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An L_2-test for comparing spatial spectral densities

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ABSTRACT

Consider Z_l, for $l = 1, \ldots, L$ a stationary spatial process. We study the asymptotic behaviour of a Cramer-von-Mises type test statistic for testing the hypothesis $H_0 : f_1 = \ldots = f_L$, where each f_l denotes the spectral density of each observed process, for $l = 1, \ldots, L$. Asymptotic distribution theory under H_0 and under local alternatives is given.

Keywords: hypothesis testing; spectral density; spatial process.

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1 Introduction

An important problem in spatial statistic is modeling the dependence structure of spatial data, both from parametric and nonparametric approaches. This problem can be focused from the spatial domain, where the variogram or the covariogram are the functions that describe the dependence. From this point of view, on goodness-of-fit testing for dependence structures, Diblasi and Bowman (2001) propose a test for independence and Maglione and Diblasi (2004) extend the former technique for choosing a valid model for a variogram, based on smoothed versions of the observed variables.

Spectral techniques constitute an alternative way for studying dependent data and this methodology has been broadly used in time series analysis (e.g. Priestley (1981)). Despite the extension of these techniques to multidimensional settings is not straightforward, this approach is gaining acceptance in spatial data analysis. From this point of view, the target function is no longer the variogram or the covariogram, but the spatial spectral density. The classical nonparametric estimator for the spectral density is the periodogram. Its extension to the spatial context has been studied by Fuentes (2002).

Under this spectral scheme, Crujeiras *et al.* (2006b) provide two testing techniques for goodness-of-fit testing, using distances on the spectral and on the log-spectral domain. These test statistics take advantage of the representation of the spatial periodogram as the response variable in a multiplicative regression model. By a logarithmic transform, the spatial log-periodogram can be written as the exogenous variable in a regression model, where the regression function is the log-spectral density.

In the goodness-of-fit for regression models literature, King *et al.* (1991) study the problem of comparing two regression curves, using linear smoothers, under independendent and Gaussian errors. The general case of comparing $L \geq 2$ regression curves is studied in Dette and Neumeyer (2001), under heterocesdastic errors. In Vilar-Fernández and González-Manteiga (2004), the authors provide a procedure for testing the equality of regression curves, under fixed design and dependent errors. Based on the ideas in Vilar-Fernández and González-Manteiga (2004), the goal of our work is to provide a test for testing the hypothesis that the spectral densities of L observed realizations of spatial random process are equal, without specifying a parametric model. For that purpose, we consider a Cramer-von-Mises type functional, as in Dette and Neumeyer (2001) and Vilar-Fernández and González-Manteiga (2004).

As a particular case, this technique allows to detect changes on the dependence structure of a process observed at different time moments. This capacity makes the technique relevant when studying spatio-temporal processes. Invariance of the spatial dependence along time makes feasible the use of stationary spatio-temporal dependence models (e.g. Fernández-Casal *et al.* (2003)).

This paper is organized as follows. In Section 2 we provide some background on spatial spectral methods and nonparametric regression. In Section 3, we study the asymptotic distribution of the test under the null hypothesis and under local alternatives.

2 Some background on spectral techniques and nonparametric regression.

Let Z_l be a zero mean second-order stationary spatial process, observed on a regular grid D_l, for $l = 1, \ldots, L$. That is, $\{Z_l(\mathbf{s}), \mathbf{s} \in D_l = \mathbf{a}_l + D\}$, with $D = \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}$. The case $\mathbf{a}_1 = \ldots = \mathbf{a}_L$ implies that the processes are observed on the same grid of locations. Denote by $N_d = d_1 d_2$ the number of points in any of the grids D_l, with $l = 1, \ldots, L$. The covariance function of the processes are defined by:

$$C_l(\mathbf{u}) = E(Z_l(\mathbf{s}), Z_l(\mathbf{s} + \mathbf{u})), \quad \mathbf{s}, \mathbf{u} \in \mathbb{Z}^2. \tag{1}$$

Assuming that $\sum_{\mathbf{u}} |C_l(\mathbf{u})| < \infty$, by Khinchin's theorem (e.g. Yaglom (1987)), the covariance function of a stationary random process can be written, for $l = 1, \ldots, L$ as:

$$C_l(\mathbf{u}) = \int_{\Pi^2} e^{-i\mathbf{u}^T \boldsymbol{\lambda}} f_l(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \quad \Pi^2 = [-\pi, \pi] \times [-\pi, \pi] \tag{2}$$

where f_l, the spectral density, is bounded and continuous for all l and T denotes the transpose operator.

The classical nonparametric estimator of the spectral density is the periodogram, which is given by:

$$I_l(\boldsymbol{\lambda}_{\mathbf{k}}) = \frac{1}{(2\pi)^2 N_d} \left| \sum_{\mathbf{s} \in D_l} Z_l(\mathbf{s}) e^{-i\mathbf{s}^T \boldsymbol{\lambda}_{\mathbf{k}}} \right|^2, \tag{3}$$

where $\mathbf{s}^T \boldsymbol{\lambda}_{\mathbf{k}}$ denotes the scalar product in $\mathbb{R}^2$. The periodogram is usually computed at the set of bidimensional Fourier frequencies, $\boldsymbol{\lambda}_{\mathbf{k}}^T = (\lambda_{k_1}, \lambda_{k_2})$:

$$\lambda_{k_i} = \frac{2\pi k_i}{d_i}, \quad k_i = 0, \pm 1, \ldots, \pm n_i = \lfloor \frac{d_i - 1}{2} \rfloor, \quad i = 1, 2 \tag{4}$$

and denote by $N = (2n_1 + 1)(2n_2 + 1)$ the number of Fourier frequencies. The periodogram (3) can be also written in terms of the sample covariances as:

$$I_l(\boldsymbol{\lambda}_{\mathbf{k}}) = \frac{1}{(2\pi)^2} \sum_{\mathbf{u} \in \mathcal{U}} \hat{C}_l(\mathbf{u}) e^{-i\mathbf{u}^T \boldsymbol{\lambda}_{\mathbf{k}}}, \quad l = 1, \ldots, L \tag{5}$$

where $\mathcal{U} = \{\mathbf{u} = (u_1, u_2); u_i = 1 - d_i, \ldots, d_i - 1, i = 1, 2\}$ and the sample covariances, for Z_l with $l = 1, \ldots, L$, are given by:

$$\hat{C}_l(\mathbf{v}) = \frac{1}{N_d} \sum_{\mathbf{s} \in D_l(\mathbf{v})} Z_l(\mathbf{s}) Z_l(\mathbf{s} + \mathbf{v}), \quad D_l(\mathbf{v}) = \{\mathbf{s} \in D_l; \mathbf{s} + \mathbf{v} \in D_l\}. \tag{6}$$

Assume that the spatial processes Z_l, $l = 1, \ldots, L$ can be represented as:

$$Z_l(\mathbf{s}) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_{jk}^l \varepsilon_l(s_1 - j, s_2 - k), \tag{7}$$

where the error variables ε_l are independent and identically distributed (i.i.d.) as $N(0, \sigma_{\varepsilon_l}^2)$, for $l = 1, \ldots, L$. Note that any Gaussian stationary process can be represented as in (7). Then, the corresponding spectral density f_l can be written as:

$$f_l(\boldsymbol{\lambda}) = |A_l(\boldsymbol{\lambda})|^2 f_{\varepsilon_l}(\boldsymbol{\lambda}), \quad \boldsymbol{\lambda} \in \Pi^2 \tag{8}$$

where $f_{\varepsilon_l}(\boldsymbol{\lambda}) = \dfrac{\sigma^2_{\varepsilon_l}}{(2\pi)^2}$ and

$$A_l(\boldsymbol{\lambda}) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \psi^l_{jk} e^{-i(j,k)\boldsymbol{\lambda}}, \quad (j,k)\boldsymbol{\lambda} = j\lambda_1 + k\lambda_2.$$

In this case, the periodogram for each process Z_l, with $l = 1, \ldots, L$, admits the following representation:

$$I_l(\boldsymbol{\lambda}_{\mathbf{k}}) = f_l(\boldsymbol{\lambda}_{\mathbf{k}}) V^l_{\mathbf{k}} + R^l_N(\boldsymbol{\lambda}_{\mathbf{k}}), \tag{9}$$

where the variables $V^l_{\mathbf{k}}$ are i.i.d. standard exponential distributed, and $V^l_{\mathbf{k}}$ and $V^{l'}_{\mathbf{k}}$, with $l \neq l'$ are also independent. The residual term $R^l_N(\boldsymbol{\lambda}_{\mathbf{k}})$ is uniformly bounded (see Crujeiras *et al.* (2006b)). Applying logarithms in (9) we have:

$$Y^l_{\mathbf{k}} = m_l(\boldsymbol{\lambda}_{\mathbf{k}}) + z^l_{\mathbf{k}} + r^l_{\mathbf{k}}, \quad l = 1, \ldots, L \tag{10}$$

where $m_l = \log f_l$ is the log-spectral density, the variables $z^l_{\mathbf{k}} = \log V^l_{\mathbf{k}}$ are i.i.d. with density function $h(x) = e^{x - e^x}$, and the residual term $r^l_{\mathbf{k}}$ is given by:

$$r^l_{\mathbf{k}} = \log\left(1 + \frac{R^l_N(\boldsymbol{\lambda}_{\mathbf{k}})}{f_l(\boldsymbol{\lambda}_{\mathbf{k}}) V^l_{\mathbf{k}}}\right).$$

Several nonparametric estimators of the spatial log-spectral density could be obtained considering a smoothed combination of log-periodogram values, that is:

$$\hat{m}_l(\boldsymbol{\lambda}_{\mathbf{k}}) = \sum_{\mathbf{i}} W^l_{\mathbf{i}}(\boldsymbol{\lambda}_{\mathbf{k}}) Y^l_{\mathbf{i}}. \tag{11}$$

The weights $W^l_{\mathbf{i}}$ can be defined as Gasser-Muller weights, for instance:

$$W^l_{\mathbf{i}}(\boldsymbol{\lambda}) = |H|^{-1/2} \int_{A_{\mathbf{i}}} K(H^{-1/2}(\boldsymbol{\lambda} - \boldsymbol{\nu})) d\boldsymbol{\nu}, \tag{12}$$

where K is a bidimensional kernel function, H is a bidimensional bandwidth matrix and the integration region is given by:

$$A_{\mathbf{i}} = [a_{i_1 - 1}, a_{i_1}] \times [a_{i_2 - 1}, a_{i_2}], \quad \boldsymbol{\lambda}_{\mathbf{i}} \in A_{\mathbf{i}}, \quad \cup_i A_i = A, \quad A_i \cap A_j = \oslash, \quad \text{for} \quad i \neq j.$$

The sets A_i in the partition of A must be Jordan measurable and $\max_i \mu(A_i) = \mathcal{O}(N^{-1})$ (see Müller (1988)). Other options are Priestley-Chao weights:

$$W^l_{\mathbf{i}}(\boldsymbol{\lambda}) = \frac{\pi^2}{N} K_H(\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\mathbf{i}}) = \frac{\pi^2}{N|H|^{1/2}} K(H^{-1/2}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\mathbf{i}})), \tag{13}$$

or Nadaraya-Watson weights. Another alternative consists of considering a local-linear estimator for the spatial log-spectral density. All these weights, under a fixed design setting, are asymptotically equivalent.

3 Asymptotic Analysis

Consider $\{Z_l(\mathbf{s}), \mathbf{s} \in D_l\}$, with $l = 1, \ldots, L$, L realizations of a spatial stochastic process (for instance, realizations taken on L time moments) or L realizations of different spatial processes. Our main purpose is to test whether the dependence structure of $\{Z_l, l = 1, \ldots, L\}$ is the same. In terms of the log-spectral densities m_l, the testing problem can be written as:

$$\begin{array}{ll} H_0: & m_1 = \ldots = m_L, \\ H_a: & m_l \neq m_j, \text{ for some } l \neq j. \end{array} \tag{14}$$

In this context, the comparison can be made by considering nonparametric estimators of the spatial log-spectral densities. Consider the following test statistic, based on a L^2-distance:

$$Q = \sum_{l=2}^{L} \left(\sum_{j=1}^{l-1} \left(\int_{\Pi^2} \left(\widehat{m_l}(\boldsymbol{\lambda}) - \widehat{m_j}(\boldsymbol{\lambda})\right)^2 \omega(\boldsymbol{\lambda}) d\boldsymbol{\lambda} \right) \right), \tag{15}$$

where ω is a positive, bounded weight function with support Π^2. This weight function is usually chosen to avoid edge-effects. In the spectral context, this function can be chosen in order to filter frequencies where the periodogram presents higher variability, as the origin or those frequencies with π-valued components.

A1 The spatial processes Z_l can be represented as in (7) and

$$\sum_i \sum_j |i|^{1/2}|j||\psi_{ij}^l| \leq \infty \quad \text{and} \quad \sum_i \sum_j |i||j|^{1/2}|\psi_{ij}^l| \leq \infty.$$

A2 The spectral densities are non-vanishing:

$$\inf_{\boldsymbol{\lambda} \in \Pi^2} f_l(\boldsymbol{\lambda}) > 0, \qquad \text{for} \qquad l = 1, \ldots, L.$$

A3 We consider Gasser-Muller type weights, given by (12), or Priestley-Chao weights, given by (13). Besides, $W_{\mathbf{i}}^1 = \ldots = W_{\mathbf{i}}^L$.

A4 The bidimensional kernel function K is continuously differentiable, with compact support and $\int K^2(\mathbf{u})d\mathbf{u} < \infty$.

A5 The bidimensional bandwidth matrix, H satisfies $N|H|^{1/2} \to \infty$, as $N \to \infty$, with $n_1, n_2 \to \infty$ and $n_1/n_2 \to c$, for some constant c.

Consider first the testing problem $H_0 : m_1 = m_2$ vs. $H_a : m_1 \neq m_2$ and assume that both Z_1 and Z_2 have been observed on grids with the same design. This implies that the corresponding Fourier frequencies are the same in both cases. By Riemann's approximation, Q can be approximated by:

$$\hat{Q} = \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \left(\widehat{m_1}(\boldsymbol{\lambda}_{\mathbf{k}}) - \widehat{m_2}(\boldsymbol{\lambda}_{\mathbf{k}})\right)^2 \omega(\boldsymbol{\lambda}_{\mathbf{k}}). \tag{16}$$

Theorem 1. *Assume conditions* (A1)-(A5) *hold. Then, under the null hypothesis that* $H_0 : m_1 = m_2$*, we have that, as* $N \to \infty$*:*

$$\sqrt{N|H|^{1/2}} \left(\hat{Q} - \frac{1}{12N|H|^{1/2}} C_K I_\omega \right) \to N(0, \sigma_{\hat{Q}}^2), \tag{17}$$

in distribution, with

$$C_K = \int K^2(\mathbf{u})d\mathbf{u}, \quad I_\omega = \int \omega(\mathbf{v})d\mathbf{v} \quad \textit{and the asymptotic variance is}$$

$$\sigma^2_{\hat{Q}} = \frac{1}{72}\int (K*K)^2(\mathbf{u})d\mathbf{u}\int \omega(\mathbf{v})d\mathbf{v},$$

where $*$ denotes the convolution operator.

Also in this context of comparing two dependence structures, consider that the null hypothesis is false and assume that:

$$m_1(\boldsymbol{\lambda}) - m_2(\boldsymbol{\lambda}) = C_N p(\boldsymbol{\lambda}), \tag{18}$$

where $p(\boldsymbol{\lambda})$ is a non-zero function. We will see that the test statistic $\hat{Q}$ allows for detecting local alternatives at a distance of order $N^{-1/2}|H|^{-1/8}$.

Theorem 2. *Assume conditions* (A1)-(A5) *hold. Then, if (18) holds and* $C_N^2 = (N^2|H|^{1/2})^{-1/2}$, *we have that, as* $N \to \infty$*:*

$$\sqrt{N|H|^{1/2}}\left(\hat{Q} - \frac{1}{12N|H|^{1/2}}C_K I_\omega\right) \to N\left(\int p^2(\mathbf{v})\omega(\mathbf{v})d\mathbf{v}, \sigma^2_{\hat{Q}}\right),$$

in distribution, with C_K, I_ω and $\sigma^2_{\hat{Q}}$ as in Theorem 1.

Theorems 1 and 2 can be generalized for stationary random fields on $\mathbb{R}^d$, under a similar asymptotic framework, assuming that the sampling grid increases at the same rate in all directions. A d-variate kernel function K satisfying condition A4 and a d-dimensional bandwidth matrix H, satisfying condition A5 must be considered. The corresponding asymptotic mean and variance in (17) are given by:

$$\frac{\pi^{2-d}}{3\cdot 2^d}\frac{1}{N|H|^{1/2}}C_K I_\omega, \quad \text{and}$$

$$\sigma^2_{\hat{Q},d} = \frac{\pi^{4-2d}}{9\cdot 2^{2d-1}}\int (K*K)^2(\mathbf{u})d\mathbf{u}\int \omega(\mathbf{v})d\mathbf{v},$$

where the weighting funcion ω is now defined on $\Pi^d = [-\pi,\pi]^d$. Thus, in the particular case of $d=1$, we provide a testing technique for comparing spectral densities in time series context. In this case, we have a scalar bandwidth parameter h, which plays the role of $H^{1/2}$ in the general dimension setting.

If the spatial process Z_l are observed on regular grids with different sizes, then the corresponding frequency spectrum is not the same. The asymptotic behaviour of $\hat{Q}$ could be determined following similar arguments to those in (Vilar-Fernández and González-Manteiga (2004), Theorem 3), under some conditions on the asymptotic rates of the samples.

In order to apply this test statistic in practice, bootstrap algorithms jointly with a cross-validation bandwidth selection method, have been provided in Crujeiras *et al.* (2006a). A completely nonparametric bootstrap procedure is proposed, although in case the spectral densities to be tested belong to the same parametric family, this algorithm can be simplified.

In the regression context, Zhang and Dette (2004) consider three major types of nonparametric regression test, among them, the Cramer-von-Mises test we consider in this work. The L_2-test result the most powerful, from a local asymptotic point of view.

4 Appendix

Let's introduce the following notation. Consider the following regression model:

$$Y_{\mathbf{k}}^{l*} = m_l(\boldsymbol{\lambda}_{\mathbf{k}}) + z_{\mathbf{k}}^{l*}, \quad l = 1, 2. \tag{19}$$

where $Y_{\mathbf{k}}^{l*} = Y_{\mathbf{k}}^{l} - C_0 - r_{\mathbf{k}}^{l}$, and $z_{\mathbf{k}}^{l*} = z_{\mathbf{k}}^{l} - C_0$, where $C_0 = E(z_{\mathbf{k}}^{l})$ is the Euler constant. Denote by $\widehat{m_l}^*$ the nonparametric estimator of m_l as in (11) for (19) equation and denote by

$$B_{\mathbf{k}}^{l} = \sum_{\mathbf{j}} W_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) r_{\mathbf{j}}^{l}, \quad l = 1, 2.$$

Lemma 1. *The test statistic $\hat{Q}$ can be decomposed in three addends:*

$$\hat{Q} = \hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3,$$

where

$$\hat{Q}_1 = \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} (\widehat{m_1}^*(\boldsymbol{\lambda}_{\mathbf{k}}) - \widehat{m_2}^*(\boldsymbol{\lambda}_{\mathbf{k}}))^2 \omega(\boldsymbol{\lambda}_{\mathbf{k}}),$$

$$\hat{Q}_2 = \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} (B_{\mathbf{k}}^1 - B_{\mathbf{k}}^2)^2 \omega(\boldsymbol{\lambda}_{\mathbf{k}}),$$

$$\hat{Q}_3 = 2\frac{(2\pi)^2}{N} \sum_{\mathbf{k}} (\widehat{m_1}^*(\boldsymbol{\lambda}_{\mathbf{k}}) - \widehat{m_2}^*(\boldsymbol{\lambda}_{\mathbf{k}}))(B_{\mathbf{k}}^1 - B_{\mathbf{k}}^2) \omega(\boldsymbol{\lambda}_{\mathbf{k}}).$$

Proof. It is straightforward from the definitions of the non parametric estimator in regression model (19) and $B_{\mathbf{k}}^{l}$, for $l = 1, 2$. □

Lemma 2. *Under conditions (A1)-(A5) and under H_0, we have that:*

$$\hat{Q}_2 = \mathcal{O}_{\mathbb{P}}\left(\frac{\log^2 N}{N^2 |H|^{1/2}}\right) \qquad \text{and} \qquad \hat{Q}_3 = \mathcal{O}_{\mathbb{P}}\left(\frac{\log^2 N}{N}\right).$$

Proof. $\hat{Q}_2$ can be decomposed as:

$$\begin{aligned}\hat{Q}_2 &= \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W_{\mathbf{j}}^2(\boldsymbol{\lambda}_{\mathbf{k}}) (r_{\mathbf{j}}^1 - r_{\mathbf{j}}^2)^2 \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\ &+ \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{i}} \sum_{\mathbf{j} \neq \mathbf{i}} W_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) W_{\mathbf{i}}(\boldsymbol{\lambda}_{\mathbf{k}}) (r_{\mathbf{j}}^1 - r_{\mathbf{j}}^2)(r_{\mathbf{i}}^1 - r_{\mathbf{i}}^2) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\ &= \hat{Q}_{2,1} + \hat{Q}_{2,2}.\end{aligned}$$

By a Taylor expansion on the residual part $r_{\mathbf{j}}^{l}$ around 0, for $l = 1, 2$:

$$r_{\mathbf{j}}^{l} = -\frac{R_N^l(\boldsymbol{\lambda}_{\mathbf{j}})}{f_l(\boldsymbol{\lambda}_{\mathbf{j}}) V_{\mathbf{j}}} - \frac{1}{2(1 + x_j)^2} \left(\frac{R_N^l(\boldsymbol{\lambda}_{\mathbf{j}})}{f_l(\boldsymbol{\lambda}_{\mathbf{j}}) V_{\mathbf{j}}}\right)^2,$$

where $x_j \in \left(0, \frac{R_N^l(\boldsymbol{\lambda}_{\mathbf{j}})}{f_l(\boldsymbol{\lambda}_{\mathbf{j}}) V_{\mathbf{j}}}\right)$. Since,

$$\max_{\mathbf{j}} |R_N^l(\boldsymbol{\lambda}_{\mathbf{j}})| = \mathcal{O}_{\mathbb{P}}(N^{-1/2} \log N),$$

for $l = 1, 2$, just following Kooperberg *et al.* (1995), the Lagrange remainder in the Taylor expansion, denoted by $LR^l_{\mathbf{j}}$ can be uniformly bounded by $\mathcal{O}_{\mathbb{P}}\left(\frac{\log^2 N}{N}\right)$. Then, under the null hypothesis, $\hat{Q}_{2,1}$ can be decomposed as $\hat{Q}_{2,1} = \hat{Q}^1_{2,1} + \hat{Q}^2_{2,1} + \hat{Q}^3_{2,1}$, where

$$\hat{Q}^1_{2,1} = \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W^2_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) \left(\frac{R^2_N(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} - \frac{R^1_N(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} \right)^2 \omega(\boldsymbol{\lambda}_{\mathbf{k}}), \tag{20}$$

$$\hat{Q}^2_{2,1} = \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W^2_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) (LR^2_{\mathbf{j}} - LR^1_{\mathbf{j}})^2 \omega(\boldsymbol{\lambda}_{\mathbf{k}}), \tag{21}$$

and

$$\hat{Q}^3_{2,1} = 2\frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W^2_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) \left(\frac{R^2_N(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} - \frac{R^1_N(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} \right) (LR^2_{\mathbf{j}} - LR^1_{\mathbf{j}}) \omega(\boldsymbol{\lambda}_{\mathbf{k}}). \tag{22}$$

Let's find a bound for $\hat{Q}^2_{2,1}$ in (21), the addend involving the Lagrange remainders.

$$\begin{aligned}
\hat{Q}^2_{2,1} &= \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W^2_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) (LR^1_{\mathbf{j}} - LR^2_{\mathbf{j}})^2 \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&= \mathcal{O}_{\mathbb{P}}\left(\frac{\log^4 N}{N^2}\right) \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} \frac{1}{N^2|H|} K^2(H^{-1/2}(\boldsymbol{\lambda}_{\mathbf{k}} - \boldsymbol{\lambda}_{\mathbf{j}})) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&= \mathcal{O}_{\mathbb{P}}\left(\frac{\log^4 N}{N^2}\right) \frac{(2\pi)^2}{N} \left(\sum_{\mathbf{k}} \sum_{\mathbf{j} \neq \mathbf{k}} \frac{1}{N^2|H|} K^2(H^{-1/2}(\boldsymbol{\lambda}_{\mathbf{k}} - \boldsymbol{\lambda}_{\mathbf{j}})) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) + \sum_{\mathbf{k}} \frac{1}{N|H|} K^2(H^{-1/2}\mathbf{0}) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \right) \\
&= \mathcal{O}_{\mathbb{P}}\left(\frac{\log^4 N}{N^4|H|^{1/2}}\right) + \mathcal{O}_{\mathbb{P}}\left(\frac{\log^4 N}{N^2}\right) \frac{(2\pi)^2}{N^3|H|} \sum_{\mathbf{k}} \sum_{\mathbf{j} \neq \mathbf{k}} K^2(H^{-1/2}(\boldsymbol{\lambda}_{\mathbf{k}} - \boldsymbol{\lambda}_{\mathbf{j}})) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&\leq \mathcal{O}_{\mathbb{P}}\left(\frac{\log^4 N}{N^4|H|^{1/2}}\right) + \mathcal{O}_{\mathbb{P}}\left(\frac{\log^4 N}{N^2}\right) \frac{(2\pi)^2}{N^2|H|} \max_{\mathbf{k}} \sum_{\mathbf{j} \neq \mathbf{k}} K^2(H^{-1/2}(\boldsymbol{\lambda}_{\mathbf{k}} - \boldsymbol{\lambda}_{\mathbf{j}})) \\
&= \mathcal{O}_{\mathbb{P}}\left(\frac{\log^4 N}{N^3|H|^{1/2}}\right),
\end{aligned}$$

where the last inequality follows from $\max_{\mathbf{k}} \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \leq c$, for some constant c. Following similar arguments as above, we will found bounds for $\hat{Q}_{2,1}$ and $\hat{Q}_{2,3}$. For $\hat{Q}^1_{2,1}$ in (20):

$$\begin{aligned}
\hat{Q}^1_{2,1} &= \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W^2_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) \left(\frac{R^2_N(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} - \frac{R^1_N(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} \right)^2 \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&= \hat{Q}^{1,1}_{2,1} + \hat{Q}^{1,2}_{2,1} + \hat{Q}^{1,3}_{2,1}
\end{aligned}$$

where

$$\hat{Q}^{1,1}_{2,1} = \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W^2_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) \left(\frac{R^1_N(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} \right)^2 \omega(\boldsymbol{\lambda}_{\mathbf{k}}),$$

$$\hat{Q}^{1,3}_{2,1} = 2\frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W^2_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) \frac{R^1_N(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} \frac{R^2_N(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} \omega(\boldsymbol{\lambda}_{\mathbf{k}})$$

and $\hat{Q}_{2,1}^{1,2}$ is the similar to $\hat{Q}_{2,1}^{1,1}$, but replacing each $R_N^1(\boldsymbol{\lambda}_{\mathbf{j}})$ for $R_N^2(\boldsymbol{\lambda}_{\mathbf{j}})$. We will find a bound for $\hat{Q}_{2,1}^{1,1}$.

$$\begin{aligned}
\hat{Q}_{2,1}^{1,1} &\leq \frac{(2\pi)^2}{N} N \max_{\mathbf{j}} \left(\frac{R_N^1(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} \right)^2 \sum_{\mathbf{j}} \sum_{\mathbf{k}} W_{\mathbf{j}}^2(\boldsymbol{\lambda}_{\mathbf{k}}) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&= \mathcal{O}_{\mathbb{P}} \left(\frac{\log^2 N}{N} \right) \sum_{\mathbf{j}} \sum_{\mathbf{k}} \frac{1}{N^2 |H|} K^2(H^{-1/2}(\boldsymbol{\lambda}_{\mathbf{k}} - \boldsymbol{\lambda}_{\mathbf{j}})) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&= \mathcal{O}_{\mathbb{P}} \left(\frac{\log^2 N}{N^2 |H|^{1/2}} \right).
\end{aligned}$$

Similar computations lead to the same bound for the other addends. Let's find a bound for the third addend, $\hat{Q}_{2,1}^3$ in (22):

$$\begin{aligned}
\hat{Q}_{2,1}^3 &= 2\frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W_{\mathbf{j}}^2(\boldsymbol{\lambda}_{\mathbf{k}}) \left(\frac{R_N^1(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} - \frac{R_N^2(\boldsymbol{\lambda}_{\mathbf{j}})}{f(\boldsymbol{\lambda}_{\mathbf{j}})V} \right) (LR_{\mathbf{j}}^2 - LR_{\mathbf{j}}^1) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&= \mathcal{O}_{\mathbb{P}} \left(\frac{\log N}{N^{1/2}} \right) \mathcal{O}_{\mathbb{P}} \left(\frac{\log^2 N}{N} \right) \frac{(2\pi)^2}{N} \sum_{\mathbf{j}} \sum_{\mathbf{k}} \frac{1}{N^2 |H|} K^2(H^{-1/2}(\boldsymbol{\lambda}_{\mathbf{k}} - \boldsymbol{\lambda}_{\mathbf{j}})) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&= \mathcal{O}_{\mathbb{P}} \left(\frac{\log^3 N}{N^{5/2} |H|^{1/2}} \right).
\end{aligned}$$

Therefore,

$$\hat{Q}_2 \leq \mathcal{O}_{\mathbb{P}} \left(\frac{\log^2 N}{N^2 |H|^{1/2}} \right).$$

Finally, $\hat{Q}_3$ can be written as:

$$\begin{aligned}
\hat{Q}_3 &= 2\frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \left(\widehat{m_1}(\boldsymbol{\lambda}_{\mathbf{k}}) - \widehat{m_2}(\boldsymbol{\lambda}_{\mathbf{k}}) \right) (B_{\mathbf{k}}^1 - B_{\mathbf{k}}^2) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&\leq 2 \max_{\mathbf{k}} |B_{\mathbf{k}}^1 - B_{\mathbf{k}}^2| \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) (Y_{\mathbf{j}}^1 - Y_{\mathbf{j}}^2) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&\leq \mathcal{O}_{\mathbb{P}} \left(\frac{\log N}{N^{1/2}} \right) \max_{\mathbf{j}} |Y_{\mathbf{j}}^1 - Y_{\mathbf{j}}^2| \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} \frac{1}{N |H|^{1/2}} K(H^{-1/2}(\boldsymbol{\lambda}_{\mathbf{k}} - \boldsymbol{\lambda}_{\mathbf{j}})) \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&= \mathcal{O}_{\mathbb{P}} \left(\frac{\log^2 N}{N} \right)
\end{aligned}$$

since $H_0 : m_1 = m_2$ implies that $Y_{\mathbf{j}}^1 - Y_{\mathbf{j}}^2 = r_{\mathbf{j}}^1 - r_{\mathbf{j}}^2$, for every Fourier frequency. $\square$

Lemma 3. *Under conditions (A1)-(A5) and under H_0, we have that*

$$\sqrt{N|H|^{1/2}} \left(\hat{Q}_1 - \frac{1}{12 N |H|^{1/2}} C_K I_\omega \right) \to N(0, \sigma_Q^2),$$

in distribution, where C_K, I_ω and $\sigma_{\hat{Q}}^2$ as in Theorem 1.

Proof. Define the following random variables $\Lambda_{\mathbf{k}} = z_{\mathbf{k}}^{1*} - z_{\mathbf{k}}^{2*}$, with $E(\Lambda_{\mathbf{k}}) = 0$, $E(\Lambda_{\mathbf{k}}^2) = \frac{\pi^2}{3}$ and $Cov(\Lambda_{\mathbf{k}}, \Lambda_{\mathbf{j}}) = 0$ for $\mathbf{j} \neq \mathbf{k}$. The statistic $\hat{Q}_1$ can be decomposed in two addends as $\hat{Q}_1 = \hat{Q}_{1,1} + \hat{Q}_{1,2}$, where :

$$\hat{Q}_{1,1} = \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} W_{\mathbf{j}}^2(\boldsymbol{\lambda}_{\mathbf{k}}) \Lambda_{\mathbf{j}}^2 \omega(\boldsymbol{\lambda}_{\mathbf{k}}), \quad \hat{Q}_{1,2} = \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} \sum_{\mathbf{j}} \sum_{\mathbf{i} \neq \mathbf{j}} W_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) W_{\mathbf{i}}(\boldsymbol{\lambda}_{\mathbf{k}}) \Lambda_{\mathbf{j}} \Lambda_{\mathbf{i}} \omega(\boldsymbol{\lambda}_{\mathbf{k}}).$$

Define

$$b_{\mathbf{i},\mathbf{j}} = \frac{(2\pi)^2}{N} \sum_{\mathbf{k}} W_{\mathbf{i}}(\boldsymbol{\lambda}_{\mathbf{k}}) W_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}}) \omega(\boldsymbol{\lambda}_{\mathbf{k}}).$$

Then:

$$\hat{Q}_{1,1} = \sum_{\mathbf{j}} b_{\mathbf{j},\mathbf{j}} \Lambda_{\mathbf{j}}^2, \quad \text{and} \quad \hat{Q}_{1,2} = \sum_{\mathbf{i} \neq \mathbf{j}} b_{\mathbf{i},\mathbf{j}} \Lambda_{\mathbf{i}} \Lambda_{\mathbf{j}}.$$

First, we will study the behaviour of $\hat{Q}_{1,1}$. For simplicity, consider Priestley-Chao weights. Taking expectations and using Riemann's approximation, it is easy to see that:

$$E(\hat{Q}_{1,1}) \approx \frac{1}{12} \frac{1}{N|H|^{1/2}} \int \omega(\mathbf{v}) d\mathbf{v} \int K^2(\mathbf{u}) d\mathbf{u}. \tag{23}$$

Let's check the order of the variance of $\hat{Q}_{1,1}$. Denote by $c_2 = Var(\Lambda_{\mathbf{j}}^2)$ This variance can be computed taking into account that:

$$Var(\hat{Q}_{1,1}) = \frac{(2\pi)^4}{N^6|H|^2} c_2 \sum_{\mathbf{j}} \alpha_{\mathbf{j}}^2, \quad \alpha_{\mathbf{j}} = \sum_{\mathbf{k}} K^2(H^{-1/2}(\boldsymbol{\lambda}_{\mathbf{k}} - \boldsymbol{\lambda}_{\mathbf{j}})) \omega(\boldsymbol{\lambda}_{\mathbf{k}}).$$

Then,

$$Var(\hat{Q}_{1,1}) = c_2 \frac{(2\pi)^4}{N^6|H|^2} \sum_{\mathbf{k}} \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \sum_{\mathbf{k}'} \omega(\boldsymbol{\lambda}_{\mathbf{k}'}) \sum_{\mathbf{j}} K^2(H^{-1/2}(\boldsymbol{\lambda}_{\mathbf{k}} - \boldsymbol{\lambda}_{\mathbf{j}})) K^2(H^{-1/2}(\boldsymbol{\lambda}_{\mathbf{k}'} - \boldsymbol{\lambda}_{\mathbf{j}}))$$

which can be approximated, using a changes of variable and Riemann's sums, by:

$$Var(\hat{Q}_{1,1}) \approx C_K^2 c_2 \frac{1}{N^4|H|} \sum_{\mathbf{k}} \omega(\boldsymbol{\lambda}_{\mathbf{k}}) \sum_{\mathbf{k}'} \omega(\boldsymbol{\lambda}_{\mathbf{k}'}) = \mathcal{O}\left(\frac{1}{N^2|H|}\right)$$

Therefore, applying Markov's inequality, it follows that:

$$\hat{Q}_{1,1} = \frac{1}{12N|H|^{1/2}} C_K I_\omega + \mathcal{O}_{\mathbb{P}}\left(\frac{1}{N|H|^{1/2}}\right).$$

Besides, $E(\hat{Q}_{1,2}) = 0$ since $\Lambda_{\mathbf{i}}$ and $\Lambda_{\mathbf{j}}$ are uncorrelated, for $\mathbf{i} \neq \mathbf{j}$ and its variance is given by:

$$Var(\hat{Q}_{1,2}) = \sum_{\mathbf{i} \neq \mathbf{j}} \sum_{\mathbf{u} \neq \mathbf{v}} b_{\mathbf{ij}} b_{\mathbf{uv}} E(\Lambda_{\mathbf{i}} \Lambda_{\mathbf{j}} \Lambda_{\mathbf{u}} \Lambda_{\mathbf{v}}), \tag{24}$$

since $\mathbf{i} \neq \mathbf{j}$, $E(\Lambda_{\mathbf{i}} \Lambda_{\mathbf{j}}) = E(\Lambda_{\mathbf{i}}) E(\Lambda_{\mathbf{j}}) = 0$. The same applies for $\mathbf{u} \neq \mathbf{j}$. For $E(\Lambda_{\mathbf{i}} \Lambda_{\mathbf{j}} \Lambda_{\mathbf{u}} \Lambda_{\mathbf{v}})$ to be different from zero, one of these two conditions must hold: $\mathbf{i} = \mathbf{u}$ and $\mathbf{j} = \mathbf{v}$ or $\mathbf{i} = \mathbf{v}$ and $\mathbf{j} = \mathbf{u}$. Then:

$$\begin{aligned} Var(\hat{Q}_{1,2}) &= \sum_{\mathbf{i}} \sum_{\mathbf{j} \neq \mathbf{i}} b_{\mathbf{ij}} \sum_{\mathbf{u}} \sum_{\mathbf{u} \neq \mathbf{v}} b_{\mathbf{uv}} E(\Lambda_{\mathbf{i}} \Lambda_{\mathbf{j}} \Lambda_{\mathbf{u}} \Lambda_{\mathbf{v}}) \\ &= 2 \sum_{\mathbf{i}} \sum_{\mathbf{j} \neq \mathbf{i}} b_{\mathbf{ij}}^2 E(\Lambda_{\mathbf{i}}^2) E(\Lambda_{\mathbf{j}}^2) = \frac{2\pi^4}{9} \sum_{\mathbf{i}} \sum_{\mathbf{j} \neq \mathbf{i}} b_{\mathbf{ij}}^2. \end{aligned}$$

Consider the following approximation for the product of two $b_{\mathbf{ij}}$ coefficients:

$$b_{\mathbf{ij}}b_{\mathbf{uv}} \approx \frac{1}{N^4|H|} K*K(H^{-1/2}(\boldsymbol{\lambda}_\mathbf{i}-\boldsymbol{\lambda}_\mathbf{j}))\omega(\boldsymbol{\lambda}_\mathbf{i})K*K(H^{-1/2}(\boldsymbol{\lambda}_\mathbf{u}-\boldsymbol{\lambda}_\mathbf{v}))\omega(\boldsymbol{\lambda}_\mathbf{u}).$$

Then,

$$\begin{aligned} Var(\hat{Q}_{1,2}) &\approx \frac{2\pi^4}{9}\sum_{\mathbf{i}}\sum_{\mathbf{j}\neq\mathbf{i}} \frac{1}{N^4|H|}(K*K)^2(H^{-1/2}(\boldsymbol{\lambda}_\mathbf{i}-\boldsymbol{\lambda}_\mathbf{j}))\omega^2(\boldsymbol{\lambda}_\mathbf{i}) \\ &\approx \frac{1}{72}\frac{1}{N|H|^{1/2}}\int (K*K)^2(\mathbf{u})d\mathbf{u}\int \omega(\mathbf{v})d\mathbf{v}. \end{aligned}$$

Therefore, the asymptotic variance of $\hat{Q}$ is given by:

$$\sigma^2_{\hat{Q}} = \lim_{N\to\infty} N^2|H|^{1/2}\sigma^2_{\hat{Q}_{1,2}} = \frac{1}{72}\int \left(K*K(\mathbf{u})\right)^2 d\mathbf{u}\int \omega^2(\mathbf{v})d\mathbf{v}.$$

In order to prove the asymptotic normal distribution of $\hat{Q}_{1,2}$, we will apply Theorem 5.2 in de Jong (1987). For that purpose, we must write $\hat{Q}_{1,2}$ as a quadratic form, namely $\hat{Q}_{1,2} = \sum_{i,j} a_{i,j}X_iX_j$, where i and j are one-dimensional indexes and X_i are i.i.d. random variables with zero mean and unit variance.

First, define a new subindex for the Fourier frequencies $\boldsymbol{\lambda}_\mathbf{k}$, with $\mathbf{k}=(k_1,k_2)$ and $k_l = 0,\pm1,\ldots,\pm n_l$, for $l=1,2$. Consider $\boldsymbol{\lambda}_\mathbf{k} = \boldsymbol{\lambda}_{\mathbf{k}'}$ where $\mathbf{k}'=(k_1',k_2')$, with $k_l' = 1,\ldots,\kappa_l = 2n_l+1$, in such a way that $k_l' = k_l + n_1 + 1$ for $l=1,2$. Denote by $\mathcal{M}_{N\times N}$ the space of square matrix with size N. The new coefficients, with one dimensional indexes, are given by the following matrix:

$$A = (a_{ij}), \quad A \in \mathcal{M}_{N\times N},$$

and each entry of this matrix is defined by $a_{ij} = \frac{\pi}{\sqrt{3}}b_{\mathbf{ij}}$ and $a_{ii}=0$, where the bidimensional indexes are given by:

$$\mathbf{i} = (i_1,i_2) = (k,k_0), \quad \text{if } (k-1)\kappa_2 \le i \le k\kappa_2 \quad \text{and } i = (k-1)\kappa_2 + k_0, \tag{25}$$

$$\mathbf{j} = (j_1,j_2) = (l,l_0), \quad \text{if } (l-1)\kappa_2 \le j \le l\kappa_2 \quad \text{and } j = (l-1)\kappa_2 + l_0. \tag{26}$$

Now, define the variables $X_i = \frac{\sqrt{3}}{\pi}\Lambda_\mathbf{i}$, where i and $\mathbf{i}$ satisfy: $i = (i_1-1)\kappa_2 + i_2$. $\hat{Q}_{1,2}$ can be written as a quadratic form with one-dimensional indexes:

$$\hat{Q}_{1,2} = \sum_{i,j} a_{i,j}X_iX_j, \quad a_{ij} = \frac{\pi}{\sqrt{3}}\frac{(2\pi)^2}{N}\sum_{\mathbf{k}} W_\mathbf{i}(\boldsymbol{\lambda}_\mathbf{k})W_\mathbf{j}(\boldsymbol{\lambda}_\mathbf{k})\omega(\boldsymbol{\lambda}_\mathbf{k}),$$

where $\mathbf{i}$ and $\mathbf{j}$ are determined by (25) and (26), respectively. Asymptotic normality is proved if the following conditions are satisfied:

(i) There exists a sequence of real numbers $k(N)$ such that $k(N)^4\sigma^2_{\hat{Q}}\max_i\sum_j a_{ij}^2 \to 0$.

(ii) The random variables X_i satisfy $\max_i E\left(X_i^2 1_{\{|X_i|>k(N)\}}\right) \to 0, \quad N\to\infty$.

(iii) The eigenvalues of the matrix $A=(a_{ij})$ are negligible: $\sigma^2_{\hat{Q}}\max_i \mu_i^2 \to 0, \quad N\to\infty$.

In order to check $(i)-(iii)$ note that:

$$
\begin{aligned}
a_{ij}^2 &= \frac{\pi^2(2\pi)^4}{3N^2}\left(\sum_{\mathbf{k}} W_{\mathbf{i}}(\boldsymbol{\lambda}_{\mathbf{k}})W_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}})\omega(\boldsymbol{\lambda}_{\mathbf{k}})\right)^2 \\
&= \frac{\pi^2(2\pi)^4}{3N^2}\sum_{\mathbf{k}} W_{\mathbf{i}}^2(\boldsymbol{\lambda}_{\mathbf{k}})W_{\mathbf{j}}^2(\boldsymbol{\lambda}_{\mathbf{k}})\omega^2(\boldsymbol{\lambda}_{\mathbf{k}}) \\
&+\frac{\pi^2(2\pi)^4}{3N^2}\sum_{\mathbf{k}} W_{\mathbf{i}}(\boldsymbol{\lambda}_{\mathbf{k}})W_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}})\omega(\boldsymbol{\lambda}_{\mathbf{k}})\sum_{\mathbf{k}\neq\mathbf{k}'} W_{\mathbf{i}}(\boldsymbol{\lambda}_{\mathbf{k}'})W_{\mathbf{j}}(\boldsymbol{\lambda}_{\mathbf{k}'})\omega(\boldsymbol{\lambda}_{\mathbf{k}'}) \\
&= a_{ij}^{2A} + a_{ij}^{2B}.
\end{aligned}
$$

Besides,

$$
\max_i \sum_j a_{ij}^{2A} \approx C_K \frac{\pi^2}{3N^4|H|^{1/2}} \max_{\mathbf{i}} (\omega^2 * K^2)(\boldsymbol{\lambda}_{\mathbf{i}}) = \mathcal{O}\left(N^{-4}|H|^{-1/2}\right). \tag{27}
$$

Following similar arguments, the same rate is obtained for $\max_i \sum_j a_{ij}^{1B}$. Therefore,

$$
k^4(N)\sigma_{\hat{Q}}^2 \max_i \sum_j a_{ij} = \mathcal{O}\left(\frac{k^4(N)}{N^6|H|}\right), \tag{28}
$$

which tends to zero if the sequence $k(N)\to\infty$ satisfies that $\dfrac{k^4(N)}{N^6|H|}\to 0$.

Condition (ii) holds since the variables X_i are i.i.d. with second order moment $E(X_i^2) = 1$. It remains to show that condition (iii) also holds. Since $\max_i \sum_j |a_{ij}| = \mathcal{O}(N^{-1})$ and taking into account that the spectral ratio of a matrix ($\max_i |\nu_i|$) is bounded by its supremun norm,

$$
\sigma_{\hat{Q}}^2 \max_i |\mu_i|^2 = \mathcal{O}\left(\frac{1}{N^4|H|^{1/2}}\right) \to 0.
$$

Then, the asymptotic convergence to a normal distribution is proved. □

Proof of Theorem 1. The theorem is proved combining the results from Lemmas 1-3. □

Proof of Theorem 2. Consider the decomposition of the test statistic given in Lemma 1: $\hat{Q} = \hat{Q}_{1a} + \hat{Q}_{2a} + \hat{Q}_{3a}$. The sketch of the proof is as follows: find bounds for $\hat{Q}_{2a}$ and $\hat{Q}_{3a}$ and $\hat{Q}_{1a}$ is decomposed in three addends $\hat{Q}_{1a1}$, $\hat{Q}_{1a2}$ and $\hat{Q}_1$. The asymptotic normality of $\hat{Q}_1$ is proved in Theorem 1. Besides, $\hat{Q}_{1a2}$ can be also bounded and

$$
E(\hat{Q}_{1a1}) \approx \int p^2(\mathbf{v})\omega(\mathbf{v})d\mathbf{v}, \tag{29}
$$

where the approximation holds fof $C_N^2 = N^{-1}|H|^{-1/4}$. □

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