphere with centre le_d and radius $s(l)$, being

$$s(l) = \sqrt{\alpha^2 - (\alpha - h_2 + l)^2}.$$

Therefore,

$$\mu(\mathcal{A}(h_2)) = \omega_{d-1} \int_0^{h_2} s(l)^{d-1} dl. \tag{28}$$

Similarly,

$$\mu(\mathcal{C}(h_1)) = \omega_{d-1} \int_0^{h_1} r(l)^{d-1} dl, \tag{29}$$

where $r(l) = \sqrt{r_n^2 - (r_n - h_1 + l)^2}$, for $0 \leq l \leq h_1$. In view of (28) and (29) and since $h_2 \leq h_1$, if we are able to prove that $s(l) \leq r(l)$ for $0 \leq l \leq h_2$, then

$$\mu(\mathcal{A}(h_2)) = \omega_{d-1} \int_0^{h_2} s(l)^{d-1} dl \leq \omega_{d-1} \int_0^{h_2} r(l)^{d-1} dl \leq \omega_{d-1} \int_0^{h_1} r(l)^{d-1} dl = \mu(\mathcal{C}(h_1)).$$

As $r(l) \geq 0$ and $s(l) \geq 0$ it suffices to show that $s(l)^2 \leq r(l)^2$ or, equivalently, $r(l)^2 - s(l)^2 \geq 0$. By construction $r(0)^2 = s(0)^2 = \lambda^2$. and an easy computation shows that $r(l)^2 - s(l)^2$ is an increasing function. Indeed,

$$r(l)^2 - s(l)^2 = 2l(\alpha - r_n + h_1 - h_2) + (h_2^2 - h_1^2 + 2r_n h_1 - 2\alpha h_2) \tag{30}$$

and the derivative of (30) with respect to l satisfies

$$2(\alpha - r_n + h_1 - h_2) \geq 0,$$

since $r_n \leq \alpha$ and $h_2 \leq h_1$. Therefore $s(l) \leq r(l)$ for $0 \leq l \leq h_2$ and $\mu(\mathcal{A}(h_2)) \leq \mu(\mathcal{C}(h_1))$. Now, if we return to the equation (27), we get

$$\mu(B(\tilde{x}, r_n) \cap S) \leq 2\mu(\mathcal{C}(h_1)). \tag{31}$$

We will thus concentrate on $\mathcal{C}(h_1)$. We get

$$\mu(\mathcal{C}(h_1)) \quad = \quad \omega_{d-1} \int_0^{h_1} \left(2r_n t - t^2\right)^{\frac{d-1}{2}} dt.$$

It is immediate that $2r_n t - t^2 \leq 2r_n t$, for $0 \leq t \leq h_1$ and hence

$$\mu(\mathcal{C}(h_1)) \leq \omega_{d-1} \int_0^{h_1} (2r_n t)^{\frac{d-1}{2}} dt = \frac{\omega_{d-1}}{d+1} 2^{\frac{d+1}{2}} r_n^{\frac{d-1}{2}} h_1^{\frac{d+1}{2}}.$$

Since $h_1 \leq d(x, \partial S)$, we have

$$\mu(\mathcal{C}(h_1)) \leq \frac{\omega_{d-1}}{d+1} 2^{\frac{d+1}{2}} r_n^{\frac{d-1}{2}} d(x, \partial S)^{\frac{d+1}{2}}. \tag{32}$$

Combine (31) and (32) to obtain

$$\mu(B(\tilde{x}, r_n) \cap S) \leq \frac{\omega_{d-1}}{d+1} 2^{\frac{d+3}{2}} r_n^{\frac{d-1}{2}} d(x, \partial S)^{\frac{d+1}{2}}.$$

As a consequence,

$$P_X(B(\tilde{x}, r_n)) \leq \frac{1}{\mu(S)} \frac{\omega_{d-1}}{d+1} 2^{\frac{d+3}{2}} r_n^{\frac{d-1}{2}} d(x, \partial S)^{\frac{d+1}{2}} = L r_n^{\frac{d-1}{2}} d(x, \partial S)^{\frac{d+1}{2}}.$$

Finally, if we apply the latter bound to (26), then we have that

$$\begin{aligned} \mathbb{E}(d_\mu(S, S_n)) &\geq \int_{\{x \in S: d(x, \partial S) \leq r_n/2\}} \exp\left(-2nLr_n^{\frac{d-1}{2}} d(x, \partial S)^{\frac{d+1}{2}}\right) \mu(dx) \\ &= \int_{\mathcal{T}^{-1}([0, r_n/2])} g(\mathcal{T}(x)) \mu(dx), \end{aligned}$$

where $\mathcal{T} : S \to \mathbb{R}$ is defined as $\mathcal{T}(x) = d(x, \partial S)$ and $g(z) = \exp(-2nLr_n^{\frac{d-1}{2}} z^{\frac{d+1}{2}})$. By the change of variables formula (see Theorem 16.12 of Billingsley (1995))

$$\int_{\mathcal{T}^{-1}([0, r_n/2])} g(\mathcal{T}(x)) \mu(dx) = \int_{[0, r_n/2]} g(\rho) \mu \mathcal{T}^{-1}(d\rho)$$

where $\rho = \mathcal{T}(x)$ and $\mu\mathcal{T}^{-1}$ is the measure on $\mathbb{R}$ defined by $\mu\mathcal{T}^{-1}(A) = \mu(\mathcal{T}^{-1}(A))$, for $A \subset \mathbb{R}$. The measure $\mu\mathcal{T}^{-1}$ is characterized by $F(z) = \mu\{x \in S : \ d(x, \partial S) \leq z\}$. We know from Federer (1959) that $F(z)$ is a polynomial of degree at most d in z. In fact, in this particular case, for $z < \alpha$, $F(z) = \omega_d(\alpha^d - (\alpha - z)^d)$. Therefore F is differentiable and

$$\begin{aligned} \mathbb{E}(d_\mu(S, S_n)) &\geq \int_{[0, r_n/2]} g(\rho) \mu \mathcal{T}^{-1}(d\rho) \\ &= \int_0^{r_n/2} \exp\left(-2nLr_n^{\frac{d-1}{2}} \rho^{\frac{d+1}{2}}\right) F'(\rho) d\rho \\ &= \int_0^{r_n/2} \exp\left(-2nLr_n^{\frac{d-1}{2}} \rho^{\frac{d+1}{2}}\right) \omega_d d(\alpha - \rho)^{d-1} d\rho. \end{aligned}$$

It is immediate to show that for $0 \leq \rho \leq r_n/2$ the function $F'(\rho) = \omega_d d(\alpha - \rho)^{d-1}$ is decreasing with $F'(\rho) \geq F'(r_n/2) = \omega_d d(\alpha - r_n/2)^{d-1} \geq \omega_d d(\alpha/2)^{d-1}$. Therefore

$$\begin{aligned} \mathbb{E}(d_\mu(S, S_n)) &\geq \int_0^{r_n/2} \exp\left(-2nLr_n^{\frac{d-1}{2}} \rho^{\frac{d+1}{2}}\right) \omega_d d \left(\frac{\alpha}{2}\right)^{d-1} d\rho \\ &= \omega_d d \left(\frac{\alpha}{2}\right)^{d-1} \int_0^{\frac{2nL}{2^{(d+1)/2}} r_n^d} \frac{1}{\frac{d+1}{2}(2L)^{2/(d+1)}} r_n^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}} e^{-v} v^{\frac{1-d}{d+1}} dv \\ &= \omega_d d \left(\frac{\alpha}{2}\right)^{d-1} \frac{1}{\frac{d+1}{2}(2L)^{2/(d+1)}} r_n^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}} \int_0^{\frac{2nL}{2^{(d+1)/2}} r_n^d} e^{-v} v^{\frac{1-d}{d+1}} dv. \end{aligned}$$

We have used the change of variables formula with $v = 2nLr_n^{\frac{d-1}{2}} \rho^{\frac{d+1}{2}}$. Therefore

$$\liminf_{n \to \infty} r_n^{\frac{d-1}{d+1}} n^{\frac{2}{d+1}} \mathbb{E}(d_\mu(S, S_n)) \geq \liminf_{n \to \infty} \frac{\omega_d d \, (\alpha/2)^{d-1}}{\frac{d+1}{2}(2L)^{2/(d+1)}} \int_0^{\frac{2nL}{2^{(d+1)/2}} r_n^d} e^{-v} v^{\frac{1-d}{d+1}} dv.$$

Since $nr_n^d \to \infty$, we have

$$\liminf_{n \to \infty} r_n^{\frac{d-1}{d+1}} n^{\frac{2}{d+1}} \mathbb{E}(d_\mu(S, S_n)) \geq \frac{\omega_d d \, (\alpha/2)^{d-1}}{\frac{d+1}{2}(2L)^{2/(d+1)}} \int_0^\infty e^{-v} v^{\frac{1-d}{d+1}} dv > 0.$$

This completes the proof of the theorem. □

Appendix. Rolling condition, reach and α-convexity

The free rolling condition, recall Definition 2.2, has useful implications which are worth noting. In this appendix we list some results relating the rolling condition to the positive reach or the α-convexity of a set are given.

We begin by making some preliminary comments. Assume that a ball of radius $\alpha > 0$ rolls freely in a nonempty closed set $A \subset \mathbb{R}^d$ and let $a \in \partial A$. By definition there exists $x \in A$ such that $a \in B(x, \alpha) \subset A$ and, necessarily, $\|x - a\| = \alpha$. Observe that if $\|x - a\| < \alpha$, then it easily follows that $a \in \mathring{B}(a, \alpha - \|x - a\|) \subset \mathring{B}(x, \alpha) \subset \mathrm{int}(A)$, yielding a contradiction since $a \in \partial A$. Define the unit vector $\eta(a) = (a - x)/\|a - x\|$. Then we can write $B(a - \alpha\eta(a), \alpha) \subset A$ since $x = a - \alpha\eta(a)$. It is important to note that the free rolling condition in A does not imply that the point x and, consequently, the vector $\eta(a)$ are unique, see Figure 3.

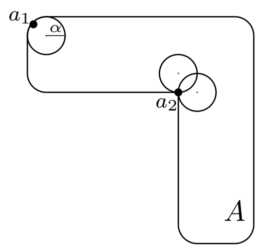

Figure 3: *A ball of radius α rolls freely in A. For the point $a_1 \in \partial A$ there exists a unique $x \in A$ such that $a_1 \in B(x, \alpha) \subset A$. However, for the point $a_2 \in \partial A$, $a_2 \in B(x, \alpha) \subset A$ for infinite $x \in A$.*

Lemma 5.2 shows that the uniqueness of the unit vector $\eta(a)$ such that $B(a - \alpha\eta(a), \alpha) \subset A$ is closely related to the existence of some $x \notin A$ such that a coincides with the metric projection of x onto A.

Lemma 5.2. *Let $A \subset \mathbb{R}^d$ be a nonempty closed set and $a \in \partial A$. Assume that there exists $x \notin A$ such that*

$$\rho = \|x - a\| = d(x, A),$$

that is, a is a metric projection of x onto A. If there exists $\alpha > 0$ and a unit vector $\eta(a)$ such that $B(a - \alpha\eta(a), \alpha) \subset A$, then

$$x = a + \rho\eta(a).$$

Proof. To see this suppose the contrary, that is, suppose that there exists x under the stated conditions such that $x \neq a + \rho\eta(a)$. Then, it can be easily seen that x, a, and $a - \alpha\eta(a)$ cannot lie on the same line and hence,

$$\|a - \alpha\eta(a) - x\| < \|a - \alpha\eta(a) - a\| + \|a - x\| = \alpha + \rho. \tag{33}$$

Now, let $z \in \partial B(a - \alpha\eta(a), \alpha) \cap [x, a - \alpha\eta(a)]$, where $[x, a - \alpha\eta(a)]$ denotes the line segment with endpoints x and $a - \alpha\eta(a)$, see Figure 4. We have

$$\|a - \alpha\eta(a) - x\| = \|a - \alpha\eta(a) - z\| + \|z - x\| = \alpha + \|z - x\|.$$

Therefore, by (33)

$$\|z - x\| = \|a - \alpha\eta(a) - x\| - \alpha < \alpha + \rho - \alpha = \rho,$$

which is a contradiction since $z \in A$ and $\rho = d(x, A)$. □

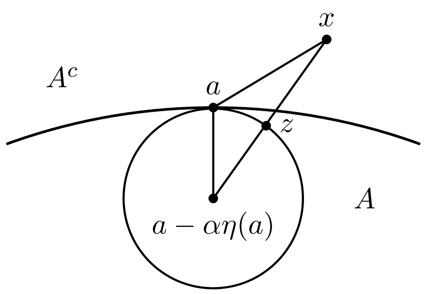

Figure 4: *Elements of Lemma 5.2.*

Remark 5.1. *A direct consequence of Lemma 5.2 is that the vector $\eta(a)$ is unique, whenever a is the metric projection of some $x \notin A$ onto A. Another interpretation is that if $a \in \partial A$ and there exists more that one ball such that $a \in B(x, \alpha) \subset A$, then a cannot be the metric projection of any point $x \notin A$, see Figure 5.*

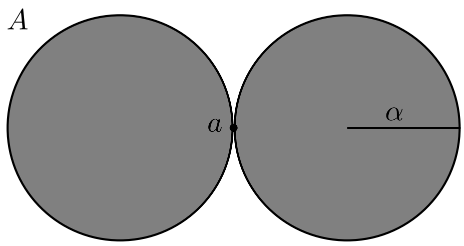

Figure 5: *For the set A in gray and the point $a \in \partial A$ we can find two unit vectors $\eta(a)$ such that $B(a - \alpha\eta(a), \alpha) \subset A$. It follows from Lemma 5.2 that a cannot be the metric projection of any $x \notin A$ onto A.*

The following lemma shows that the rolling condition guarantees some regularity on the boundary of the set.

Lemma 5.3. *Let $A \subset \mathbb{R}^d$ be a nonempty closed set. Assume that a ball of radius $\alpha > 0$ rolls freely in A. Then,*

$$\mathrm{int}(\overline{A^c}) = A^c \quad \textit{and} \quad \partial A = \partial \overline{A^c}.$$

Proof. First we prove that $\mathrm{int}(\overline{A^c}) = A^c$. It is straightforward to see that $A^c \subset \mathrm{int}(\overline{A^c})$ by using that A^c is open. Now we prove that $\mathrm{int}(\overline{A^c}) \subset A^c$. Suppose the contrary, that is, suppose that there exists $x \in \mathrm{int}(\overline{A^c})$ such that $x \notin A^c$. Then, $x \in A \cap \overline{A^c} = \partial A$. By the free rolling condition in A, there exists $p \in A$ such that $x \in B(p, \alpha) \subset A$. Moreover, as we have seen $\|x - p\| = \alpha$. Since $x \in \mathrm{int}(\overline{A^c})$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \overline{A^c}$. Assume that $\varepsilon < \alpha$ and consider the point

$$y_\lambda = x + \lambda \frac{p - x}{\|p - x\|}, \quad \lambda \in (0, \varepsilon).$$

We have $y_\lambda \in \mathring{B}(p, \alpha) \subset \mathrm{int}(A)$. We get a contradiction since $y_\lambda \in B(x, \varepsilon) \subset \overline{A^c}$. The proof for $\partial A = \partial \overline{A^c}$ is now straightforward if we use that the boundary of a set can be written as the adherence of the set minus its interior. Since A^c is open and $\mathrm{int}(\overline{A^c}) = A^c$, we obtain

$$\partial \overline{A^c} = \overline{\overline{A^c}} \setminus \mathrm{int}(\overline{A^c}) = \overline{A^c} \setminus A^c = \overline{A^c} \setminus \mathrm{int}(A^c) = \partial A^c = \partial A.$$

$\square$

An immediate consequence of Lemma 5.3 is given in the following lemma.

Lemma 5.4. *Let $A \subset \mathbb{R}^d$ be a nonempty closed set. Assume that a ball of radius $\alpha > 0$ rolls freely in A. Then,*

$$A = \overline{\overline{A^c}^c}.$$

Proof. The result is a straightforward application of Lemma 5.3. Use that $\mathrm{int}(\overline{A^c}) = A^c$ to obtain

$$\overline{\overline{A^c}^c} = \mathrm{int}(\overline{A^c})^c = A.$$

$\square$

Remark 5.2. *The set $\overline{\overline{A^c}^c}$ can also be written as $\overline{\mathrm{int}(A)}$. Since A is closed, it is straightforward to verify that $\overline{\mathrm{int}(A)} \subset A$. We then deduce that the rolling condition in A is essential in order to guarantee that $A \subset \overline{\mathrm{int}(A)}$, since in general this is not true, see Figure 6.*

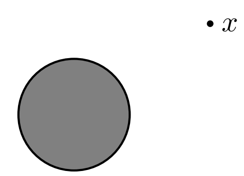

Figure 6: *For $A = B \cup \{x\}$, we have that $\overline{\overline{A^c}^c} = \overline{\mathrm{int}(A)} = B$. Note that A does not fulfill the free rolling condition in A.*

From here on, we will assume that $A \subset \mathbb{R}^d$ is a nonempty closed set such that a ball of radius $\alpha > 0$ rolls freely not only in A but also in $\overline{A^c}$. The implications of this assumption are established in Lemmas 5.6, 5.7, 5.8, and 5.9. First, we would like to comment on the symmetric roles that A and $\overline{A^c}$ play in this assumption. It can be proved that the roles of A and $\overline{A^c}$ are interchangeable in the sense that if a ball of radius $\alpha > 0$ rolls freely in A and in $\overline{A^c}$, then we also have that a ball of radius $\alpha > 0$ rolls freely in $\overline{A^c}$ and in $\overline{\overline{A^c}^c}$. The precise statement is given in Lemma 5.5, which relies on Lemma 5.4.

Lemma 5.5. *Let $A \subset \mathbb{R}^d$ be a nonempty closed set. Assume that a ball of radius $\alpha > 0$ rolls freely in A and in $\overline{A^c}$. Then, a ball of radius $\alpha > 0$ rolls freely in $\overline{A^c}$ and in $\overline{\overline{A^c}^c}$.*

Proof. The result is a direct consequence of Lemma 5.4 which states that $\overline{\overline{A^c}^c} = A$. □

Lemma 5.6. *Let $A \subset \mathbb{R}^d$ be a nonempty closed set. Assume that a ball of radius $\alpha > 0$ rolls freely in A and in $\overline{A^c}$. Then, for all $a \in \partial A$ there exists a unique unit vector $\eta(a)$ such that*

$$B(a - \alpha\eta(a), \alpha) \subset A \quad \text{and} \quad B(a + \alpha\eta(a), \alpha) \subset \overline{A^c}.$$

Proof. Let $a \in \partial A$. By the free rolling condition in A, there exists $x \in A$ such that $a \in B(x, \alpha) \subset A$. Moreover, x can be written as $x = a - \alpha\eta(a)$, where $\eta(a) = (a - x)/\|a - x\|$. By Lemma 5.3, $\partial A = \partial\overline{A^c}$ and hence $a \in \partial\overline{A^c}$. The free rolling condition in $\overline{A^c}$ yields that there exists $y \in \overline{A^c}$ such that $a \in B(y, \alpha) \subset \overline{A^c}$ and then $\|y - a\| = d(y, A) = \alpha$, that is, a is the metric projection of $y \notin A$ onto A. It follows from Lemma 5.2 that

$$y = a + \alpha\eta(a),$$

and therefore $B(a + \alpha\eta(a), \alpha) \subset \overline{A^c}$. □

Remark 5.3. *Note that by Lemma 5.3 we can conclude that if $B(a + \alpha\eta(a), \alpha) \subset \overline{A^c}$, then $\mathring{B}(a + \alpha\eta(a), \alpha) \subset A^c$, since* $\mathrm{int}(\overline{A^c}) = A^c$.

Next we focus on the relation between the free rolling condition and the positive reach of a set. The reach of a nonempty set A, reach(A), is defined as the largest α, possibly infinity, such that if $x \in \mathbb{R}^d$ and $d(x, A) < \alpha$, then the metric projection of x onto A is unique. Lemma 5.7 states that if A is a nonempty closed subset of $\mathbb{R}^d$ such that a ball of radius α rolls freely in A and in $\overline{A^c}$, then ∂A has positive reach, being reach$(\partial A) \geq \alpha$. As a consequence every point whose distance to ∂A is lower than α has a unique metric projection onto ∂A.

Lemma 5.7. *Let $A \subset \mathbb{R}^d$ be a nonempty closed set. Assume that a ball of radius $\alpha > 0$ rolls freely in A and in $\overline{A^c}$. Then, for all $x \in \mathbb{R}^d$ such that $\rho = d(x, \partial A) < \alpha$ there exists a unique point $a \in \partial A$ such that $\|x - a\| = d(x, \partial A)$. That is, the reach of ∂A is greater or equal to α.*

Proof. Let $x \in \mathbb{R}^d$ such that $\rho = d(x, \partial A) < \alpha$. We can assume that $x \notin \partial A$ since the result is trivial otherwise. First, suppose that $x \notin A$. If there exist two metric projections of x onto ∂A, namely a_1 and a_2, then by the free rolling condition in A and by Lemmas 5.2 and 5.6, we have that

$$x = a_1 + \rho\eta(a_1) = a_2 + \rho\eta(a_2),$$

where $\eta(a_1)$ and $\eta(a_2)$ are the unique unit vector such that

$$B(a_i - \alpha\eta(a_i), \alpha) \subset A \quad \text{and} \quad B(a_i + \alpha\eta(a_i), \alpha) \subset \overline{A^c}, \quad i = 1, 2.$$

The points $x, a_2 + \alpha\eta(a_2)$, and a_1 cannot lie on the same line. Otherwise

$$a_1 = a_2 + \lambda\eta(a_2)$$

for some $\lambda \in \mathbb{R}$. But by assumption $a_1 = a_2 + \rho\eta(a_2) - \rho\eta(a_1)$ and hence $|\lambda - \rho| = \rho$, that is, $\lambda = 0$ or $\lambda = 2\rho$. None of these two values is valid. First, $\lambda = 0$ yields $a_1 = a_2$ which is a contradiction since we are assuming that both points are different. Second, $\lambda = 2\rho < 2\alpha$ yields

$$\|a_1 - (a_2 + \alpha\eta(a_2))\| = |2\rho - \alpha| < \alpha,$$

and hence $a_1 \in \mathring{B}(a_2 + \alpha\eta(a_2), \alpha) \subset A^c$, which is another contradiction since $a_1 \in \partial A$. Therefore, x, $a_2 + \alpha\eta(a_2)$, and a_1 do not lie on the same line. Finally, using the strict triangle inequality and $\rho \leq \alpha$ we have that

$$\|a_1 - (a_2 + \alpha\eta(a_2))\| < \|a_1 - x\| + \|x - (a_2 + \alpha\eta(a_2))\| = \rho + (\alpha - \rho) = \alpha.$$

This is again a contradiction since $a_1 \in \partial A$. Therefore, the projection onto ∂A of $x \notin A$ such that $\rho = d(x, \partial A) < \alpha$ is unique. Now suppose that $x \in A$. Since we are assuming that $x \notin \partial A$ it can be easily seen that $x \notin \overline{A^c}$. Moreover, $\partial\overline{A^c} = \partial A$ by Lemma 5.3 and hence $d(x, \partial\overline{A^c}) < \alpha$. The result is now straightforward if we repeat the same steps as before and use that, by Lemma 5.5, the roles of A and $\overline{A^c}$ are interchangeable.

□

Therefore, Lemma 5.7 proves that a sufficient condition for $\partial A \subset \mathbb{R}^d$ to have positive reach is that a ball of radius $\alpha > 0$ rolls freely in A and in $\overline{A^c}$. It is convenient to note, as it is shown in Figure 7, that it is not enough that a ball of radius α rolls freely in A in order to guarantee that $\text{reach}(\partial A) \geq \alpha$. The same occurs if a ball of radius α only rolls freely in $\overline{A^c}$, see Figure 8. In Lemma 5.8 we state a useful application of Lemma 5.7.

Lemma 5.8. *Let $A \subset \mathbb{R}^d$ be a nonempty closed set. Assume that a ball of radius $\alpha > 0$ rolls freely in A and in $\overline{A^c}$. Then A and $\overline{A^c}$ are both sets with positive reach, being* $\text{reach}(A)$ *and* $\text{reach}(\overline{A^c})$ *greater or equal to α.*

Proof. The result is an immediate consequence of Lemma 5.7. □

Finally, it remains to establish the relation between the rolling condition and the α-convexity, recall Definition 2.1. Lemma 5.9 states the result.

Lemma 5.9. *Let $A \subset \mathbb{R}^d$ be a nonempty closed set. Assume that a ball of radius $\alpha > 0$ rolls freely in A and in $\overline{A^c}$. Then A and $\overline{A^c}$ are both α-convex.*

Proof. First we shall prove that $A = C_\alpha(A)$. Since by definition $A \subset C_\alpha(A)$, it suffices to show that if $x \in A^c$ then $x \notin C_\alpha(A)$. Thus, let $x \in A^c$ and $\rho = d(x, A)$. If $\rho \geq \alpha$, then $x \in \mathring{B}(x, \alpha) \subset A^c$ and therefore $x \notin C_\alpha(A)$. If $\rho < \alpha$, then by Lemmas 5.8 and 5.6 there exists a unique point $a \in \partial A$ and a unique unit vector $\eta(a)$ such that $x = a + \rho\eta(a)$ and

$$x \in \mathring{B}(a + \alpha\eta(a), \alpha) \subset A^c,$$

which yields $x \notin C_\alpha(A)$. It remains to proof that $\overline{A^c}$ is α-convex. The result is an immediate consequence of the latter and Lemma 5.4. □

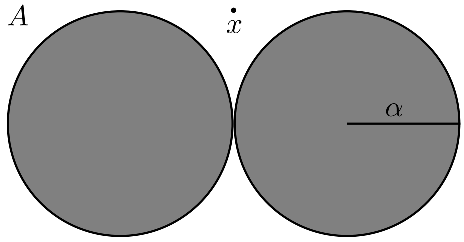

Figure 7: *A ball of radius α rolls freely in A, $d(x, \partial A) < \alpha$, and the metric projection of x onto ∂A is not unique.*

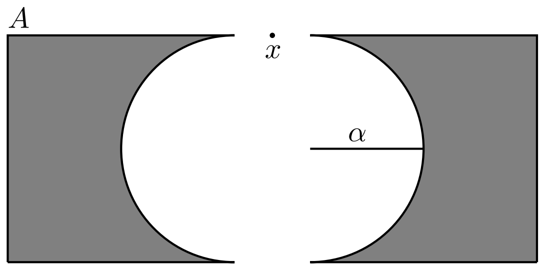

Figure 8: *A ball of radius α rolls freely in $\overline{A^c}$, $d(x, \partial A) < \alpha$, and the metric projection of x onto ∂A is not unique.*

Remark 5.4. *The converse of Lemma 5.9 may fail, that is, we may find sets A such that A and $\overline{A^c}$ are both α-convex but do not satisfy the rolling condition in A and in $\overline{A^c}$. See for example Figure 6, where the sets $A = B \cup \{x\}$ and $\overline{A^c} = \mathbb{R}^2 \setminus \mathring{B}$ are both α-convex for $\alpha = 1$. However, a ball of radius 1 does not roll freely in A because of the point x.*

Acknowledgments. This work was supported by grants MTM2005-00820 from the Ministerio de Educación y Ciencia and PGIDIT06PXIB207009PR from the Xunta de Galicia.

References

Billingsley, P. (1995). *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, third ed. A Wiley-Interscience Publication.

Chevalier, J. (1976). Estimation du support et du contour du support d'une loi de probabilité. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, vol. 12(4), pp. 339–364.

DEVROYE, L. (1983). The equivalence of weak, strong and complete convergence in L_1 for kernel density estimates. *Ann. Statist.*, vol. 11(3), pp. 896–904.

DEVROYE, L. AND WISE, G. L. (1980). Detection of abnormal behavior via non-parametric estimation of the support. *SIAM J. Appl. Math.*, vol. 38(3), pp. 480–488.

FEDERER, H. (1959). Curvature measures. *Trans. Amer. Math. Soc.*, vol. 93, pp. 418–491.

GEFFROY, J. (1964). Sur un problème d'estimation géométrique. *Publ. Inst. Statist. Univ. Paris*, vol. 13, pp. 191–210.

KOROSTELËV, A. P. AND TSYBAKOV, A. B. (1993). *Minimax theory of image reconstruction*, vol. 82 of *Lecture Notes in Statistics*. Springer-Verlag, New York.

PATEIRO-LÓPEZ, B. (2008). *Set estimation under convexity type restrictions*. Ph.D. thesis, Universidad de Santiago de Compostela.

RÉNYI, A. AND SULANKE, R. (1963). Über die konvexe Hülle von n zufällig gewählten Punkten. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, vol. 2, pp. 75–84 (1963).

RÉNYI, A. AND SULANKE, R. (1964). Über die konvexe Hülle von n zufällig gewählten Punkten. II. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, vol. 3, pp. 138–147 (1964).

RODRÍGUEZ-CASAL, A. (2007). Set estimation under convexity type assumptions. *Annales de l'I.H.P.- Probabilités & Statistiques*, vol. 43, pp. 763–774.

WALTHER, G. (1999). On a generalization of Blaschke's rolling theorem and the smoothing of surfaces. *Math. Methods Appl. Sci.*, vol. 22(4), pp. 301–316.

Reports in Statistics and Operations Research

2004

04-01 Goodness of fit test for linear regression models with missing response data. *González Manteiga, W., Pérez González, A.*
Canadian Journal of Statistics (to appear).

04-02 Boosting for Real and Functional Samples. An Application to an Environmental Problem. *B. M. Fernández de Castro and W. González Manteiga*.

04-03 Nonparametric classification of time series: Application to the bank share prices in Spanish stock market. *Juan M. Vilar, José A. Vilar and Sonia Pértega*.

04-04 Boosting and Neural Networks for Prediction of Heteroskedatic Time Series. *J. M. Matías, M. Febrero, W. González Manteiga and J. C. Reboredo.*

04-05 Partially Linear Regression Models with Farima-Garch Errors. An Application to the Forward Exchange Market. *G. Aneiros Pérez, W. González Manteiga and J. C. Reboredo Nogueira.*

04-06 A Flexible Method to Measure Synchrony in Neuronal Firing. *C. Faes, H. Geys, G. Molenberghs, M. Aerts, C. Cadarso-Suárez, C. Acuña and M. Cano.*

04-07 Testing for factor-by-curve interactions in generalized additive models: an application to neuronal activity in the prefrontal cortex during a discrimination task. *J. Roca-Pardiñas, C. Cadarso-Suárez, V. Nacher and C. Acuña.*

04-08 Bootstrap Estimation of the Mean Squared Error of an EBLUP in Mixed Linear Models for Small Areas. *W. González Manteiga, M. J. Lombardía, I. Molina, D. Morales and L. Santamaría.*

04-09 Set estimation under convexity type assumptions. *A. Rodríguez Casal.*

2005

05-01 SiZer Map for Evaluating a Bootstrap Local Bandwidth Selector in Nonparametric Additive Models. *M. D. Martínez-Miranda, R. Raya-Miranda, W. González-Manteiga and A. González-Carmona.*

05-02 The Role of Commitment in Repeated Games. *I. García Jurado, Julio González Díaz*.

05-03 Project Games. *A. Estévez Fernández, P. Borm, H. Hamers*

05-04 Semiparametric Inference in Generalized Mixed Effects Models. *M. J. Lombardía, S. Sperlich*

2006

06-01 A unifying model for contests: effort-prize games. J. González Díaz

06-02 The Harsanyi paradox and the "right to talk" in bargaining among coalitions. J. J. Vidal Puga

06-03 A functional analysis of NOx levels: location and scale estimation and outlier detection. M. Febrero, P. Galeano, W. González-Manteiga

06-04 Comparing spatial dependence structures. R. M. Crujeiras, R. Fernández-Casal, W. González-Manteiga

06-05 On the spectral simulation of spatial dependence structures. R. M. Crujeiras, R. Fernández-Casal

06-06 An L_2-test for comparing spatial spectral densities. R. M. Crujeiras, R. Fernández-Casal, W. González-Manteiga.

2007

07-01 Goodness-of-fit tests for the spatial spectral density. R. M. Crujeiras, R. Fernández-Casal, W. González-Manteiga.

07-02 Presmothed estimation with left truncated and right censores data. M. A. Jácome, M. C. Iglesias-Pérez

07-03 Robust nonparametric estimation with missing data. G. Boente, W. González-Manteiga, A. Pérez-González

07-04 k-Sample test based on the common area of kernel density estimators, P. Martínez-Camblor, J. de Uña Álvarez, N. Corral-Blanco

07-05 A bootstrap based model checking for selection-biased data, J. L. Ojeda, W. González-Manteiga, J . A. Cristobal

07-06 The Gaussian mixture dynamic conditional correlation model: Bayesian estimation, value at risk calculation and portfolio selection, P. Galeano, M. C. Ausín

2008

08-01 ROC curves in nonparametric location-scale regression models, W. González-Manteiga, J. C. Pardo Fernández, I. Van Keilegom

08-02 On the estimation of α-convex sets, B. Pateiro-López, A. Rodríguez-Casal.

Previous issues (2001 – 2003):
http://eio.usc.es/pub/reports.html