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On the core of an airport game and the properties of its center

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Abstract

The airport problems (Littlechild and Owen, 1973) are a well known class of cost allocation problems. The game-theoretic approach consists of transforming these problems into coalitional games, finding a payoff vector that solves the game and studying the corresponding rule. Since the core of the game is nonempty and it has many allocations at which agents' payoffs differ, we study the payoff that is the average expectation of all the stable allocations: the core-center (González-Díaz and Sánchez-Rodríguez, 2007). The structure of the core is exploited to derive insights on the core-center and its properties. The results in this paper build upon explicit integral formulae for the core-center of the airport game. A thorough analysis of these integrals allows not only to study the monotonicity properties of the core-center, and many other axioms discussed in the survey by Thomson (2007), but also to compute in a relatively easy way the core-center of an airport game.

Keywords: cooperative TU games, monotonicity, core, core-center, airport games.

1 Introduction

The airport problem, introduced by Littlechild and Owen (1973), is a classic cost allocation problem that has been widely studied. To get a better idea of the attention it has generated one can refer to the survey by Thomson (2007). One standard approach to study this problem consists of associating a cooperative game with it and take advantage of all the machinery developed for cooperative games to gain insights in the original problem. The core, introduced by Gillies (1953), stands as one of the most studied solution concepts in the theory of cooperative games. Its properties have been thoroughly analyzed and, when a new class of games is studied, one of the first questions to ask is whether or not the games in that class have an nonempty core. This is because of the desirable stability requirements that underly core allocations.

Importantly, the cooperative game associated with an airport problem has a special structure that can be exploited to facilitate the analysis of different solutions. In particular, $2^n - 1$ parameters are needed to define a general n-player cooperative game, whereas for an airport game one just needs n. This special structure simplifies the geometry of the core of such games, since they turn to be defined by $2n - 1$ inequality constraints instead of the usual $2^n - 2$. A related property of airport games is that their Shapley value can be computed in polynomial time, whereas in general the worth of all $2^n - 1$ coalitions must be used to calculate it.

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When the core of a game is nonempty, there is a set of alternatives at which agents' payoffs differ that are coalitionally stable. Studying the center of gravity of such set may be interesting in some cases. The core-center (González-Díaz and Sánchez-Rodríguez, 2007) selects the mathematical expectation of the uniform distribution over the core of the game. The intuition provided by its definition is a good reason to be interested in it and to justify the study of its properties.

In this paper we try to exploit the aforementioned structure of the core of an airport game to gain insights in the monotonicity properties of its core-center.

The formal definition of the core-center is given in terms of integrals over the core of the game. Therefore, in order to study the core and its center of gravity we must extensively employ tools of integral calculus. In fact, our main results build upon particular integral formulae obtained for the core-center of an airport game.

Apart from the results on the standard monotonicity properties, our approach also leads to some findings that may be of independent interest and that we enumerate below.

First, we show that, to each player i, we can associate a face of the core of an airport game in such a way that the derivative of the volume of the core with respect to the cost associated with player i is proportional to the volume of his associated face. This result, which is crucial to study the core-center, may be of independent interest.

Second, we establish that each component of the core-center of an airport game is the ratio of the volumes of the cores of two airport games: the core of the original game and the core of the airport game obtained by replicating agent j. This unexpected result allows to use volume computation algorithms for convex polytopes to develop a method to compute the core-center.

Third, the machinery we develop to study the monotonicity properties of the core-center facilitates the study of other axioms. To illustrate, we have studied the behavior of the core-center not only with respect to monotonicity properties, but also with respect to all the axioms listed in the survey by Thomson (2007). It turns out that the core-center satisfies, among others, those properties which are, arguably, the most important ones.

Finally, as a byproduct of our analysis we get two new natural monotonicity properties that impose conditions on how a change on a cost parameter of a given agent affects the payoffs of the other agents. They are called *higher cost decreasing monotonicity* and *lower cost increasing monotonicity*. The first one says that if a cost c_i increases while the others are held constant, then the payoff decreases for all the players with costs higher than c_i. The second property is a kind of reciprocal, the payoff increases for all the players with costs lower than c_i.

The paper is structured as follows. In Section 2 we present the basic concepts and notations. Then, in Section 3 we obtain an integral representation of the core-center that exploits the special structure of airport games. In Section 4 we develop our main mathematical results, that build upon thorough exploration of the derivatives of the volumes of the core of an airport game. In sections 5 and 6 we study the properties of the core-center and in Section 7 we present a summary of this analysis, comparing the behavior of the core-center with the behavior of the Shapley value and the nucleolus. We have relegated the most technical results to the Appendix.

2 Preliminaries

We assume that there is an infinite set of potential players, indexed by the natural numbers. Then, in each given problem only a finite number of them are present. Let $\mathcal{N}$ be the set of all finite subsets of $\mathbb{N} = \{1, 2, \dots\}$.

A *cost game* with transferable utility is a pair (N, c), where $N \in \mathcal{N}$ and $c\colon 2^N \to \mathbb{R}$ is a function assigning, to each coalition S, its cost $c(S)$. By convention $c(\emptyset) = 0$. Given a coalition of players S, $|S|$ denotes its cardinality. Given $N \in \mathcal{N}$ and $S \subseteq N$, a vector $x \in \mathbb{R}^N$ is referred to as an *allocation* and $x(S) = \sum_{i\in S} x_i$; also, $e_S \in \{0,1\}^N$ is defined as $e_S^i = 1$ if $i \in S$ and $e_S^i = 0$ otherwise. An allocation is *efficient* for (N, c) if $x(N) = c(N)$. A cost game (N, c) is *concave* if, for each $i \in N$ and each S and T such that $S \subseteq T \subseteq N \backslash \{i\}$, $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$.

For most of the discussion and results, we have a fixed n-player set $N = \{1, 2, \dots, n\}$. A *solution* is a correspondence ψ defined on some subdomain of cost games that associates to each game (N, c) in the subdomain a subset $\psi(N, c)$ of efficient allocations. If a solution is single-valued then it is referred to as an *allocation rule.*

Given a cost game (N, c), the *imputation set*, $I(N, c)$, consists of the *individually rational* and efficient allocations, *i.e.*, $I(N, c) = \{x \in \mathbb{R}^N : x(N) = c(N) \text{ and, for each } i \in N,\ x_i \le c(\{i\})\}$. The *core* (Gillies, 1953), is defined as $C(N, c) = \{x \in I(N, c) : \text{ for each } S \subset N,\ x(S) \le c(S)\}$.

An *airport problem* (Littlechild and Owen, 1973) with set of agents N is a positive vector $c \in \mathbb{R}^N$, with $c_i \ge 0$ for each $i \in N$. Throughout the paper, given an airport problem $c \in \mathbb{R}^N$, we make the standard assumption that for each pair of agents i and j, if $i < j$, then $c_i \le c_j$. An allocation for an airport problem is given by a non-negative vector $x \in \mathbb{R}^N$ such that $x(N) = c_n$. An allocation rule selects an allocation for each airport problem in a given subdomain. A complete survey on airport problems is Thomson (2007).

Given an allocation x, the difference $c_i - x_i$ between agent i's cost parameter and her contribution can be seen as her profit at x. A basic requirement is that at an allocation x no group $N' \subset N$ of agents should contribute more that what it would have to pay on its own, $\max\{c_i, i \in N'\}$. Otherwise, the group would unfairly "subsidize" the other agents. The constraints $\sum_{j \le i} x_j \le c_i$ are called the *no-subsidy constraints.*

To each airport problem $c \in \mathbb{R}^N$ one can associate a cost game (N, c) defined, for each $S \subseteq N$, by setting $c(S) = \max_{i \in S}\{c_i\}$; such a game is called an *airport game.* Airport games are concave and their core coincides with the set of allocations satisfying the no-subsidy constraints. We slightly abuse notation and use $C(N, c)$ to refer to both the core of the airport problem $c \in \mathbb{R}^N$, hereafter called *airport core*, and the core of the associated airport game, (N, c),

$$C(N, c) = \big\{x \in \mathbb{R}^n : x \ge 0,\ x(N) = c_n, \text{ and, for each } i < n,\ \sum_{j \le i} x_j \le c_i\big\}.$$

The core of the airport game is contained in the efficient hyperplane $x(N) = c_n$ and it is defined by, at most, $2n - 2$ inequality constraints, instead of the maximum number of $2^n - 2$ inequality constraints of an arbitrary coalitional game. This makes the structure of the core of an airport game more tractable. In particular, whenever $c_1 > 0$, the core of an airport game is a $(n-1)$-dimensional convex polytope. Further, because of the no-subsidy constraints, any core payoff for the highest cost agent (agent n) can be obtained by adding the incremental cost $c_n - c_{n-1}$ to any core allocation of the airport game where agents $n-1$ and n have the same cost c_{n-1}. Therefore,

$$C(N, c) = (c_n - c_{n-1})e_{\{n\}} + C(N, c - (c_n - c_{n-1})e_{\{n\}}).$$

Now, suppose that the agent with the lowest cost leaves the game paying x_1, with $0 \le x_1 \le c_1$. Since $0 \le c_2 - x_1 \le c_3 - x_1 \le \dots \le c_n - x_1$, we have a new airport problem $c^{1,x_1} = (c_2 - x_1, \dots, c_n - x_1) = c_{N\backslash\{1\}} - x_1 e_{N\backslash\{1\}}$ with an associated reduced cost game $(N\backslash\{1\}, c^{1,x_1})$. The problem c^{1,x_1} is known as the *downstream-substraction reduced problem* of c with respect to $N\backslash\{1\}$ and x_1 (Thomson, 2007). In general, given $i \in N$, and $0 \le x_i \le c_i$, the downstream-substraction reduced problem of c with respect to $N\backslash\{i\}$ and x_i, c^{i,x_i}, is defined by

$$c_k^{i,x_i} = \begin{cases} c_k - x_i & k > i \\ \min\{c_i - x_i, c_k\} & k < i. \end{cases}$$

Similarly, the *uniform-substraction reduced problem* of c with respect to $N\backslash\{i\}$ and x_i is defined by:

$$uc_k^{i,x_i} = \begin{cases} c_k - x_i & k > i \\ \max\{c_k - x_i, 0\} & k < i. \end{cases}$$

The next proposition, whose proof is straightforward, relates the core of the airport game (N, c) and the core of the $(N\backslash\{1\}, c^{1,x_1})$ reduced games.

Proposition 1. *Let (N,c) be an airport game. Then,*

$$C(N,c) = \left\{(x_1, x_{N\backslash\{1\}}) \in \mathbb{R}^n : 0 \le x_1 \le c_1, x_{N\backslash\{1\}} \in C(N\backslash\{1\}, c^{1,x_1})\right\} = \bigcup_{0\le x_1 \le c_1} \{x_1\} \times C(N\backslash\{1\}, c^{1,x_1}).$$

Now, if we repeatedly apply the above decomposition, the core of the airport game can be covered with the cores of reduced games of s agents, $1 \le s \le n-1$. In particular, we have the following result for $s = n-1$.

Corollary 1. *Given an airport game (N,c), the allocation $(x_1,\dots,x_n)$ belongs to $C(N,c)$ if and only if, for each $j \in N\backslash\{n\}$, $0 \le x_j \le c_j - \sum_{i<j} x_i$, and $x_n = c_n - \sum_{i<n} x_i$.*

González-Díaz and Sánchez-Rodríguez (2008) associate a *face game* with each face of the core and show that these games have interesting properties for the class of (strictly) convex games. In terms of cost games, given a coalition $T \subset N$, the face F_T of the core contains the allocations that are worst for T and best for $N\backslash T$. Geometrically, when $0 < x_1 < c_1$, the core $C(N\backslash\{1\}, c^{1,x_1})$ is a cross-section of the airport core, which is also the core of a (reduced) airport game. The reduced games for the cases $x_1 = c_1$ and $x_1 = 0$ are the face games for agent 1 and coalition $N\backslash\{1\}$, respectively.

Remark 1. *In general, the core of an airport game (N,c) can be described using reduced airport games with respect to other players. Let $i \in N$. Then,*

$$C(N,c) = \bigcup_{0 \le x_i \le c_i} \{x_i\} \times C\left(N\backslash\{i\}, c^{i,x_i}\right).$$

3 The core-center and its integral representation

As we said in the Introduction, one of the main goals of this paper is to gain insights on the monotonicity of the core of an airport problem using its center of gravity as a proxy. Given a cooperative game (N,c), its *core-center*, $\mu(N,c)$, is defined as the center of gravity of the core (González-Díaz and Sánchez-Rodríguez, 2007). In this section and the next we develop some analytic tools that exploit the structure of the core of an airport game to facilitate the study of the properties of the core-center. Given a convex set A we sometimes use the notation $\mu(A)$ to denote its center of gravity.

In the case of an airport game (N,c) with $c_1 > 0$, the core is an $(n-1)$-manifold contained in the efficient hyperplane, $x(N) = c_n$. The latter is, therefore, the tangent space at each point of the manifold. The vector $(1,1,\dots,1) \in \mathbb{R}^n$ is normal to the manifold at each point and it has length $\sqrt{n}$. The transformation $g\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$, $g(x_1,\dots,x_{n-1}) = (x_1,\dots,x_{n-1}, c_n - x_1 - \dots - x_{n-1})$ defines a coordinate system for $C(N,c)$, so that $g^{-1}(C(N,c))$ is the projection of the core onto $\mathbb{R}^{n-1}$ that simply "drops" the n-th coordinate. Let $\hat{C}(N,c) = g^{-1}(C(N,c)) \subset \mathbb{R}^{n-1}$. This transformation is illustrated in Figure 1.

Given $1 \le r \le n$, let m_r be the r-dimensional Lebesgue measure. Then, the $(n-1)$-dimensional measure of the core is given by

$$m_{n-1}(C(N,c)) = \int_{C(N,c)} dm_{n-1} = \int_{g^{-1}(C(N,c))} \sqrt{n}\; dm_{n-1} = \sqrt{n}\; m_{n-1}(\hat{C}(N,c)).$$

Hence, the volume of the core as a subset of $\mathbb{R}^n$ is $\sqrt{n}$ times the volume of its projection onto $\mathbb{R}^{n-1}$. Analogously, for each $i \in N$, the corresponding component of the core-center, $\mu_i(N,c)$, is given by

$$\frac{1}{m_{n-1}(C(N,c))} \int_{C(N,c)} x_i \; dm_{n-1} = \frac{1}{m_{n-1}(C(N,c))} \int_{\hat{C}(N,c)} \sqrt{n} x_i \; dm_{n-1} = \frac{1}{m_{n-1}(\hat{C}(N,c))} \int_{\hat{C}(N,c)} x_i \; dm_{n-1}.$$

Example 1. *Consider the airport problem with $N = \{1,2\}$ and $0 < c_1 \le c_2$. Clearly, the core of the airport game is the segment $[(0,c_2),(c_1,c_2-c_1)] \subset \mathbb{R}^2$, then $\mu_1(N,c) = \frac{\int_0^{c_1} x_1 dx_1}{\int_0^{c_1} dx_1} = \frac{c_1}{2}$ and $\mu_2(N,c) = c_2 - \frac{c_1}{2}$.*

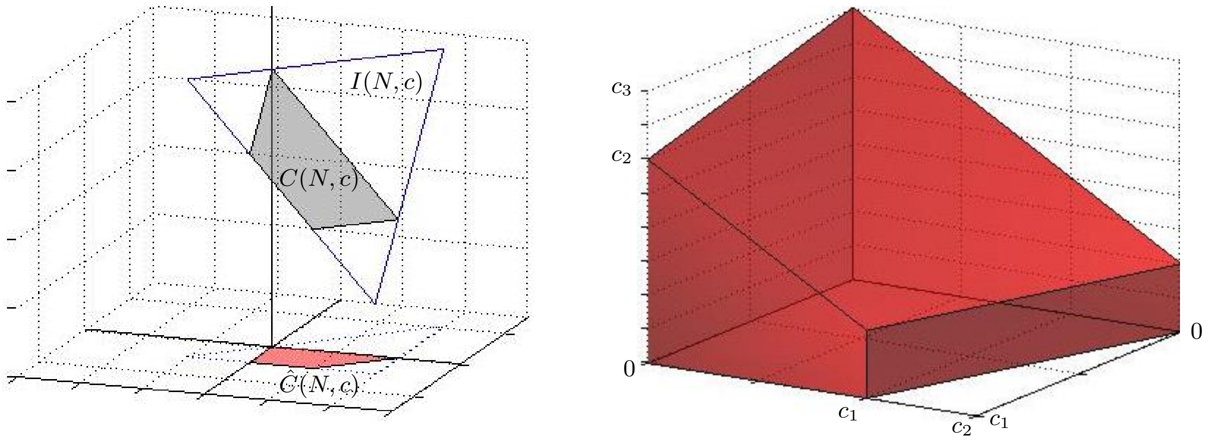

Figure 1: (left) $C(N,c)$ and $\hat{C}(N,c)$ for a 3-player airport game. (right) $\hat{C}(N,c)$ for a 4-player airport game.

The above integral expression for the core-center can be written in terms of iterated integrals. First, we introduce some notation. Given $0 < c_1 \leq c_2 \leq \cdots \leq c_{n-1} \leq c_n$ and $j \in \{1,\ldots,n-1\}$, we define

$$U^j_{n-1}(c_1,\ldots,c_{n-1}) = \int_0^{c_1}\int_0^{c_2-x_1}\cdots\int_0^{c_{n-1}-\sum\limits_{k=1}^{n-2}x_k} x_j \; dx_{n-1}\ldots dx_2 dx_1,$$

$$V_{n-1}(c_1,\ldots,c_{n-1}) = \int_0^{c_1}\int_0^{c_2-x_1}\cdots\int_0^{c_{n-1}-\sum\limits_{k=1}^{n-2}x_k} dx_{n-1}\ldots dx_2 dx_1, \text{ and}$$

$$\hat{\mu}_j(c_1,\ldots,c_{n-1}) = \frac{U^j_{n-1}(c_1,\ldots,c_{n-1})}{V_{n-1}(c_1,\ldots,c_{n-1})},$$

with the convention that $V_0 = 1$. Clearly, U^j_{n-1} is a homogeneous function of degree n, V_{n-1} is a homogeneous function of degree $n-1$, and $\hat{\mu}_j$ is a homogeneous function of degree 1. Now, applying Corollary 1, it is easy to derive the following result.

Theorem 1. *Let (N,c) be an airport game such that $0 < c_1 \leq c_2 \leq \cdots \leq c_n$. Then, $m_{n-1}(\hat{C}(N,c)) = V_{n-1}(c_1,\ldots,c_{n-1})$. Moreover, for each $j \in \{1,\ldots,n-1\}$,*

$$\mu_j(N,c) = \hat{\mu}_j(c_1,\ldots,c_{n-1}) \quad \textit{and} \quad \mu_n(N,c) = \hat{\mu}_{n-1}(c_1,\ldots,c_{n-1}) + (c_n - c_{n-1}).$$

Note that all the coordinates of the core-center, except the last one, are independent of c_n. In addition, all the coordinates μ_j are homogeneous functions of degree 1.

Remark 2. *The decompositions in Remark 1 give rise to alternative integral expressions for the core-center.*

4 On the differentiability of the core-center

Suppose that $0 < c_1 \leq c_2 \leq \cdots \leq c_n$. From the integral expressions derived in the previous section it is clear that the functions V_{n-1} and U^j_{n-1}, $j \in N\backslash\{n\}$, can be differentiated with respect to the c_i costs, with $i \in N\backslash\{n\}$. As a first consequence, we obtain a result that is fundamental for the analysis in this paper, namely, a representation of the core-center as a ratio of volumes of airport cores. We relegate its proof to the Appendix.

Theorem 2. *Let (N,c) be an airport game such that $0 < c_1 \leq c_2 \leq \cdots \leq c_n$ and fix $j \in N \setminus \{n\}$. Then:*

1. $U_{n-1}^j(c_1,\ldots,c_{n-1}) = V_n(c_1,\ldots,c_j,c_j,\ldots,c_{n-1})$

2. $\hat{\mu}_j(c_1,\ldots,c_{n-1}) = \dfrac{V_n(c_1,\ldots,c_j,c_j,\ldots c_{n-1})}{V_{n-1}(c_1,\ldots,c_{n-1})}$

Remark 3. *Apart from being a key tool for the ensuing analysis, Theorem 2 is interesting on its own. There are no known efficient deterministic algorithms for computing the centroid of a convex body. Therefore, the issue of computing the core-center of a general balanced game is very complex. The second statement in Theorem 2 says that the center of gravity of the core of an airport game can be computed just by using the volume of its core and the volume of the core of the airport game obtained by replicating agent j. Then, Theorem 2 opens the door to implementing volume computation algoritms for convex polytopes to compute, in a relatively easy way, the core-center of an airport game.*

Now, we turn our attention the derivatives of the functions V_{n-1} and U_{n-1}^j with respect to the costs c_i.

Proposition 2. *For all $i \in N\backslash\{n\}$,*

$$\frac{\partial V_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1}) = V_{i-1}(c_1,\ldots,c_{i-1})V_{n-1-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i).$$

Proof. A direct computation using Leibnitz's rule shows that

$$\frac{\partial V_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1}) = \int_0^{c_1}\cdots\int_0^{c_{i-1}-\sum\limits_{k=1}^{i-2}x_k}\int_0^{c_{i+1}-c_i}\cdots\int_0^{c_{n-1}-c_i-\sum\limits_{k=i+1}^{n-2}x_k} dx_{n-1}\ldots dx_{i+1}dx_{i-1}\ldots dx_1.$$

This iterated integral can be split as the product of the following two integrals,

$$\int_0^{c_1}\cdots\int_0^{c_{i-1}-\sum\limits_{k=1}^{i-2}x_k} dx_{i-1}\ldots dx_1 \quad\text{and}\quad \int_0^{c_{i+1}-c_i}\cdots\int_0^{c_{n-1}-c_i-\sum\limits_{k=i+1}^{n-2}x_k} dx_{n-1}\ldots dx_{i+1}.$$

The first one coincides with $V_{i-1}(c_1,\ldots,c_{i-1})$ while the second is equal to $V_{n-1-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i)$. □

Proposition 3. *Let $i,j \in N\backslash\{n\}$. Then,*

$$\frac{\partial U_{n-1}^j}{\partial c_i}(c_1,\ldots,c_{n-1}) = \begin{cases} V_{i-1}(c_1,\ldots,c_{i-1})U_{n-1-i}^{j-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i) & i<j \\ U_{i-1}^j(c_1,\ldots,c_{i-1})V_{n-1-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i) & i>j \\ V_i(c_1,\ldots,c_i)V_{n-1-i}(c_{i+1},\ldots,c_{n-1})) & i=j. \end{cases}$$

Moreover, $V_i(c_1,\ldots,c_i)V_{n-1-i}(c_{i+1},\ldots,c_{n-1})$ can be equivalently written as

$$\Big(c_iV_{i-1}(c_1,\ldots,c_{i-1}) - \sum_{k=1}^{i-1}U_{i-1}^k(c_1,\ldots,c_{i-1})\Big)V_{n-1-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i)$$

.

Proof. The computations when either $i \neq j$ are straightforward from the chain rule, Theorem 2 and Proposition 2. The same applies to the first equality in the case $j = i$. The alternative expression is obtained by directly applying Leibnitz's rule to the integral formulation of U_{n-1}^j. □

Next, we present a series of results that relate the partial derivatives of the functions V_{n-1} and U^j_{n-1} to the faces of $\hat{C}(N,c)$ and its centroids. In particular, the derivative of V_{n-1} with respect to a cost c_i is proportional to the volume of the corresponding face of the core. Let (N,c) be an airport game such that $0 < c_1 \leq c_2 \leq \cdots \leq c_n$. Denote by F_i, $i \in N \backslash \{n\}$, the i-th face of $\hat{C}(N,c)$, *i.e.*,

$$F_i = \hat{C}(N,c) \cap \{x \in \mathbb{R}^{n-1} : x_1 + \cdots + x_i = c_i\} \subset \mathbb{R}^{n-1}.$$

Let $V(F_i)$ be its $(n-2)$-measure and $\mu(F_i)$ be its centroid. The next decomposition of F_i is easily derived.

Proposition 4. *For all $i \in N\backslash\{n\}$, we have that*

$$F_i = C(\{1,\ldots,i\},(c_1,\ldots,c_i)) \times \hat{C}(\{i+1,\ldots,n\},(c_{i+1}-c_i,\ldots,c_n-c_i)).$$

The coordinates of the centroid $\mu(F_i)$ are

$$\mu_j(F_i) = \begin{cases} \mu_j(\{1,\ldots,i\},(c_1,\ldots,c_i)) & \text{if } i \geq j \\ \hat{\mu}_{j-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i) & \text{if } i < j \leq n-1 \end{cases}.$$

We now show that, whenever the $(n-2)$-measure of the face F_i is positive, its centroid can be obtained using the partial derivatives computed above.

Proposition 5. *Let (N,c) be an airport game with $0 < c_1 \leq c_2 \leq \cdots \leq c_n$. For all $i,j \in N\backslash\{n\}$ such that $V(F_i) > 0$,*

1. $\dfrac{\partial V_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1}) = \frac{1}{\sqrt{i}} V(F_i)$.

2. $\mu_j(F_i) = \dfrac{\frac{\partial U^j_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})}{\frac{\partial V_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})}$.

Proof. Assume that $V(F_i) > 0$. Recall that $C(\{1,\ldots,i\},(c_1,\ldots,c_i))$ is an $(i-1)$-dimensional polytope contained in the hyperplane $x_1 + \cdots + x_i = c_i$, so

$$\begin{aligned} m_{i-1}(C(\{1,\ldots,i\},(c_1,\ldots,c_i)) &= \sqrt{i}\; m_{i-1}(\hat{C}(\{1,\ldots,i\},(c_1,\ldots,c_i))) \\ &= \sqrt{i}\; V_{i-1}(c_1,\ldots,c_{i-1}). \end{aligned}$$

On the other hand, the measure of $\hat{C}(\{i+1,\ldots,n\},(c_{i+1}-c_i,\ldots,c_n-c_i))$ as a subset of $\mathbb{R}^{n-1-i}$ is $V_{n-1-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i)$. Therefore, the first assertion follows immediately from Proposition 2 and the decomposition of Proposition 4.

The proof of the second property is divided in three cases. First, assume that $i < j$. Then, by Proposition 4,

$$\mu_j(F_i) = \hat{\mu}_{j-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i) = \frac{U^{j-i}_{n-1-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i)}{V_{n-1-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i)} = \frac{\frac{\partial U^j_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})}{\frac{\partial V_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})},$$

where the last equality follows from Propositions 2 and 3.

If $i > j$, then, as above, by Propositions 2, 3 and 4, we have

$$\begin{aligned} \mu_j(F_i) &= \mu_j(\{1,\ldots,i\},(c_1,\ldots,c_i)) \\ &= \hat{\mu}_j(c_1,\ldots,c_{i-1}) = \frac{U^j_{i-1}(c_1,\ldots,c_{i-1})}{V_{i-1}(c_1,\ldots,c_{i-1})} = \frac{\frac{\partial U^j_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})}{\frac{\partial V_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})}. \end{aligned}$$

Finally, when $j = i$, by Proposition 4

$$\mu_i(F_i) = \mu_i(\{1,\ldots,i\},(c_1,\ldots,c_i)) = c_i - \sum_{k=1}^{i-1} \mu_k(\{1,\ldots,i\},(c_1,\ldots,c_i))$$
$$= c_i - \sum_{k=1}^{i-1} \hat{\mu}_k(c_1,\ldots,c_{i-1}) = c_i - \sum_{k=1}^{i-1} \frac{U_{i-1}^k(c_1,\ldots,c_{i-1})}{V_{i-1}(c_1,\ldots,c_{i-1})}$$

Therefore,

$$\mu_i(F_i)V_{i-1}(c_1,\ldots,c_{i-1}) = c_i V_{i-1}(c_1,\ldots,c_{i-1}) - \sum_{k=1}^{i-1} U_{i-1}^k(c_1,\ldots,c_{i-1}).$$

Substituting this expression in Proposition 3 and using Proposition 2, the result follows. □

Remark 4. *The first equality of Proposition 5 admits a generalization for any convex polyhedron (Lasserre, 1983).*

5 Properties of the core-center

Following Thomson (2007), we recall a list of properties of allocation rules for airport problems. We distinguish between fixed-population axioms and variable-population ones.

Fixed population

Let ψ be a rule and (N, c) an airport game. We say that ψ satisfies:

- *Non-negativity* if, for each $i \in N$, $\psi_i(N,c) \geq 0$.
- *Cost boundedness* if, for each $i \in N$, $\psi_i(N,c) \leq c_i$.
- *Efficiency* if $\sum_{i\in N} \psi_i(N,c) = c_n$.
- *No-subsidy* if, for each $S \subset N$, $\sum_{i \in S} \psi_i(N,c) \leq \max_{i\in S} c_i$.
- *Anonymity* if, for each permutation π of N and each $i \in N$, $\psi_i(\pi(N,c)) = \psi_{\pi_i^{-1}}(N,c)$.
- *Equal treatment of equals* if, for each $i,j \in N$ with $c_i = c_j$, then $\psi_i(N,c) = \psi_j(N,c)$.
- *Order preservation for contributions* if, for each pair $i,j \in N$ with $c_i \leq c_j$, then $\psi_i(N,c) \leq \psi_j(N,c)$.
- *Order preservation for benefits* if, for each pair $i,j \in N$ with $c_i \leq c_j$, then $c_i - \psi_i(N,c) \leq c_j - \psi_j(N,c)$.

We now turn to relational requirements on rules. Let ψ be a rule and (N,c), (N,c') and (N,c'') airport games. We say that ψ satisfies:

- *Homogeneity* if, for each $\alpha > 0$, $\psi(N,\alpha c) = \alpha\psi(N,c)$.
- *Continuity* if, for each sequence $\{(N,c^\nu)\}_{\nu\in\mathbb{N}}$ of airport problems such that $c^\nu \to c$, then $\psi(N,c^\nu) \to \psi(N,c)$.
- *Independence of at-least-as-large costs* if for each $i \in N$ such that
  - $c'_i = c_i$,
  - $c'_j = c_j$, for each $j \in N\backslash\{i\}$ such that $c_j < c_i$ and

- $c'_j \geq c_i$, for each $j \in N \backslash \{i\}$ such that $c_j \geq c_i$,

then $\psi_i(N, c') = \psi_i(N, c)$.

- *Last-agent cost additivity* if for each $i \in N$ such that $c_i = \max_{j \in N} c_j$ whenever $c'_{N\backslash\{i\}} = c_{N\backslash\{i\}}$ and $c'_i = c_i + \gamma$, then $\psi_{N\backslash\{i\}}(N, c') = \psi_{N\backslash\{i\}}(N, c)$ and $\psi_i(N, c') = \psi_i(N, c) + \gamma$.
- *weak last-agent cost additivity* is property weaker than *last-agent cost additivity*, that demands that the payment required to the last agent should increase by an amount equal to the increase in her cost parameter, nothing being said about the payments required of the others.
- *Conditional cost additivity* if $\psi(N, c + c') = \psi(N, c) + \psi(N, c')$ whenever the agents of both airport games are ordered in the same way, .
- *Individual cost monotonicity* if for each $i \in N$ such that $c'_i \geq c_i$ and, for all $j \in N\backslash\{i\}$, $c'_j = c_j$, then $\psi_i(N, c') \geq \psi_i(N, c)$.
- *Downstream cost monotonicity* if for each $i \in N$ such that for each $j \in N$ with $c_j < c_i$, $c'_j = c_j$ and for each $j \in N$ with $c_j \geq c_i$, $c'_j - c_j = c'_i - c_i \geq 0$, then for each $j \in N$ such that $c_j \geq c_i$, $\psi_j(N, c') \geq \psi_j(N, c)$.
- *Marginalism* if, under the hypotheses of downstream cost monotonicity, for each $j \in N$ such that $c_j < c_i$, $\psi_j(N, c') = \psi_j(N, c)$.
- *Strong cost monotonicity* if, for each pair (N, c) and (N, c') with $c \leq c'$, then $\psi(N, c) \leq \psi(N, c')$.
- *Weak cost monotonicity* if $c' = c + c''$, then $\psi(N, c') \geq \psi(N, c)$.
- *Incremental no subsidy* if $c = c' + c''$ then, for each $i \in N$, $\sum_{c_j \leq c_i} (\psi_j(N, c') - \psi_j(N, c)) \leq c'_i - c_i$.
- *Reciprocity* if for each $i \in N$ such that
  - $\sum_{j \leq i} \psi_i(N, c) = c_i$,
  - there is an airport problem c'' such that $c' = c + c''$, and
  - $c_i - \sum_{j \leq i} \psi_i(N, c) \geq c'_n - c'_i - (c_n - c_i)$,

  then there is a pair $\{j, k\} \subset N$ such that $c_j \leq c_i < c_k$ and $\psi_j(N, c') - \psi_j(N, c) \geq \psi_k(N, c') - \psi_k(N, c)$.
- *Others-oriented cost monotonicity* if, under the assumptions of individual cost monotonicity, for each $j \in N\backslash\{i\}$, $\psi_j(N, c') \leq \psi_j(N, c)$.

Variable population

Let ψ be a rule and (N, c) an airport game. We say that ψ satisfies:

- *Population monotonicity* if, for each N and N' with $N' \subset N$, $\psi_{N'}(N, c) \leq \psi(N', c_{N'})$.
- *First-agent consistency* if, for each (N, c) and $j \in N$ with $j > 1$, then $\psi_j(N, c) = \psi_j(N\backslash\{1\}, c^{1,\psi_1(N,c)})$.
- *Downstream-subtraction consistency* if, for each (N, c), each $i \in N$ and each $j \neq i$, then $\psi_j(N, c) = \psi_j(N\backslash\{i\}, c^{i,\psi_i(N,c)})$.
- *Last-agent consistency* if, for each (N, c) and each $j < n$, $\psi_j(N, c) = \psi_j(N\backslash\{n\}, c^{n,\psi_n(N,c)})$.
- *Uniform-substraction consistency* if, for each (N, c) with $c_{n-1} = c_n$, each $i \in N$ with $c_i = c_n$ and each $j \neq i$, then $\psi_j(N, c) = \psi_j(N\backslash\{i\}, uc^{i,\psi_i(N,c)})$.

Let us begin our analysis of which properties the core-center rule satisfies. A complete recapitulation of our findings will be presented in Section 7.

Proposition 6. *The core-center satisfies non-negativity, cost-boundedness, efficiency, no-subsidy, anonymity, homogeneity, equal treatment of equals and continuity.*

Proof. The first six properties follow from the fact that any core allocation satisfies them. A couple of comments on the last two properties are needed. I González-Díaz and Sánchez-Rodríguez (2007) prove that the core-center treats symmetric players equally and that it is a continuous function of the values of the characteristic function. In our context, equal treatment of equals holds because agents with the same cost parameter are symmetric players in the associated airport game. Similarly, continuity holds because the values of the characteristic function are continuous with respect to the cost parameters. Then, the core-center satisfies continuity since it is a composition of continuous functions. □

Proposition 7. *The core-center satisfies order preservation for contributions.*

Proof. Trivially, by Theorem 1, $\mu_{n-1}(N,c) \leq \mu_n(N,c)$. Now, take two consecutive agents i and $i+1$ where $i < n-1$. Then, by Theorem 2, $\mu_i(N,c) \leq \mu_{i+1}(N,c)$ if and only if

$$V_n(c_1,\ldots,c_i,c_i,c_{i+1},\ldots,c_{n-1}) \leq V_n(c_1,\ldots,c_i,c_{i+1},c_{i+1},\ldots,c_{n-1}),$$

which is immediate since $c_i \leq c_{i+1}$. □

Proposition 8. *The core-center satisfies order preservations for benefits.*

Proof. Recall that $\mu_{n-1}(N,c)-\mu_n(N,c) = c_{n-1}-c_n$. Let $i < n-1$. We have to prove that $\mu_{i+1}(N,c)-\mu_i(N,c) \leq c_{i+1}-c_i$. Applying Theorem 2, the difference $\mu_{i+1}(N,c)-\mu_i(N,c)$ can be written as follows:

$$\mu_{i+1}(N,c)-\mu_i(N,c) = \frac{\int_0^{c_1}\int_0^{c_2-x_1}\cdots\int_0^{c_i-\sum_{j=1}^{i-1}x_j}\int_{c_i-\sum_{j=1}^{i}x_j}^{c_{i+1}-\sum_{j=1}^{i}x_j}\cdots\int_0^{c_{n-1}-\sum_{j=1}^{n-1}x_j} dx_n\ldots dx_2dx_1}{V_{n-1}(c_1,\ldots,c_{n-1})}$$

$$= \frac{\int_0^{c_1}\int_0^{c_2-x_1}\cdots\int_{c_i-\sum_{j=1}^{i}x_j}^{c_{i+1}-\sum_{j=1}^{i}x_j} V_{n-1-i}\big(c_{i+1}-\sum_{j=1}^{i+1}x_j,\ldots,c_{n-1}-\sum_{j=1}^{i+1}x_j\big)dx_{i+1}dx_i\ldots dx_1}{V_{n-1}(c_1,\ldots,c_{n-1})}$$

Now, by the mean-value theorem for integrals, there exists a point $\xi \in (c_i-\sum_{j=1}^{i}x_j, c_{i+1}-\sum_{j=1}^{i}x_j)$ such that

$$\int_{c_i-\sum_{j=1}^{i}x_j}^{c_{i+1}-\sum_{j=1}^{i}x_j} V_{n-1-i}\big(c_{i+1}-\sum_{j=1}^{i+1}x_j,\ldots,c_{n-1}-\sum_{j=1}^{i+1}x_j\big)dx_{i+1} =$$

$$(c_{i+1}-c_i)V_{n-1-i}\big(c_{i+1}-\sum_{j=1}^{i}x_j-\xi,\ldots,c_{n-1}-\sum_{j=1}^{i}x_j-\xi\big).$$

But, since $c_s-\sum_{j=1}^{i}x_j-\xi \leq c_s-c_i$ for all $i+1 \leq s \leq n-1$, we have that

$$V_{n-1-i}\big(c_{i+1}-\sum_{j=1}^{i}x_j-\xi,\ldots,c_{n-1}-\sum_{j=1}^{i}x_j-\xi\big) \leq V_{n-1-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i).$$

Therefore,

$$\begin{aligned}
\mu_{i+1}(N,c)-\mu_i(N,c) &\le (c_{i+1}-c_i)\frac{\displaystyle\int_0^{c_1}\cdots\int_0^{c_i-\sum\limits_{j=1}^{i-1}x_j} V_{n-1-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i)dx_i\ldots dx_1}{V_{n-1}(c_1,\ldots,c_{n-1})}\\
&=(c_{i+1}-c_i)\frac{\displaystyle\int_0^{c_1}\cdots\int_0^{c_i-\sum\limits_{j=1}^{i-1}x_j} V_{n-1-i}(c_{i+1}-c_i,\ldots,c_{n-1}-c_i)dx_i\ldots dx_1}{\displaystyle\int_0^{c_1}\cdots\int_0^{c_i-\sum\limits_{j=1}^{i-1}x_j} V_{n-1-i}(c_{i+1}-\sum_{j=1}^{i}x_j,\ldots,c_{n-1}-\sum_{j=1}^{i}x_j)dx_i\ldots dx_1}\\
&\le c_{i+1}-c_i
\end{aligned}$$

where the last inequality holds because, $\sum\limits_{j=1}^{i} x_i \le c_i$ implies that $c_s - c_i \le c_s - \sum\limits_{j=1}^{i} x_i$ for $i+1 \le s \le n-1$. $\square$

Example 2. *The core-center does not satisfy independence of at least-as-large costs. Consider the airport problem with $N=\{1,2,3\}$ and $0<c_1\le c_2\le c_3$. A simple computation shows that $\mu_1(N,c)=\frac{\int_0^{c_1}\int_0^{c_2-x} xdydx}{\int_0^{c_1}\int_0^{c_2-x} dydx}=\frac{c_1}{3}\frac{3c_2-2c_1}{2c_2-c1}$. Let $c=(1,2,3)$ and $c'=(1,3,3)$. Then, $\mu_1(N,c)=\frac{4}{9}<\mu_1(N,c')=\frac{7}{15}$.*

Proposition 9. *The core-center satisfies last-agent cost additivity.*

Proof. Let (N,c) and (N,c') be airport games satisfying the hypothesis of last-agent cost additivity. Then,

$$(x_1,\ldots,x_{n-1},c_n-\sum_{i=1}^{n-1}x_i)\in C(N,c)\Leftrightarrow(x_1,\ldots,x_{n-1},c'_n-\sum_{i=1}^{n-1}x_i)\in C(N,c').$$

Hence, $\hat{C}(N,c)=\hat{C}(N,c')$ and $\mu_{N\setminus\{n\}}(N,c')=\mu_{N\setminus\{n\}}(N,c)$. By Theorem 1, $\mu_n(N,c')=\mu_{n-1}(N,c')+(c'_n-c'_{n-1})=\mu_{n-1}(N,c)+(c_n+\gamma-c_{n-1})=\mu_n(N,c)+\gamma$. $\square$

Remark 5. *If an allocation rule ψ satisfies efficiency and last-agent cost additivity, then*

$$\psi_{n-1}(N,c)=\frac{1}{2}\big(c_{n-1}-\sum_{i=1}^{n-2}\psi_i(N,c)\big).$$

Example 3. *The core-center does not satisfy conditional cost additivity. From Example 2, we know that for an airport problem with $N=\{1,2,3\}$ and $0<c_1\le c_2\le c_3$, $\mu_1(N,c)=\frac{c_1}{3}\frac{3c_2-2c_1}{2c_2-c1}$. Again, let $c=(1,2,3)$ and $c'=(1,3,3)$. Then, $c+c'=(2,5,6)$ and $\mu_1(N,c)+\mu_1(N,c')=\frac{4}{7}+\frac{7}{15}=\frac{41}{45}\ne\mu_1(N,c+c')=\frac{11}{12}$.*

Proposition 10. *The core-center satisfies neither last-agent consistency, nor uniform substraction consistency, nor downstream substraction consistency.*

Proof. The result follows from the following characterizations and the fact that the core-center satisfies equal treatment of equals, homogeneity and last-agent cost additivity:

- The slack maximizer rule is the only rule satisfying equal treatment of equals, weak last-agent cost additivity and last-agent consistency (Yeh, 2003).
- The constrained equal benefits rule is the only rule satisfying equal treatment of equals, homogeneity, last-agent cost additivity and uniform-subtraction consistency (Potters and Sudhölter, 2005).

- The slack maximizer rule is the only rule satisfying equal treatment of equals, homogeneity, last-agent cost additivity and downstream-subtraction consistency (Potters and Sudhölter, 2005). □

Example 4. *The core-center does not satisfy first-agent consistency. Let* $N = \{1,2,3,4\}$ *and* $c = (1,2,3,4)$*. Then* $\mu_1(N,c) = \frac{27}{64}$*,* $\mu_2(N,c) = \frac{43}{64}$ *and*

$$c^{1,\mu_1(c)} = (c_2 - \mu_1(c), c_3 - \mu_1(c), c_4 - \mu_1(c)) = (\tfrac{101}{64}, \tfrac{165}{64}, \tfrac{229}{64}).$$

An easy computation shows that $\mu_2(N \setminus \{1\}, c^{1,\mu_1(c)}) = \frac{667}{991} \neq \mu_2(N,c)$*.*

Nevertheless, the core-center satisfies first-agent consistency for 3*-player airport games. Indeed, if* $(N,c) = (\{1,2,3\},(c_1,c_2,c_3))$*,* $c_1 \leq c_2 \leq c_3$*, then*

$$\begin{aligned}
\mu_2(N,c) &= \frac{c_2 - \mu_1(N,c)}{2} = \mu_2\big(\{2,3\},(c_2 - \mu_1(N,c), c_3 - \mu_1(N,c))\big)\\
\mu_3(N,c) &= c_3 - c_2 + \mu_2(N,(c_1,c_2,c_2)) = c_3 - c_2 + \mu_2(N,(c_1,c_2,c_3))\\
&= c_3 - c_2 + \frac{c_2 - \mu_1(N,c)}{2} = \mu_3\big(\{2,3\},(c_2 - \mu_1(N,c), c_3 - \mu_1(N,c))\big).
\end{aligned}$$

Example 5. *The core-center does not satisfy strong cost monotonicity. Indeed, let* $N = \{1,2,3\}$ *and* $c = (1,2,4)$*. Then* $\mu(N,c) = (\frac{4}{9}, \frac{7}{9}, \frac{25}{9})$*. Now, for the airport problem with costs* $c' = (1,3,4)$*,* $\mu(N,c') = (\frac{7}{15}, \frac{19}{15}, \frac{34}{15})$*. Thus,* $c \leq c'$ *but* $\mu_3(N,c') < \mu_3(N,c)$*.*

Example 6. *The core-center does not satisfy marginalism. Consider the airport problems with players* $N = \{1,2,3\}$ *and costs* $c = (1,2,3)$ *and* $c' = (1,3,4)$ *that satisfy the hypothesis of downstream cost monotonicity (with* $i = 2$*). Their respective core-centers are* $\mu(N,c) = (\frac{4}{9}, \frac{7}{9}, \frac{16}{9})$ *and* $\mu(N,c') = (\frac{7}{15}, \frac{19}{15}, \frac{34}{15})$ *so, in particular,* $\mu_1(N,c) \neq \mu_1(N,c')$*.*

6 On the monotonicity of the core-center

In this section we discuss the behavior of the core-center with respect to well known monotonicity properties. Moreover, we also introduce two natural monotonicity properties that have not been studied before in the literature, *higher cost decreasing monotonicity* and *lower cost increasing monotonicity.* Throughout this section, when no confusion arises we often use notations such as $\mu_j(c_1,\ldots,c_p)$ instead of $\mu_j(\{1,\ldots,p\},(c_1,\ldots,c_p))$. Some of the forthcoming results rely on the following connection between the monotonicity of the core-center components and the centroid of the F_i faces.

Proposition 11. *Let* (N,c) *be an airport game with* $0 < c_1$ *and* $i,j \in N \backslash \{n\}$*. Then,* $\mu_j(N,c)$ *is increasing with respect to* c_i *if and only if* $\mu_j(N,c) \leq \mu_j(F_i)$*. Analogously, it is decreasing if and only if* $\mu_j(N,c) \geq \mu_j(F_i)$*.*

Proof. Recall that $\mu_j(N,c) = \hat{\mu}_j(c_1,\ldots,c_{n-1})$ and, by Theorem 2,

$$\frac{\partial \hat{\mu}_j}{\partial c_i}(c_1,\ldots,c_{n-1}) = \frac{\frac{\partial U^j_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})V_{n-1}(c_1,\ldots,c_{n-1}) - U^j_{n-1}(c_1,\ldots,c_{n-1})\frac{\partial V_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})}{(V_{n-1}(c_1,\ldots,c_{n-1}))^2}.$$

Thus, $\mu_j(N,c)$ is increasing with respect to c_i if and only if the numerator is positive. Now, by Proposition 5 the latter is equivalent to

$$\mu_j(F_i) = \frac{\frac{\partial U^j_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})}{\frac{\partial V_{n-1}}{\partial c_i}(c_1,\ldots,c_{n-1})} \geq \frac{U^j_{n-1}(c_1,\ldots,c_{n-1})}{V_{n-1}(c_1,\ldots,c_{n-1})} = \mu_j(N,c). \qquad \square$$

We now present two new monotonicity properties for a rule. The first one states that if one single cost c_i increases while the others are held constant, then the rule decreases its value for all the players with costs higher than c_i. The second property is a kind of reciprocal, the allocation rule increases its value for all the players with costs lower than c_i.

Definition 1. *Let ψ be a rule. Suppose that we have two airport games (N, c) and (N, c'), and $i \in N$ such that $c'_i \geq c_i$ and for all $j \in N \backslash \{i\}$, $c'_j = c_j$. Then,*

- *ψ satisfies* higher cost decreasing monotonicity *if $\psi_j(N, c') \leq \psi_j(N, c)$ whenever $c_j > c_i$.*
- *ψ satisfies* lower cost increasing monotonicity *if $\psi_j(N, c') \geq \psi_j(N, c)$ whenever $c_j \leq c_i$.*

Proposition 12. *Let (N, c) be an airport game with $0 < c_1$ and $i, j \in N \backslash \{n\}$, $i < j$. The component $\mu_j(N, c)$ is decreasing with respect to c_i if and only if $\hat{\mu}_j(c_1, \ldots, c_{n-1}) \geq \hat{\mu}_{j-i}(c_{i+1} - c_i, \ldots, c_{n-1} - c_i)$.*

Proof. According to Proposition 11, $\mu_j(N, c)$ is decreasing with respect to c_i if and only if $\mu_j(N, c) \geq \mu_j(F_i)$. Now, the result is a direct application of Proposition 4 when $i < j$. □

The HCDM property for the core-center is a consequence of Theorem 4, which is a particular case of Theorem 3 below, whose proof is relegated to the Appendix.

Theorem 3. *For all $p, k \in \mathbb{N}$ such that $k \geq p$, and all $0 < \delta \leq d_1 \cdots \leq d_p \leq \cdots \leq d_k$, we have that*

$$\hat{\mu}_p(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta) \leq \hat{\mu}_{p+1}(\delta, d_1, \ldots, d_p, \ldots, d_k).$$

Theorem 4. *Given $j \in \{2, \ldots, n-1\}$ and costs $0 < c_1 \leq \cdots \leq c_{n-1}$, we have that*

$$\hat{\mu}_j(c_1, \ldots, c_{n-1}) \geq \hat{\mu}_{j-1}(c_2 - c_1, \ldots, c_{n-1} - c_1) \geq \cdots \geq \hat{\mu}_1(c_j - c_{j-1}, \ldots, c_{n-1} - c_{j-1}).$$

Proof. This is a particular case of Theorem 3. Given $j \in \{2, \ldots, n-1\}$ and $r \in \{0, \ldots, j-2\}$, the inequality

$$\hat{\mu}_{j-r}(c_{r+1} - c_r, \ldots, c_{n-1} - c_r) \geq \hat{\mu}_{j-r-1}(c_{r+2} - c_{r+1}, \ldots, c_{n-1} - c_{r+1})$$

follows by taking $k = n - r - 2$, $p = j - r - 1$ and $(\delta, d_1, \ldots, d_k) = (c_{r+1} - c_r, c_{r+2} - c_r, \ldots, c_{n-1} - c_r)$. □

Proposition 13. *The core-center satisfies higher cost decreasing monotonicity.*

Proof. As a corollary of Proposition 12 and Theorem 4 we have that if $i, j \in N \backslash \{n\}$, $i < j$, then $\mu_j(N, c)$ is decreasing with respect to c_i. □

We now move to LCIM. First, note that it implies individual monotonicity. The LCIM property for the core-center is a consequence of Theorem 5 below, whose proof is relegated to the Appendix.

Theorem 5. *Given $p, s \in \mathbb{N}$ and costs $0 < c_1 \leq \cdots \leq c_p \leq c_{p+1} \leq \cdots \leq c_{p+s}$, we have that*

$$\mu_p(c_1, \ldots, c_p) \geq \mu_p(c_1, \ldots, c_p, c_{p+1}) \geq \cdots \geq \mu_p(c_1, \ldots, c_p, c_{p+1}, \ldots, c_{p+s})$$

Proposition 14. *The core-center satisfies lower cost increasing monotonicity.*

Proof. Let (N, c) be an airport game such that $0 < c_1 \leq c_2 \leq \cdots \leq c_n$. Then the core-center satisfies LCIM if and only if $\mu_j(N, c)$ is increasing with respect to c_i for all $j \leq i \leq n$. First, assume that $j \leq i < n$. By Proposition 4, $\mu_j(F_i) = \mu_j(\{1, \ldots, i\}, (c_1, \ldots, c_i))$. Now, according to Theorem 5,

$$\mu_j(c_1, \ldots, c_j) \geq \mu_j(c_1, \ldots, c_j, \ldots, c_i) \geq \mu_j(c_1, \ldots, c_n) = \mu_j(N, c),$$

and LCIM is now a direct consequence of Proposition 11. As for the case $j \leq i = n$, we already know that $\mu_j(N, c)$, $j = 1, \ldots, n-1$, is independent of c_n, and that $\frac{\partial \mu_n}{\partial c_n}(N, c) = 1$. □

Corollary 2. *The core-center satisfies individual cost monotonicity.*

Example 7. *The core-center does not satisfy others-oriented cost monotonicity. To see this one can consider, for instance, the two problems in Example 2, where an increase in the cost of player 2 results in a lower core-center payoff for player 1.*

The next monotonicity property, *downstream cost monotonicity*, is a consequence of Theorem 6 below, whose proof is relegated to the Appendix.

Theorem 6. *Given indices $i, j \in \mathbb{N}$, $j \geq i$, a value $\gamma > 0$ and costs $0 < c_1 \leq \cdots \leq c_k$, we have that*

$$\mu_j(c_1, c_2, \ldots, c_i + \gamma, \ldots, c_k + \gamma) \geq \mu_j(c_1, c_2, \ldots, c_k).$$

Proposition 15. *The core-center satisfies downstream cost monotonicity.*

Proof. Let (N, c) be an airport game such that $0 < c_1 \leq c_2 \leq \cdots \leq c_n$. Observe that downstream cost monotonicity can be rewritten as follows. If for each pair (N, c) and (N, c') and each $i \in N$, if for each $j \in N$ such that $c_j < c_i$, $c'_j = c_j$ and each $j \in N$ such that $c_j \geq c_i$, $c'_j = c_j + \gamma$ ($\gamma \geq 0$), then for each $j \in N$ such that $c_j \geq c_i$, $\psi_j(N, c') \geq \psi_j(N, c)$. Thus,

$$\begin{array}{lllllllll} (N,c): & c_1 & c_2 & \ldots & c_{i-1} & c_i & \ldots & c_n \\ (N,c'): & c_1 & c_2 & \ldots & c_{i-1} & c_i+\gamma & \ldots & c_n+\gamma, \end{array}$$

and the result is a direct consequence of Theorem 6. □

Proposition 16. *Let ψ a rule satisfying downstream cost monotonicity and LCIM. Then, ψ satisfies weak cost monotonicity.*

Proof. We have to prove that, for each pair (N, c) and (N, c'), if there exists (N, c'') such that $c' = c + c''$, then $\psi(N, c') \geq \psi(N, c)$. Consider the following airport problems:

Problem	Costs						
$c^0 = c$	c_1	c_2	$\ldots$	c_i	$\ldots$	c_n
c^1	$c_1 + c''_1$	$c_2 + c''_1$	$\ldots$	$c_i + c''_1$	$\ldots$	$c_n + c''_1$
c^2	$c_1 + c''_1$	$c_2 + c''_1 + c''_2 - c''_1$	$\ldots$	$c_i + c''_1 + c''_2 - c''_1$	$\ldots$	$c_n + c''_1 + c''_2 - c''_1$
			$\vdots$		$\vdots$	
c^i	$c_1 + c''_1$	$c_2 + c''_2$	$\ldots$	$c_i + c''_i$	$\ldots$	$c_n + c''_i$
			$\vdots$		$\vdots$	
$c^n = c'$	$c_1 + c''_1$	$c_2 + c''_2$	$\ldots$	$c_i + c''_i$	$\ldots$	$c_n + c''_n$

Now, noting that $c' = c^n$ and combining downstream cost monotonicity and LCIM we have

$$\begin{array}{lllllllllll} \psi_1(N,c^n) & \geq^{\text{LCIM}} & \ldots & \geq^{\text{LCIM}} & \psi_1(N,c^i) & \geq^{\text{LCIM}} & \ldots & \geq^{\text{LCIM}} & \psi_1(N,c^1) & \geq^{\text{DOWN}} & \psi_1(N,c) \\ \psi_2(N,c^n) & \geq^{\text{LCIM}} & \ldots & \geq^{\text{LCIM}} & \psi_2(N,c^i) & \geq^{\text{LCIM}} & \ldots & \geq^{\text{DOWN}} & \psi_2(N,c^1) & \geq^{\text{DOWN}} & \psi_2(N,c) \\ \vdots & & & & & & & & & & \\ \psi_i(N,c^n) & \geq^{\text{LCIM}} & \ldots & \geq^{\text{LCIM}} & \psi_i(N,c^i) & \geq^{\text{DOWN}} & \ldots & \geq^{\text{DOWN}} & \psi_i(N,c^1) & \geq^{\text{DOWN}} & \psi_i(N,c) \\ \vdots & & & & & & & & & & \\ \psi_n(N,c^n) & \geq^{\text{DOWN}} & \ldots & \geq^{\text{DOWN}} & \psi_n(N,c^i) & \geq^{\text{DOWN}} & \ldots & \geq^{\text{DOWN}} & \psi_n(N,c^1) & \geq^{\text{DOWN}} & \psi_n(N,c). \end{array}$$ □

Corollary 3. *The core-center satisfies weak cost monotonicity.*

Proposition 17. *The core-center satisfies neither reciprocity nor incremental no subsidy.*

Proof. The result follows from the following characterizations in Aadland and Kolpin (1998) and the fact that the core-center satisfies no-subsidy, order preservation for contributions and weak cost monotonicity:

- The constrained equal contribution rule is the only selection from the no-subsidy correspondence satisfying order preservation for contributions, weak cost monotonicity, and reciprocity.

- The sequential equal contributions rule is the only rule satisfying order preservation for contributions, weak cost monotonicity, and incremental no subsidy. □

Proposition 18. *The core-center satisfies population monotonicity.*

Proof. We prove the result for the case in which there is $k \in N$ such that $N = N' \cup \{k\}$. The general case follows from repeated application of that property.

Thus, given $N' = N \backslash \{k\}$, we prove that $\mu_{N'}(N, c) \leq \mu(N', c_{N'})$. We distinguish three cases.

Case 1: $\boldsymbol{c_k = c_n}$. So, for each $i \in N'$, $c_i \leq c_k = c_n$. By Theorem 5, for each $i \in N'$,

$$\mu_i(N', c_{N'}) = \mu_i(c_1, \dots, c_{n-1}) \geq \mu_i(c_1, \dots, c_{n-1}, c_n) = \mu_i(N, c).$$

Case 2: $\boldsymbol{c_k = c_1}$. So, for each $i \in N'$, $c_i \geq c_k = c_1$. Now, for each $\varepsilon \geq 0$, let $c^\varepsilon = (\varepsilon, c_2, \dots, c_n)$. Clearly, $\mu(N', c_{N'}) = \mu_{N'}(N, c^0)$. By HCDM, for each $\varepsilon \in (0, c_1]$, $\mu_{N'}(N, c) \leq \mu_{N'}(N, c^\varepsilon)$ and, by continuity, $\mu_{N'}(N, c) \leq \mu_{N'}(N, c^0)$. Therefore,

$$\mu_{N'}(N, c) \leq \mu_{N'}(N, c^0) = \mu(N', c_{N'}).$$

Case 3: $\boldsymbol{c_1 < c_k < c_n}$. Let $i \in N'$. We distinguish two subcases.

$\boldsymbol{c_i > c_k}$**:** Consider the airport problems (N, c^ε), with $c^\varepsilon = (\varepsilon, c_1, \dots, c_{k-1}, c_{k+1}, c_{k+2}, \dots, c_n)$ and $\varepsilon \in (0, c_1]$. Similarly to Case 2, HCDM and continuity ensure that $\mu_i(N', c_{N'}) = \mu_i(N, c^0) \geq \mu_i(N, c^\varepsilon)$. Combining this with a repeated application of HCDM, we have

$$\begin{aligned}
\mu_i(N', c_{N'}) &= \mu_i(N, c^0) \\
&\geq \mu_i(c_1, c_1, c_2, \dots, c_{k-1}, c_{k+1}, c_{k+2}, \dots, c_n) \\
&\geq \mu_i(c_1, c_2, c_2, \dots, c_{k-1}, c_{k+1}, c_{k+2}, \dots, c_n) \\
&\vdots \\
&\geq \mu_i(c_1, c_2, c_3, \dots, c_{k-1}, c_k, c_{k+1}, \dots, c_n) = \mu_i(N, c).
\end{aligned}$$

$\boldsymbol{c_i \leq c_k}$**:** In the case in which $c_i = c_k$ we assume, without loss of generality, that $i < k$. Now, applying LCIM repeatedly,

$$\mu_i(N', c_{N'}) = \mu_i(c_1, \dots, c_{k-1}, c_{k+1}, c_{k+2}, \dots, c_n) \geq \mu_i(c_1, \dots, c_{k-1}, c_k, c_{k+1}, \dots, c_{n-1}),$$

and, by Case 1,

$$\mu_i(c_1, \dots, c_{k-1}, c_k, c_{k+1}, \dots, c_{n-1}) \geq \mu_i(c_1, \dots, c_{k-1}, c_k, c_{k+1}, \dots, c_{n-1}, c_n) = \mu_i(N, c).$$

Combining the inequalities in both equations we get that $\mu_i(N', c_{N'}) \geq \mu_i(N, c)$. □

7 Summary of properties

To conclude, we present a table that summarizes the behavior of the core-center with respect to the properties we have studied.

	Rules		
Properties	Shapley	Nucleolus	Core-center
Fixed population			
Non negativity	✓	✓	✓
Cost boundedness	✓	✓	✓
Efficiency	✓	✓	✓
No-subsidy	✓	✓	✓
Anonymity	✓	✓	✓
Equal treatment of equals	✓	✓	✓
Order preservation for contributions	✓	✓	✓
Order preservation for benefits	✓	✓	✓
Homogeneity	✓	✓	✓
Continuity	✓	✓	✓
Independence at-least-as-large costs	✓	-	-
Last-agent cost additivity	✓	✓	✓
Weak last-agent cost additivity	✓	✓	✓
Conditional cost additivity	✓	-	-
Individual cost monotonicity	✓	✓	✓
Downstream cost monotonicity	✓	✓	✓
Marginalism	✓	-	-
Strong cost monotonicity	-	-	-
Weak cost monotonicity	✓	✓	✓
Incremental no-subsidy	✓	✓	-
Reciprocity	-	-	-
Others-oriented cost monotonicity	✓	-	-
Variable population			
Population monotonicity	✓	✓	✓
First agent consistency	✓	✓	-
Downstream substraction consistency	-	✓	-
Last-agent consistency	-	✓	-
Uniform substraction consistency	-	-	-

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Appendix

In the first part of this Appendix we present the proof of Theorem 2. This result is then crucial to prove also Theorems 3, 5, and 6, which establish the main monotonicity properties satisfied by the core center.

Theorem 2. *Let (N,c) be an airport game such that $0 < c_1 \leq c_2 \leq \cdots \leq c_n$ and fix $j \in N \setminus \{n\}$. Then:*

1. $U^j_{n-1}(c_1,\ldots,c_{n-1}) = V_n(c_1,\ldots,c_j,c_j,\ldots,c_{n-1})$
2. $\hat{\mu}_j(c_1,\ldots,c_{n-1}) = \dfrac{V_n(c_1,\ldots,c_j,c_j,\ldots c_{n-1})}{V_{n-1}(c_1,\ldots,c_{n-1})}$

Before engaging in the proof of Theorem 2 we need some preliminary results. Recall the convention $V_0 = 1$.

Lemma 1. *Given $0 < c_1 \leq \cdots \leq c_k$, $k \in \mathbb{N}$, we have that $V_1(c_1) = c_1$, $V_2(c_1,c_2) = \frac{c_2^2}{2} - \frac{(c_2-c_1)^2}{2}$, and, if $k \geq 3$,*

$$V_k(c_1,\ldots,c_k) = \frac{c_k^k}{k!} - \frac{(c_k-c_1)^k}{k!} - \sum_{i=2}^{k-1} \frac{(c_k-c_i)^{k-i+1}}{(k-i+1)!} V_{i-1}(c_1,\ldots,c_{i-1}).$$

Proof. The expressions for $V_1(c_1)$ and $V_2(c_1,c_2)$ are straightforward. The result also holds for $k = 3$, since $V_3(c_1,c_2,c_3) = \int_0^{c_1} V_2(c_2-x_1,c_3-x_1)dx_1 = \int_0^{c_1}\Big(\frac{(c_3-x_1)^2}{2} - \frac{(c_3-c_2)^2}{2}\Big)dx_1 = \frac{c_3^3}{3!} - \frac{(c_3-c_1)^3}{3!} - \frac{(c_3-c_2)^2}{2}c_1$. We proceed by induction. Let $k \in \mathbb{N}$, $k > 3$, and assume that the result holds for $k-1$. Then,

$$\begin{aligned} V_k(c_1,\ldots,c_k) &= \int_0^{c_1} V_{k-1}(c_2-x_1,\ldots,c_k-x_1)dx_1 \\ &= \int_0^{c_1}\Big(\frac{(c_k-x_1)^{k-1}}{(k-1)!} - \frac{(c_k-c_2)^{k-1}}{(k-1)!} - \sum_{i=2}^{k-2}\frac{(c_k-c_{i+1})^{k-i}}{(k-i)!}V_{i-1}(c_2-x_1,\ldots,c_i-x_1)\Big)dx_1. \end{aligned}$$

We compute separately the integral of each addend:

$$\int_0^{c_1} \frac{(c_k-x_1)^{k-1}}{(k-1)!}dx_1 = \frac{c_k^k}{k!} - \frac{(c_k-c_1)^k}{k!}, \quad \int_0^{c_1} \frac{(c_k-c_2)^{k-1}}{(k-1)!}dx_1 = \frac{(c_k-c_2)^{k-1}}{(k-1)!}V_1(c_1), \quad \text{and}$$
$$\int_0^{c_1} V_{i-1}(c_2-x_1,\ldots,c_i-x_1)dx_1 = V_i(c_1,\ldots,c_i)$$

By the linearity of the integral,

$$\begin{aligned} V_k(c_1,\ldots,c_k) &= \frac{c_k^k}{k!} - \frac{(c_k-c_1)^k}{k!} - \frac{(c_k-c_2)^{k-1}}{(k-1)!}V_1(c_1) - \sum_{i=2}^{k-2}\frac{(c_k-c_{i+1})^{k-i}}{(k-i)!}V_i(c_1,\ldots,c_i) \\ &= \frac{c_k^k}{k!} - \frac{(c_k-c_1)^k}{k!} - \sum_{i=1}^{k-2}\frac{(c_k-c_{i+1})^{k-i}}{(k-i)!}V_i(c_1,\ldots,c_i), \end{aligned}$$

which, after a simple rearrangement of the indices, coincides with the desired expression. □

Lemma 2. *Let $0 < c_1 \leq \cdots \leq c_k$, $k \in \mathbb{N}$, $x_1 \leq c_1$, and denote $u_k(x_1) = V_k(c_1 - x_1, \ldots, c_k - x_1)$. Then,*

$$\frac{du_k}{dx_1}(x_1) = -V_{k-1}(c_2 - x_1, \ldots, c_k - x_1).$$

Proof. The proof is by induction. The property holds for $k = 1$, since $u_1(x_1) = V_1(c_1 - x_1) = c_1 - x_1$ and $\frac{du_1}{dx_1}(x_1) = -1$. It also holds for $k = 2$, because $u_2(x_1) = \frac{(c_2-x_1)^2}{2} - \frac{(c_2-c_1)^2}{2}$ and $\frac{du_2}{dx_1}(x_1) = -(c_2 - x_1)$. Now, let $k \geq 3$ and suppose that the property is true for u_i, $1 \leq i \leq k-1$. Then, according to Lemma 1,

$$u_k(x_1) = V_k(c_1 - x_1, \ldots, c_k - x_1) = \frac{(c_k - x_1)^k}{k!} - \frac{(c_k-c_1)^k}{k!} - \sum_{i=2}^{k-1}\frac{(c_k-c_i)^{k-i+1}}{(k-i+1)!}u_{i-1}(x_1).$$

Differentiating with respect to x_1,

$$\frac{du_k}{dx_1}(x_1) = -\frac{(c_k-x_1)^{k-1}}{(k-1)!} - \sum_{i=2}^{k-1}\frac{(c_k-c_i)^{k-i+1}}{(k-i+1)!}\frac{du_{i-1}}{dx_1}(x_1).$$

By the induction hypothesis, if $i \geq 2$, $\frac{du_{i-1}}{dx_1}(x_1) = -V_{i-2}(c_2 - x_1, \ldots, c_{i-1} - x_1)$. Therefore,

$$\frac{du_k}{dx_1}(x_1) = -\frac{(c_k-x_1)^{k-1}}{(k-1)!} + \frac{(c_k-c_2)^{k-1}}{(k-1)!} + \sum_{i=3}^{k-1}\frac{(c_k-c_i)^{k-i+1}}{(k-i+1)!}V_{i-2}(c_2 - x_1, \ldots, c_{i-1} - x_1).$$

Renumbering the terms,

$$\begin{aligned} \frac{du_k}{dx_1}(x_1) &= -\left(\frac{(c_k-x_1)^{k-1}}{(k-1)!} - \frac{(c_k-c_2)^{k-1}}{(k-1)!} - \sum_{i=2}^{k-2}\frac{(c_k-c_{i+1})^{k-i}}{(k-i)!}V_{i-1}(c_2 - x_1, \ldots, c_i - x_1)\right) \\ &= -V_{k-1}(c_2 - x_1, \ldots, c_k - x_1). \end{aligned}$$

where the last equality follows directly from Lemma 1. □

Proof of Theorem 2. First observe that

$$V_n(c_1, \ldots, c_j, c_j, \ldots c_{n-1}) = \int_0^{c_1} \cdots \int_0^{c_j - \sum_{k=1}^{j-1} x_k} V_{n-j}\big(c_j - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\big)\, dx_j \ldots dx_1.$$

If $u_{n-j}(x_j) = V_{n-j}\Big(c_j - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\Big)$ then, by Lemma 2,

$$\frac{du_{n-j}}{dx_j}(x_j) = -V_{n-j-1}\Big(c_{j+1} - \sum_{k=1}^{j} x_k, \ldots, c_{n-1} - \sum_{k=1}^{j} x_k\Big).$$

Integrating by parts,

$$\int_0^{c_j-\sum_{k=1}^{j-1}x_k} V_{n-j}\big(c_j-\sum_{k=1}^{j}x_k,\dots,c_{n-1}-\sum_{k=1}^{j}x_k\big)\ dx_j =$$

$$\Big[x_jV_{n-j}\big(c_j-\sum_{k=1}^{j}x_k,\dots,c_{n-1}-\sum_{k=1}^{j}x_k\big)\Big]_0^{c_j-\sum_{k=1}^{j-1}x_k}+\int_0^{c_j-\sum_{k=1}^{j-1}x_k} x_jV_{n-j-1}\big(c_{j+1}-\sum_{k=1}^{j}x_k,\dots,c_{n-1}-\sum_{k=1}^{j}x_k\big)\ dx_j.$$

The bracketed expression vanishes, since $V_{n-j}(0,c_{j+1}-c_j,\dots,c_{n-1}-c_j)=0$. Consequently,

$$\int_0^{c_j-\sum_{k=1}^{j-1}x_k} V_{n-j}\Big(c_j-\sum_{k=1}^{j}x_k,\dots,c_{n-1}-\sum_{k=1}^{j}x_k\Big)\ dx_j =$$

$$\int_0^{c_j-\sum_{k=1}^{j-1}x_k} x_jV_{n-j-1}\Big(c_{j+1}-\sum_{k=1}^{j}x_k,\dots,c_{n-1}-\sum_{k=1}^{j}x_k\Big)\ dx_j. \quad (1)$$

Finally,

$$\begin{aligned}V_n(c_1,\dots,c_j,c_j,\dots c_{n-1}) &= \int_0^{c_1}\dots\int_0^{c_j-\sum_{k=1}^{j-1}x_k} x_jV_{n-j-1}\Big(c_{j+1}-\sum_{k=1}^{j}x_k,\dots,c_{n-1}-\sum_{k=1}^{j}x_k\Big)\ dx_j\dots dx_1\\ &= U^j_{n-1}(c_1,\dots,c_{n-1}).\end{aligned}$$

This last result follows from the previous property and the definition of $\hat{\mu}$. □

The proofs of the remaining theorems in this Appendix follow the same basic structure. We know, by Proposition 11, that the monotonicity of the core-center with respect to costs can be established by checking the relative position of the core-center of the game and the centroid of the faces of the core. So, we have to prove inequalities of the type $\hat{\mu}_p(c_1,\dots,c_k)\le\hat{\mu}_{p+1}(d_1,\dots,d_{k+1})$. But, by Theorem 2, that is equivalent to proving an inequality such as $\Delta=V_{k+1}(c_1,\dots,c_p,c_p,\dots,c_k)V_{k+1}(d_1,\dots,d_{k+1})-V_k(c_1,\dots,c_k)V_{k+2}(d_1,\dots,d_{p+1},d_{p+1},\dots,d_{k+1})\le 0$. Then, we try to decompose each of the four volumes in Δ in terms of volumes of certain "manageable" types. Finally, we rearrange Δ as a sum of expressions involving these types of volumes and study, by induction, their sign. It turns out that the "manageable" volumes for Theorem 3 and Theorem 6 are of the same type (see Proposition 21) while for Theorem 5 different volumes are needed (see Proposition 24). Therefore, we develop the proof of Theorem 3 in full detail and just sketch the one for Theorem 6. We finish with the proof of Theorem 5.

An expression like $V_{p+s-1}(c_1,\dots,c_p,\overset{s}{\dots},c_p)$ means that cost c_p is repeated s times. When all the costs are the same we write $V_k(c_1,\dots,c_1)$ instead of $V_k(c_1,\overset{k}{\dots},c_1)$. In order to prove Theorem 3, our first step is to derive some results involving volumes of this type.

Lemma 3. *For all $k\in\mathbb{N}$ and $\alpha\ge 0$, $V_k(\alpha,\dots,\alpha)=\frac{\alpha^k}{k!}$.*

Proof. Clearly, the property holds for $k=1$. Assume that the result is true for $k-1$ and proceed by induction. Then,

$$V_k(\alpha,\dots,\alpha)=\int_0^{\alpha}V_{k-1}(\alpha-x_1,\dots,\alpha-x_1)dx_1=\int_0^{\alpha}\frac{(\alpha-x_1)^{k-1}}{(k-1)!}dx_1=\frac{\alpha^k}{k!}.$$ □

Lemma 4. *Given $k\in\mathbb{N}$ and $0<\alpha\le\beta\le c_1\le\dots\le c_k$, we have that*

$$\int_{\alpha}^{\beta}V_k(c_1-x_1,\dots,c_k-x_1)dx_1=V_{k+1}(\beta-\alpha,c_1-\alpha,\dots,c_k-\alpha).$$

Proof. The result is true for $k=1$ since $\int_\alpha^\beta (c_1-x_1)dx_1 = \frac{(c_1-\alpha)^2}{2} - \frac{(c_1-\beta)^2}{2}$. Assume that the equality holds for all $i<k$. Then,

$$\begin{aligned}
&\int_\alpha^\beta V_k(c_1-x_1,\dots,c_k-x_1)dx_1\\
&\quad=\int_\alpha^\beta \frac{(c_k-x_1)^k}{k!}dx_1 - \int_\alpha^\beta \frac{(c_k-c_1)^k}{k!}dx_1 - \sum_{i=2}^{k-1}\frac{(c_k-c_i)^{k-i+1}}{(k-i+1)!}\int_\alpha^\beta V_{i-1}(c_1-x_1,\dots,c_{i-1}-x_1)dx_1\\
&\quad=\frac{(c_k-\alpha)^{k+1}}{(k+1)!} - \frac{(c_k-\beta)^{k+1}}{(k+1)!} - \frac{(c_k-c_1)^k}{k!}(\beta-\alpha) - \sum_{i=2}^{k-1}\frac{(c_k-c_i)^{k-i+1}}{(k-i+1)!}V_i(\beta-\alpha,c_1-\alpha\dots,c_{i-1}-\alpha),
\end{aligned}$$

where the first equality holds by Lemma 1 and the second by the induction hypothesis. Again by Lemma 1, the last expression equals $V_{k+1}(\beta-\alpha,c_1-\alpha,\dots,c_k-\alpha)$. □

The following result allows to decompose any given volume in terms of volumes with repeated costs.

Proposition 19. *If $0<\alpha\le c_1\le\dots\le c_k$, $k\in\mathbb{N}$, then*

1. $V_k(c_1,\dots,c_k) = \sum_{i=0}^{k} V_i(\alpha,\dots,\alpha)V_{k-i}(c_{i+1}-\alpha,\dots,c_k-\alpha)$,

2. $V_k(c_1-\alpha,\dots,c_k-\alpha) = \sum_{i=1}^{k} V_i(c_1-\alpha,\dots,c_1-\alpha)V_{k-i}(c_{i+1}-c_1,\dots,c_k-c_1)$, *and*

3. $V_k(c_1-\alpha,\dots,c_k-\alpha) = \sum_{i=2}^{k} V_i(c_1-\alpha,c_2-\alpha,\overset{i-1}{\dots},c_2-\alpha)V_{k-i}(c_{i+1}-c_2,\dots,c_k-c_2)$.

Proof. 1. Let $s\in\mathbb{N}$ such that $1\le s\le k-2$ and denote

$$I_s = \int_0^{c_{s+1}-\sum_{j=1}^s x_j}\dots\int_0^{c_k-\sum_{j=1}^{k-1}x_j} dx_k\dots dx_{s+1} = V_{k-s}\Big(c_{s+1}-\sum_{j=1}^s x_j,\dots,c_k-\sum_{j=1}^s x_j\Big).$$

We claim that

$$\int_0^\alpha\dots\int_0^{\alpha-\sum_{j=1}^{s-1}x_j} I_s\,dx_s\dots dx_1 = \\ V_s(\alpha,\dots,\alpha)V_{k-s}(c_{s+1}-\alpha,\dots,c_k-\alpha) + \int_0^\alpha\dots\int_0^{\alpha-\sum_{j=1}^s x_j} I_{s+1}\,dx_{s+1}\dots dx_1 \quad (2)$$

Indeed,

$$\int_0^\alpha\dots\int_0^{\alpha-\sum_{j=1}^{s-1}x_j} I_s\,dx_s\dots dx_1 = \int_0^\alpha\dots\int_0^{\alpha-\sum_{j=1}^s x_j} I_{s+1}\,dx_{s+1}\dots dx_1 + \int_0^\alpha\dots\int_{\alpha-\sum_{j=1}^s x_j}^{c_{s+1}-\sum_{j=1}^s x_j} I_{s+1}\,dx_{s+1}\dots dx_1.$$

Then, in order to prove the claim, we just have to decompose the second addend of the last expression. But, since $I_{s+1} = V_{k-s-1}\Big(c_{s+2}-\sum_{j=1}^{s+1}x_j,\dots,c_k-\sum_{j=1}^{s+1}x_j\Big)$, we have, by Lemma 4,

$$\int_{\alpha-\sum_{j=1}^s x_j}^{c_{s+1}-\sum_{j=1}^s x_j} I_{s+1}\,dx_{s+1}\dots dx_1 = V_{k-s}\big(c_{s+1}-\alpha,c_{s+2}-\alpha\dots,c_k-\alpha\big)\,dx_s\dots dx_1.$$

Consequently,

$$\int_0^{\alpha} \cdots \int_{\alpha-\sum_{j=1}^{s} x_j}^{c_{s+1}-\sum_{j=1}^{s} x_j} I_{s+1}\, dx_{s+1} \dots dx_1 = \int_0^{\alpha} \cdots \int_0^{\alpha-\sum_{j=1}^{s-1} x_j} V_{k-s}\big(c_{s+1}-\alpha, \dots, c_k-\alpha\big)\, dx_s \dots dx_1$$

$$= V_{k-s}\big(c_{s+1}-\alpha, \dots, c_k-\alpha\big) \int_0^{\alpha} \cdots \int_0^{\alpha-\sum_{j=1}^{s-1} x_j} dx_s \dots dx_1$$

$$= V_{k-s}(c_{s+1}-\alpha, \dots, c_k-\alpha) V_s(\alpha, \dots, \alpha).$$

Then, Equation (2) holds. Now, we make repeated use of Equation (2) to obtain the first equality of the result. Observe that

$$\begin{aligned} V_k(c_1, \dots, c_k) &= \int_0^{c_1} I_1 dx_1 = \int_0^{\alpha} I_1 dx_1 + \int_{\alpha}^{c_1} I_s dx_1 \\ &= \int_0^{\alpha} I_1 dx_1 + \int_{\alpha}^{c_1} V_{k-1}(c_2-x_1, \dots, c_k-x_1) dx_1 = \int_0^{\alpha} I_1 dx_1 + V_k(c_1-\alpha, \dots, c_k-\alpha), \end{aligned}$$

where the last equality holds by Lemma 4. But, now, according to Equation (2),

$$\int_0^{\alpha} I_1 dx_1 = \int_0^{\alpha} \int_0^{\alpha-x_1} I_2 dx_2 dx_1 + V_{k-1}(c_2-\alpha, \dots, c_k-\alpha) V_1(\alpha).$$

Then,

$$V_k(c_1, \dots, c_k) = V_k(c_1-\alpha, \dots, c_k-\alpha) + V_1(\alpha) V_{k-1}(c_2-\alpha, \dots, c_k-\alpha) + \int_0^{\alpha} \int_0^{\alpha-x_1} I_2 dx_2 dx_1.$$

Next, decompose $\int_0^{\alpha} \int_0^{\alpha-x_1} I_2 dx_2 dx_1$ by applying Equation (2), and repeat the process until the intended equality is reached.

2. To prove the second equality, just take $A = c_1 - \alpha$, so that $(c_i - \alpha) - A = c_i - c_1$, and apply statement 1.

$$V_k(c_1-\alpha, \dots, c_k-\alpha) = \sum_{i=0}^{k} V_i(c_1-\alpha, \dots, c_1-\alpha) V_{k-i}(c_{i+1}-c_1, \dots, c_k-c_1).$$

Now, simply observe that, for $i = 0$, $V_k(c_1-c_1, c_2-c_1, \dots, c_k-c_1) = 0$.

3. From Lemma 4 and statement 1.

$$\begin{aligned} V_k(c_1-\alpha, \dots, c_k-\alpha) &= \int_{\alpha}^{c_1} V_{k-1}(c_2-x_1, \dots, c_k-x_1) dx_1 \\ &= \sum_{i=2}^{k} V_{k-i}(c_{i+1}-c_2, \dots, c_k-c_2) \int_{\alpha}^{c_1} V_{i-1}(c_2-x_1, \dots, c_2-x_1) dx_1. \end{aligned}$$

But, by Lemma 4,

$$\int_{\alpha}^{c_1} V_{i-1}(c_2-x_1, \dots, c_2-x_1) dx_1 = V_i(c_1-\alpha, c_2-\alpha, \overset{i-1}{\dots}, c_2-\alpha). \quad \square$$

Now we need some extra notation. Given $k \in \mathbb{N}$ and $0 < c_1 \leq \cdots \leq c_k$, let $Z_0 = 1$ and

$$Z_s^\alpha = V_s(c_{k-s+1} - \alpha, \ldots, c_k - \alpha), \;\; s = 1, \ldots, k, \;\; \alpha < c_{k-s+1}.$$

When no confusion arises, we write Z_s instead of Z_s^α.

Remark 6. *Let $q, k \in \mathbb{N}$, $q < k$, $0 < c_1 \leq \cdots \leq c_k$, and fix $\alpha < c_{k-q}$. Clearly, $Z_1^\alpha = V_1(c_k - \alpha) = c_k - \alpha$. Now, let $A_0 = 1$ and*

$$A_r = Z_r^{c_{k-q+1}} = V_r(c_{k-r+1} - c_{k-q+1}, \ldots, c_k - c_{k-q+1}), \;\; r = 1, \ldots, q-1.$$

Then, by statement 2 of Proposition 19,

$$Z_q^\alpha = \sum_{i=1}^{q} \mathcal{V}_i A_{q-i}, \;\; \textit{with } \mathcal{V}_i = V_i(c_{k-q+1} - \alpha, \ldots, c_{k-q+1} - \alpha), \;\; i = 1, \ldots, q,$$

and by statement 3 of Proposition 19,

$$Z_{q+1}^\alpha = \sum_{i=2}^{q+1} \bar{\mathcal{V}}_i A_{q+1-i}, \;\; \textit{with } \bar{\mathcal{V}}_j = V_j(c_{k-q} - \alpha, c_{k-q+1} - \alpha, \overset{j-1}{\ldots}, c_{k-q+1} - \alpha), \;\; j = 2, \ldots, q+1.$$

Lemma 5. *For all $q, k \in \mathbb{N}$, $q < k$, and $0 < c_1 \leq \cdots \leq c_k$, fix $\alpha < c_{k-q}$. Then, $Z_1^\alpha Z_q^\alpha - Z_{q+1}^\alpha \geq 0$.*

Proof. We use the notation and decompositions of Remark 6. Clearly, by Lemma 3,

$$\mathcal{V}_i = \frac{(c_{k-q+1} - \alpha)^i}{i!}, \;\; i = 1, \ldots, q. \tag{3}$$

Besides, applying the definition of $\bar{\mathcal{V}}_i$ and Lemma 3,

$$\begin{aligned} \bar{\mathcal{V}}_i &= \int_0^{c_{k-q}} \frac{(c_{k-q-1} - \alpha - x_1)^{i-1}}{(i-1)!} \, dx_1 = \frac{1}{i!}\big((c_{k-q+1} - \alpha)^i - (c_{k-q+1} - c_{k-q})^i\big) \\ &= \mathcal{V}_i - X_i, \quad \text{where } X_i = \frac{1}{i!}(c_{k-q+1} - c_{k-q})^i, \;\; i = 2, \ldots, q+1. \end{aligned} \tag{4}$$

In order to prove that

$$Z_1 Z_q - Z_{q+1} = \sum_{i=0}^{q-1} \big((c_k - \alpha)\mathcal{V}_{q-i} - \bar{\mathcal{V}}_{q-i+1}\big) A_i \geq 0,$$

we check that, for all $i = 0, \ldots, q-1$, $(c_k - \alpha)\mathcal{V}_{q-i} - \bar{\mathcal{V}}_{q-i+1} \geq 0$. Certainly,

$$(c_k - \alpha)\mathcal{V}_{q-i} - \bar{\mathcal{V}}_{q-i+1} = \frac{(c_k - \alpha)(c_{k-q+1} - \alpha)^{q-i}}{(q-i)!} - \frac{(c_{k-q+1} - \alpha)^{q-i+1}}{(q-i+1)!} + \frac{(c_{k-q+1} - c_{k-q})^{q-i+1}}{(q-i+1)!} \geq 0,$$

since $(c_k - \alpha) \geq (c_{k-q+1} - \alpha)$ and $(q-i+1)! \geq (q-i)!$. $\square$

Proposition 20. *Let $t, q, k \in \mathbb{N}$ be such that $t \leq q < k$ and $c_1 \leq \cdots \leq c_k$. Fix $\alpha < c_{k-q}$. Then, $Z_t^\alpha Z_q^\alpha - Z_{t-1}^\alpha Z_{q+1}^\alpha \geq 0$.*

Proof. We proceed by induction on $t \in \mathbb{N}$. The case $t = 1$ has been proved in Lemma 5. Now, assume that the result holds for any $i \leq t-1$, *i.e.*,

$$Z_i^\beta Z_j^\beta - Z_{i-1}^\beta Z_{j+1}^\beta \geq 0, \; i \leq j < k, \; \beta < c_{k-j}, \tag{5}$$

and then, we prove that it also holds for $t < k$. According to the notation and decompositions of Remark 6,

$$\begin{aligned} Z_t Z_q - Z_{t-1} Z_{q+1} &= \Big(\sum_{i=0}^{t} \mathcal{V}_i A_{t-i}\Big)\Big(\sum_{i=1}^{q} \mathcal{V}_i A_{q-i}\Big) - \Big(\sum_{i=0}^{t-1} \mathcal{V}_i A_{t-1-i}\Big)\Big(\sum_{i=2}^{q+1} \bar{\mathcal{V}}_i A_{q+1-i}\Big) \\ &= \sum_{s=0}^{t-1}\sum_{r=0}^{q-1} A_s A_r \big(\mathcal{V}_{t-s}\mathcal{V}_{q-r} - \mathcal{V}_{t-1-s}\bar{\mathcal{V}}_{q+1-r}\big) + A_t \sum_{r=1}^{q} A_{q-r}\mathcal{V}_r. \end{aligned}$$

Certainly, $A_t \sum_{r=1}^{q} A_{q-r}\mathcal{V}_r \geq 0$. Then, it suffices to prove that

$$S = \sum_{s=0}^{t-1}\sum_{r=0}^{q-1} A_s A_r \Delta_{s,r} \geq 0, \; \text{where } \Delta_{s,r} = \mathcal{V}_{t-s}\mathcal{V}_{q-r} - \mathcal{V}_{t-s-1}\bar{\mathcal{V}}_{q-r+1}. \tag{6}$$

First, we claim that

$$\mathcal{V}_i \mathcal{V}_j - \mathcal{V}_{i-1}\bar{\mathcal{V}}_{j+1} \geq 0, \; \text{if } i \leq j+1. \tag{7}$$

Indeed, applying Equality (4) in Lemma 5, we have that $\mathcal{V}_i\mathcal{V}_j - \mathcal{V}_{i-1}\mathcal{V}_{j+1} + \mathcal{V}_{i-1}X_{j+1}$ and $\mathcal{V}_{i-1}X_{j+1} \geq 0$. Then, it suffices to prove that $\mathcal{V}_i\mathcal{V}_j - \mathcal{V}_{i-1}\mathcal{V}_{j+1} \geq 0$ whenever $i \leq j+1$. Let $B = (c_{k-q+1} - \alpha)$ and apply Equation (3) in Lemma 5,

$$\mathcal{V}_i\mathcal{V}_j - \mathcal{V}_{i-1}\mathcal{V}_{j+1} = \frac{B^i}{i!}\frac{B^j}{j!} - \frac{B^{i-1}}{(i-1)!}\frac{B^{j+1}}{(j+1)!} = \Big(\frac{1}{i!j!} - \frac{1}{(i-1)!(j+1)!}\Big)B^{i+j}.$$

Now Equation (7) is straightforward, since $\frac{1}{i!j!} - \frac{1}{(i-1)!(j+1)!} \geq 0$ if and only if $i \leq j+1$.

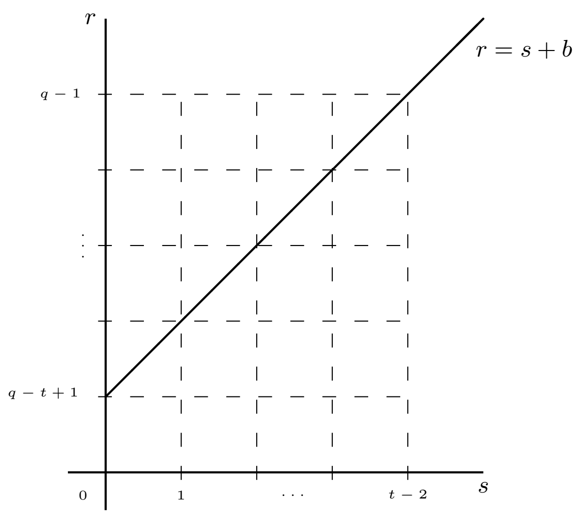

Figure 2: The straight line $r = s + b$, with $b = q - t + 1$.

Let $b = q - t + 1 > 0$ and $T = \{(s, r) \in [0, t-2] \times [0, q-1] : r > s + b\} \subset \mathbb{N}^2$ be the set depicted in Figure 2. According to Equation (7), if $(s, r) \in [0, t-1] \times [0, q-1]$ but $(s, r) \notin T$ then $A_s A_r \Delta_{s,r} \geq 0$. Now, take $(r, s) \in T$ such that $A_s A_r \Delta_{s,r} \leq 0$, then $h = (r - s) - b > 0$ and $(s + h, r - h) \notin T$, because $(r - h) \leq (s + h) + b$. In addition, $t - s - h = q - r + 1$ and $q - r + h = t - s - 1$. Therefore, each negative addend $A_s A_r \Delta_{s,r} \leq 0$ in Equation (6) can be paired with the corresponding $A_{s+h} A_{r-h} \Delta_{s+h,r-h} \geq 0$ in the following way

$$
\begin{gathered}
A_s A_r \Delta_{s,r} + A_{s+h} A_{r-h} \Delta_{s+h,r-h} = \\
A_s A_r \Big(\mathcal{V}_{t-s} \mathcal{V}_{q-r} - \mathcal{V}_{t-s-1} \bar{\mathcal{V}}_{q-r+1} \Big) + A_{s+h} A_{r-h} \Big(\mathcal{V}_{t-s-h} \mathcal{V}_{q-r+h} - \mathcal{V}_{t-s-h-1} \bar{\mathcal{V}}_{q-r+h+1} \Big) = \\
A_s A_r \Big(\mathcal{V}_{t-s} \mathcal{V}_{q-r} - \mathcal{V}_{t-s-1} \big(\mathcal{V}_{q-r+1} - X_{q-r+1} \big) \Big) + A_{s+h} A_{r-h} \Big(\mathcal{V}_{q-r+1} \mathcal{V}_{t-s-1} - \mathcal{V}_{q-r} \big(\mathcal{V}_{t-s} - X_{t-s} \big) \Big) = \\
A_s A_r \big(\mathcal{V}_{t-s} \mathcal{V}_{q-r} - \mathcal{V}_{t-s-1} \mathcal{V}_{q-r+1} \big) + A_{s+h} A_{r-h} \big(\mathcal{V}_{q-r+1} \mathcal{V}_{t-s-1} - \mathcal{V}_{q-r} \mathcal{V}_{t-s} \big) + \\
A_s A_r \mathcal{V}_{t-s-1} X_{q-r+1} + A_{s+h} A_{r-h} \mathcal{V}_{q-r} X_{t-s} = \\
\big(A_{s+h} A_{r-h} - A_s A_r \big) \big(\mathcal{V}_{t-s-1} \mathcal{V}_{q-r+1} - \mathcal{V}_{q-r} \mathcal{V}_{t-s} \big) + A_s A_r \mathcal{V}_{t-s-1} X_{q-r+1} + A_{s+h} A_{r-h} \mathcal{V}_{q-r} X_{t-s}.
\end{gathered}
$$

Therefore, if we prove that the last expression is positive whenever $(s, r) \in T$, then $S \geq 0$. Clearly, the last two terms are positive. Since $(q - r + 1) < (t - s - 1) + 1$ then, applying Equation (7), we get that $\mathcal{V}_{t-s-1} \mathcal{V}_{q-r+1} - \mathcal{V}_{q-r} \mathcal{V}_{t-s} \geq 0$. It remains to show that for all $(s, r) \in T$, $A_{s+h} A_{r-h} - A_s A_r \geq 0$. Now, if $(s, r) \in T$, then

1. $s \leq r - h \leq s + h \leq r$, since $s \leq s + h = r - b \leq r$, $s \leq s + b = r - h \leq r$ and $(r - h) < (s + h) + b \leq (s + h)$.
2. $s + h \leq t - 2$, since $s + h = r - b \leq q - 1 - b = t - 2$.

Thus,

$$
\begin{gathered}
A_{r-h} A_{s+h} - A_s A_r = \\
\big(A_{r-h} A_{s+h} - A_{r-h-1} A_{s+h+1} \big) + \big(A_{r-h-1} A_{s+h+1} - A_{r-h-2} A_{s+h+2} \big) + \cdots + \big(A_{s+1} A_{r-1} - A_s A_r \big).
\end{gathered}
$$

All the expressions in parentheses are of the form $A_i A_j - A_{i-1} A_{j+1} = Z_i^\beta Z_j^\beta - Z_{i-1}^\beta Z_{j+1}^\beta$, with $i \leq t - 2$, $i \leq j$ and $\beta = c_{k-q+1}$. Therefore, we can apply Equation (5), the induction hypotheses, and conclude that all the addends $A_i A_j - A_{i-1} A_{j+1} \geq 0$ are positive and then $A_{s+h} A_{r-h} - A_s A_r \geq 0$ as well. ☐

The next step consists of providing a way to decompose any given volume in terms of volumes involving only the costs up to a fixed c_p.

Proposition 21. *Let $p, k \in \mathbb{N}$ be such that $p < k$ and $0 < c_1 \leq \cdots \leq c_k$. Then,*

$$
V_k(c_1, \ldots, c_k) = \sum_{i=0}^{k-p} V_{k-p-i}(c_{p+1+i} - c_p, \ldots, c_k - c_p) V_{p+i}(c_1, \ldots, c_p, \overset{i+1}{\ldots}, c_p).
$$

Proof. First we prove that

$$
V_k(c_1, \ldots, c_k) = V_{k-p}(c_{p+1} - c_p, \ldots, c_k - c_p) V_p(c_1, \ldots, c_p) + V_k(c_1, \ldots, c_p, c_p, c_{p+2}, \ldots, c_k). \tag{8}
$$

Indeed, we know that

$$
V_k(c_1, \ldots, c_k) = \int_0^{c_1} \cdots \int_0^{c_p - \sum_{j=1}^{p-1} x_j} V_{k-p}\Big(c_{p+1} - \sum_{j=1}^{p} x_j, \ldots, c_k - \sum_{j=1}^{p} x_j \Big) dx_p \ldots dx_1. \tag{9}
$$

If $u_{k-p}(x_p) = V_{k-p}\Big(c_{p+1} - \sum_{j=1}^{p} x_j, \ldots, c_k - \sum_{j=1}^{p} x_j\Big)$ then, applying Lemma 2, we have that

$$\frac{du(x_p)}{dx_p} = -V_{k-p-1}\Big(c_{p+2} - \sum_{j=1}^{p} x_j, \ldots, c_k - \sum_{j=1}^{p} x_j\Big).$$

Integrating by parts,

$$\int_0^{c_p - \sum_{j=1}^{p-1} x_j} V_{k-p}\Big(c_{p+1} - \sum_{j=1}^{p} x_j, \ldots, c_k - \sum_{j=1}^{p} x_j\Big) dx_p =$$

$$V_{k-p}(c_{p+1} - c_p, \ldots, c_k - c_p)\Big(c_p - \sum_{j=1}^{p-1} x_j\Big) + \int_0^{c_p - \sum_{j=1}^{p-1} x_j} x_p V_{k-p-1}\Big(c_{p+2} - \sum_{j=1}^{p} x_j, \ldots, c_k - \sum_{j=1}^{p} x_j\Big)\, dx_p. \tag{10}$$

Now, applying equality in Equation (1) in Theorem 2, the integral in the last addend can be written as

$$\int_0^{c_p - \sum_{j=1}^{p-1} x_j} V_{k-p}\Big(c_p - \sum_{j=1}^{p} x_j, c_{p+2} - \sum_{j=1}^{p} x_j, \ldots, c_k - \sum_{j=1}^{p} x_j\Big) dx_p. \tag{11}$$

Then, combining equations (9), (10), and (11),

$$\begin{aligned} V_k(c_1, \ldots, c_k) &= V_{k-p}(c_{p+1} - c_p, \ldots, c_k - c_p) \int_0^{c_1} \cdots \int_0^{c_{p-1} - \sum_{j=1}^{p-2} x_j} \Big(c_p - \sum_{j=1}^{p-1} x_j\Big) dx_{p-1} \ldots dx_1 \\ &+ \int_0^{c_1} \cdots \int_0^{c_p - \sum_{j=1}^{p-1} x_j} V_{k-p}\Big(c_p - \sum_{j=1}^{p} x_j, c_{p+2} - \sum_{j=1}^{p} x_j, \ldots, c_k - \sum_{j=1}^{p} x_j\Big) dx_p \ldots dx_1 \\ &= V_{k-p}(c_{p+1} - c_p, \ldots, c_k - c_p) V_p(c_1, \ldots, c_p) + V_k(c_1, \ldots, c_p, c_p, c_{p+2}, \ldots, c_k). \end{aligned}$$

Therefore, Equation (8) holds. Now, applying Equation (8) to $V_k(c_1, \ldots, c_p, c_p, c_{p+2}, \ldots, c_k)$, we find that

$$\begin{aligned} &V_k(c_1, \ldots, c_p, c_p, c_{p+2}, \ldots, c_k) = \\ &\qquad V_{k-p+1}(c_{p+2} - c_p, \ldots, c_k - c_p) V_{p+1}(c_1, \ldots, c_p, c_p) + V_k(c_1, \ldots, c_p, c_p, c_p, c_{p+3}, \ldots, c_k). \end{aligned}$$

Repeating this process, the result eventually follows. □

Lemma 6. *For all $s \in \mathbb{N}$ and $0 < \delta \le d_1$, iwe have $\hat{\mu}_1(d_1 - \delta, \overset{s}{\ldots}, d_1 - \delta) \le \hat{\mu}_2(\delta, d_1, \overset{s}{\ldots}, d_1)$.*

Proof. According to Lemma 3, $\hat{\mu}_1(d_1 - \delta, \overset{s}{\ldots}, d_1 - \delta) = \frac{d_1 - \delta}{s+1}$. We have to prove that,

$$\Lambda = \tfrac{(d_1 - \delta)}{s+1} V_{s+1}(\delta, d_1, \overset{s}{\ldots}, d_1) - V_{s+2}(\delta, d_1, \overset{s+1}{\ldots}, d_1) \le 0.$$

Now, by Proposition 21 with $p = 1$ and $k = s + 1$, and Lemma 3,

$$\tfrac{(d_1 - \delta)}{s+1} V_{s+1}(\delta, d_1, \overset{s}{\ldots}, d_1) = \tfrac{(d_1 - \delta)}{s+1} \sum_{i=0}^{s} \frac{(d_1 - \delta)^{s-i}}{(s-i)!} \frac{\delta^{i+1}}{(i+1)!} = \sum_{i=0}^{s} \frac{(d_1 - \delta)^{s-i+1}}{(s+1)(s-i)!} \frac{\delta^{i+1}}{(i+1)!}$$

and

$$V_{s+2}(\delta, d_1, \overset{s+1}{\ldots}, d_1) = \sum_{i=0}^{s+1} \frac{(d_1-\delta)^{s-i+1}}{(s-i+1)!} \frac{\delta^{i+1}}{(i+1)!}.$$

Then, since $(s-i+1)! = (s-i+1)(s-i)!$ and $s+1 \geq s+1-i$, for all $0 \leq i \leq s$,

$$\Lambda = \sum_{i=0}^{s} \frac{(d_1-\delta)^{s-i+1}\delta^{i+1}}{(i+1)!} \Big(\frac{1}{(s+1)(s-i)!} - \frac{1}{(s-i+1)!} \Big) - \frac{\delta^{s+2}}{(s+2)!} \leq 0.$$

$\square$

Lemma 7. *Let $m \in \mathbb{N}$. Given real numbers H^j, G^j, $j = 1, \ldots, m+2$, and Z_i, $i = 1, \ldots, m+1$, then*

$$\Big(\sum_{i=0}^{m+1} G^{i+1} Z_{m+1-i}\Big)\Big(\sum_{i=0}^{m} H^{i+1} Z_{m-i}\Big) - \Big(\sum_{i=0}^{m} G^{i+1} Z_{m-i}\Big)\Big(\sum_{i=0}^{m+1} H^{i+1} Z_{m+1-i}\Big) =$$
$$= \sum_{i=0}^{m}\sum_{j=i}^{m} \big(G^{m+2-i}H^{m+1-j} - G^{m+1-j}H^{m+2-i}\big)\big(Z_iZ_j - Z_{i-1}Z_{j+1}\big).$$

where $G^r = H^r = 0$ for all $r \neq 1, \ldots, m+2$, $Z_0 = 1$ and $Z_r = 0$ for all $r \neq 0, \ldots, m+1$.

Proof. Let

$$D = \Big(\sum_{i=0}^{m+1} G^{i+1} Z_{m+1-i}\Big)\Big(\sum_{i=0}^{m} H^{i+1} Z_{m-i}\Big) - \Big(\sum_{i=0}^{m} G^{i+1} Z_{m-i}\Big)\Big(\sum_{i=0}^{m+1} H^{i+1} Z_{m+1-i}\Big).$$

Straightforward computations show that

$$D = \sum_{i=0}^{m} \big(G^{m+2-i}H^{m+1-i} - G^{m+1-i}\ H^{m+2-i}\big) Z_i Z_i + \sum_{i=0}^{m} \big(G^1 H^{m+1-i} - G^{m+1-i}\ H^1\big) Z_i Z_{m+1}$$
$$+ \sum_{i=0}^{m-1}\sum_{j=i+1}^{m} \big(G^{m+2-i}H^{m+1-j} + G^{m+2-j}H^{m+1-i} - G^{m+1-i}H^{m+2-j} - G^{m+1-j}H^{m+2-i}\big) Z_i Z_j.$$

Next, we group all terms of the type $\Delta(q,t) = G^qH^t - G^tH^q$, with $t < q$. Let $A(i,j)$, $i \leq j$, be the coefficient of Z_iZ_j in the last expression and let $A^+(i,j) = \Delta(m+2-i, m+1-j)$ and $A^-(i,j) = \Delta(m+1-i, m+2-j)$. For all $i = 0, \ldots, m$, $A^-(i,i+1) = 0$, so, in particular, $A(m,m+1) = A^-(m,m+1) = 0$. Then,

$$A(i,j) = \begin{cases} A^+(i,i) & \text{if } i = j \in \{0,\ldots,m\} \\ A^+(i,i+1) & \text{if } j = i+1 \in \{1,\ldots,m\} \\ -A^-(i,m+1) & \text{if } j = m+1, i \in \{0,\ldots,m-1\} \\ A^+(i,j) - A^-(i,j) & \text{if } i \in \{0,\ldots,m-2\},\ i+2 \leq j \leq m. \end{cases}$$

Observe that $m+2-i \geq m+1-j$ whenever $i \leq j$ and also $m+1-i \geq m+2-j$ whenever $j \geq i+2$. All the coefficients $A^+(i,j)$ and $A^-(i,j)$ involved are of the type $\Delta(q,t)$ with $t < q$. But, clearly, $A^-(i,j) = A^+(i+1,j-1)$. TheN,

$$D = \sum_{i=0}^{m} A^+(0,i)Z_i + \sum_{i=0}^{m-1}\sum_{j=i+2}^{m+1} A^+(i+1,j-1)(Z_{i+1}Z_{j-1} - Z_iZ_j).$$

Rearranging the indices and setting, if necessary, $G^r = H^r = 0$ for all $r \neq 1, \ldots, m+2$ and $Z_r = 0$ for all $r \neq 0, \ldots, m+1$, we obtain the expression of the statement of the theorem. $\square$

We need some extra notation. Given $p, s \in \mathbb{N}$ and $0 < \delta \leq d_1 \leq \cdots \leq d_p$, we write:

$$g_p^s = (d_1 - \delta, \ldots, d_p - \delta, \overset{s}{\ldots}, d_p - \delta), \qquad G_p^s = V_{p+s-1}(g_p^s)$$
$$h_p^s = (\delta, d_1, \ldots, d_p, \overset{s}{\ldots}, d_p), \qquad H_p^s = V_{p+s}(h_p^s)$$

Next, we establish a particular case of Theorem 3.

Proposition 22. *For all $p, s \in \mathbb{N}$ and $0 < \delta \leq d_1 \leq \cdots \leq d_p$ it holds that*

$$\hat{\mu}_p(d_1 - \delta, \ldots, d_p - \delta, \overset{s}{\ldots}, d_p - \delta) \leq \hat{\mu}_{p+1}(\delta, d_1, \ldots, d_p, \overset{s}{\ldots}, d_p).$$

Proof. We proceed by induction on $p \in \mathbb{N}$. Lemma 6 solves the case $p = 1$. Next, fix $p > 1$, and assume that for all $s \in \mathbb{N}$, $\hat{\mu}_{p-1}(d_1 - \delta, \ldots, d_{p-1} - \delta, \overset{s}{\ldots}, d_{p-1} - \delta) \leq \hat{\mu}_p(\delta, d_1, \ldots, d_{p-1}, \overset{s}{\ldots}, d_{p-1})$, or, equivalently, for all $s \in \mathbb{N}$, $\frac{G_{p-1}^{s+1}}{G_{p-1}^s} \leq \frac{H_{p-1}^{s+1}}{H_{p-1}^s}$. We claim that

$$G_{p-1}^q H_{p-1}^t - G_{p-1}^t H_{p-1}^q \leq 0, \text{ for all } t < q. \tag{12}$$

Indeed $\frac{G_{p-1}^q}{G_{p-1}^t} \leq \frac{H_{p-1}^q}{H_{p-1}^t}$ because of the induction hypothesis and the fact that

$$\frac{G_{p-1}^q}{G_{p-1}^t} = \frac{G_{p-1}^q}{G_{p-1}^{q-1}} \frac{G_{p-1}^{q-1}}{G_{p-1}^{q-2}} \cdots \frac{G_{p-1}^{t+1}}{G_{p-1}^t} \qquad \text{and} \qquad \frac{H_{p-1}^q}{H_{p-1}^t} = \frac{H_{p-1}^q}{H_{p-1}^{q-1}} \frac{H_{p-1}^{q-1}}{H_{p-1}^{q-2}} \cdots \frac{H_{p-1}^{t+1}}{H_{p-1}^t}.$$

In order to establish the result for $p > 1$, we have to prove that for all $s \in \mathbb{N}$, $G_p^{s+1} H_p^s - G_p^s H_p^{s+1} \leq 0$. From Proposition 21,

$$G_p^{s+1} = \sum_{i=0}^{s+1} G_{p-1}^{i+1} Z_{s+1-i}, \qquad H_p^s = \sum_{i=0}^{s} H_{p-1}^{i+1} Z_{s-i},$$
$$G_p^s = \sum_{i=0}^{s} G_{p-1}^{i+1} Z_{s-i}, \qquad H_p^{s+1} = \sum_{i=0}^{s+1} H_{p-1}^{i+1} Z_{s+1-i},$$

where $Z_0 = 1$ and $Z_r = V_r(d_p - d_{p-1}, \ldots, d_p - d_{p-1})$, for all $r = 1, \ldots, s+1$. Applying Lemma 7,

$$G_p^{s+1} H_p^s - G_p^s H_p^{s+1} = \sum_{i=0}^{s} \sum_{j=i}^{s} \big(G_{p-1}^{s+2-i} H_{p-1}^{s+1-j} - G_{p-1}^{s+1-j} H_{p-1}^{s+2-i}\big)\big(Z_i Z_j - Z_{i-1} Z_{j+1}\big).$$

where $Z_r = 0$ for all $r \neq 0, \ldots, s+1$ and $G_{p-1}^r = H_{p-1}^r = 0$ for all $r \neq 1, \ldots, s+2$. Applying Equation (12), the induction hypothesis, and Proposition 20, we obtain that indeed, $G_p^{s+1} H_p^s - G_p^s H_p^{s+1} \leq 0$. $\square$

We can easily generalize the property stated in Proposition 22.

Proposition 23. *Given $p, t, q \in \mathbb{N}$ such that $t < q$, we have that $G_p^q H_p^t - G_p^t H_p^q \leq 0$.*

Proof. In Proposition 22 we proved that for all $s \in \mathbb{N}$, $G_p^{s+1} H_p^s - G_p^s H_p^{s+1} \leq 0$ or, equivalently, $\frac{G_p^{s+1}}{G_p^s} \leq \frac{H_p^{s+1}}{H_p^s}$. Now, given $t < q$, we have that $G_p^q H_p^t - G_p^t H_p^q \leq 0$ if and only if $\frac{G_p^q}{G_p^t} \leq \frac{H_p^q}{H_p^t}$. Again, this inequality follows directly from the hypothesis and the decomposition

$$\frac{G_p^q}{G_p^t} = \frac{G_p^q}{G_p^{q-1}} \frac{G_p^{q-1}}{G_p^{q-2}} \cdots \frac{G_p^{t+1}}{G_p^t}, \qquad \frac{H_p^q}{H_p^t} = \frac{H_p^q}{H_p^{q-1}} \frac{H_p^{q-1}}{H_p^{q-2}} \cdots \frac{H_p^{t+1}}{H_p^t}. \qquad \square$$

Finally, we can proceed with the proof of Theorem 3.

Theorem 3. *For all $p, k \in \mathbb{N}$ such that $k \geq p$, and all $0 < \delta \leq d_1 \leq \cdots \leq d_p \leq \cdots \leq d_k$,*

$$\hat{\mu}_p(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta) \leq \hat{\mu}_{p+1}(\delta, d_1, \ldots, d_p, \ldots, d_k).$$

Proof. According to Theorem 2,

$$\hat{\mu}_p(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta) = \frac{V_{k+1}(d_1 - \delta, \ldots, d_p - \delta, d_p - \delta, \ldots, d_k - \delta)}{V_k(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta)}$$

and

$$\hat{\mu}_{p+1}(\delta, d_1, \ldots, d_p, \ldots, d_k) = \frac{V_{k+2}(\delta, d_1, \ldots, d_p, d_p \ldots, d_k)}{V_{k+1}(\delta, d_1, \ldots, d_p, \ldots, d_k)}.$$

Therefore, $\hat{\mu}_p(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta) \leq \hat{\mu}_{p+1}(\delta, d_1, \ldots, d_p, \ldots, d_k)$ if and only if

$$\begin{aligned} \Delta = {} & V_{k+1}(d_1 - \delta, \ldots, d_p - \delta, d_p - \delta, \ldots, d_k - \delta) V_{k+1}(\delta, d_1, \ldots, d_p, \ldots, d_k) \\ & - V_k(d_1 - \delta, \ldots, d_p - \delta, \ldots, d_k - \delta) V_{k+2}(\delta, d_1, \ldots, d_p, d_p, \ldots, d_k) \leq 0. \end{aligned} \tag{13}$$

Now, applying Proposition 21, we decompose each of the four factors in the last inequality as sums involving volumes of the types G_p^s and H_p^s. Then,

$$V_k(d_1 - \delta, \ldots, d_k - \delta) = \sum_{i=0}^{k-p} G_p^{i+1} Z_{k-p-i}, \qquad V_{k+1}(d_1 - \delta, \ldots, d_p - \delta, d_p - \delta, \ldots, d_k - \delta) = \sum_{i=0}^{k-p} G_p^{i+2} Z_{k-p-i},$$

$$V_{k+1}(\delta, d_1, \ldots, d_k) = \sum_{i=0}^{k-p} H_p^{i+1} Z_{k-p-i}, \qquad V_{k+2}(\delta, d_1, \ldots, d_p, d_p, \ldots, d_k) = \sum_{i=0}^{k-p} H_p^{i+2} Z_{k-p-i},$$

where $Z_0 = 1$ and $Z_t = V_t(d_{k-t+1} - \delta, \ldots, d_k - \delta)$, $t = 1, \ldots, k - p$. Therefore, applying Lemma 7,

$$\Delta = \sum_{i=0}^{k-p} \sum_{j=i}^{k-p} \big(G_p^{k-p+2-i} H_p^{k-p+1-j} - G_p^{k-p+1-j} H_p^{k-p+2-i}\big)\big(Z_i Z_j - Z_{i-1} Z_{j+1}\big).$$

where $Z_r = 0$ for all $r \neq 0, \ldots, k - p$ and $G_p^r = H_p^r = 0$ for all $r \neq 1, \ldots, k - p + 2$. Therefore, for $\Delta \leq 0$ it is sufficient to establish that $\Delta(q, t) \leq 0$ whenever $t < q$ and that $Z_t Z_q - Z_{t-1} Z_{q+1} \geq 0$ if $t \leq q$. These two properties were already proved in Propositions 23 and 20, respectively. $\square$

As already pointed out, the proof of Theorem 6 has the same structure as that of Theorem 3. Hence we just provide an outline.

Theorem 6. *Let $k, i, j \in \mathbb{N}$, $i \leq j \leq k$, a value $\gamma > 0$ and costs $0 < c_1 \leq \cdots \leq c_k$. Then*

$$\mu_j(c_1, c_2, \ldots, c_i + \gamma, \ldots, c_k + \gamma) \geq \mu_j(c_1, c_2, \ldots, c_k).$$

Proof. Let us examine some simple situations. If $i = j = k$, then $\mu_k(c_1, \ldots, c_{k-1}, c_k + \gamma) = \gamma + \mu_k(c_1, \ldots, c_k) \geq \mu_k(c_1, \ldots, c_k)$. If $i < k$ and $j = k$ then

$$\mu_k(c_1, \ldots, c_i + \gamma, \ldots, c_k + \gamma) = (c_k - c_{k-1}) + \hat{\mu}_{k-1}(c_1, \ldots, c_i + \gamma, \ldots, c_{k-1} + \gamma)$$

and

$$\mu_k(c_1, \ldots, c_k) = (c_k - c_{k-1}) + \hat{\mu}_{k-1}(c_1, \ldots, c_{k-1}).$$

Then, $\mu_k(c_1,\ldots,c_i+\gamma,\ldots,c_k+\gamma) \geq \mu_k(c_1,\ldots,c_k)$ if and only if $\hat{\mu}_{k-1}(c_1,\ldots,c_i+\gamma,\ldots,c_{k-1}+\gamma) \geq \hat{\mu}_{k-1}(c_1,\ldots,c_{k-1})$. Ovbiously, if $i \leq j < k$ then $\mu_j(c_1,c_2,\ldots,c_i+\gamma,\ldots,c_k+\gamma) = \hat{\mu}_j(c_1,\ldots,c_i+\gamma,\ldots,c_{k-1}+\gamma)$ and $\mu_j(c_1,\ldots,c_k) = \hat{\mu}_j(c_1,\ldots,c_{k-1})$. It suffices to prove that for all $i \leq j \leq k$,

$$\hat{\mu}_j(c_1,\ldots,c_{i-1},c_i+\gamma,\ldots,c_j+\gamma,\ldots,c_k+\gamma) \geq \hat{\mu}_j(c_1,\ldots,c_k).$$

According to Theorem 2, this is equivalent to establishing that

$$\begin{aligned}\Delta = {} & V_{k+1}(c_1,\ldots,c_{i-1},c_i+\gamma,\ldots,c_j+\gamma,c_j+\gamma,\ldots,c_k+\gamma)V_k(c_1,\ldots,c_k)\\ & - V_k(c_1,\ldots,c_{i-1},c_i+\gamma,\ldots,c_j+\gamma,\ldots,c_k+\gamma)V_{k+1}(c_1,\ldots,c_j,c_j,\ldots,c_k) \geq 0.\end{aligned}$$

Denote $Z_0 = 1$ and

$$\begin{aligned}G_j^s &= V_{j+s-1}(c_1,\ldots,c_{i-1},c_i+\gamma,\ldots,c_j+\gamma,\overset{s}{\ldots},c_j+\gamma),\ s=1,\ldots,k-j+2\\ H_j^s &= V_{j+s-1}(c_1,\ldots,c_j,\overset{s}{\ldots},c_j),\ s=1,\ldots,k-j+2\\ Z_t &= V_t(c_{k-t+1}-c_j,\ldots,c_k-c_j),\ t=1,\ldots,k-j.\end{aligned}$$

Applying Proposition 21, $V_k(c_1,\ldots,c_k) = \sum_{r=0}^{k-j} H_j^{r+1} Z_{k-j-r}$, $V_{k+1}(c_1,\ldots,c_j,c_j,\ldots,c_k) = \sum_{r=0}^{k-j} H_j^{r+2} Z_{k-j-r}$, and

$$\begin{aligned}V_{k+1}(c_1,\ldots,c_i+\gamma,\ldots,c_j+\gamma,c_j+\gamma,\ldots,c_k+\gamma) &= \sum_{r=0}^{k-j} G_j^{r+2} Z_{k-j-r}\\ V_k(c_1,\ldots,c_i+\gamma,\ldots,c_j+\gamma,\ldots,c_k+\gamma) &= \sum_{r=0}^{k-j} G_j^{r+1} Z_{k-j-r}\end{aligned}$$

Therefore, applying Lemma 7,

$$\Delta = \sum_{r=0}^{k-j}\sum_{t=r}^{k-j}\big(G_j^{k-j+2-r}H_j^{k-j+1-t} - G_j^{k-j+1-t}H_j^{k-j+2-r}\big)\big(Z_rZ_t - Z_{r-1}Z_{t+1}\big).$$

where $Z_r = 0$ for all $r \neq 0,\ldots,k-j$ and $G_j^r = H_j^r = 0$ for all $r \neq 1,\ldots,k-j+2$. Then, in order to prove that $\Delta \geq 0$ it is sufficient to establish that $\Delta(q,t) \geq 0$ whenever $t < q$ and that $Z_rZ_t - Z_{r-1}Z_{t+1} \geq 0$ if $r \leq t$. The first property can be established, with very few adjustments, as in Proposition 23, and the second holds by Proposition 20. $\square$

The proof of Theorem 5 follows a similar technique but with a significant difference: the decomposition of a given volume provided by Proposition 21 has to be changed by the one given in Proposition 24. We need some final notations. Given $p,k,s,t,q \in \mathbb{N}$ such that $k \geq p$ and $0 < c_1 \leq \cdots \leq c_p \leq \cdots \leq c_k$, define

$$\begin{aligned}A_{k,s}^p &= V_{k+s-1}(c_1,\ldots,c_p,\ldots,c_k,\overset{s}{\ldots},c_k)\\ \hat{A}_{k,s}^p &= V_{k+s}(c_1,\ldots,c_p,c_p\ldots,c_k,\overset{s}{\ldots},c_k)\\ \Delta_k^p(t,q) &= \hat{A}_{k,t}^p A_{k,q}^p - A_{k,t}^p \hat{A}_{k,q}^p\end{aligned}$$

The superscript in $A_{k,s}^p$, though somehow unnecessary or ambiguous in cases like $\hat{A}_{p,s}^p = A_{p,s+1}^p$, is helpful to refer to a particular coordinate of the core-center. It is also worth noting that $\Delta_k^p(t,t) = 0$.

Proposition 24. *Given $p,k,s \in \mathbb{N}$ such that $k \geq p > 1$ and $0 < c_1 \leq \cdots \leq c_p \leq \cdots \leq c_k$, let $\delta(p,k) = p-1$ if $p = k$ and $\delta(p,k) = p$ if $p < k$. Then,*

$$A_{k,s}^p = \sum_{i=0}^{s} \frac{1}{i!} A_{k-1,s+1-i}^{\delta(p,k)}\,(c_k - c_{k-1})^i,\ \ \hat{A}_{k,s}^p = \sum_{i=0}^{s} \frac{1}{i!} \hat{A}_{k-1,s+1-i}^{\delta(p,k)}\,(c_k - c_{k-1})^i.$$

Proof. Let $p, k, s \in \mathbb{N}$ with $k > p > 1$. Then

$$\begin{aligned}
A^p_{k,s} &= V_{k+s-1}(c_1, \dots, c_p, \dots, c_k, \overset{s}{\dots}, c_k) \\
&= \int_0^{c_1} \dots \int_0^{c_{k-1}-\sum_{j=1}^{k-2} x_j} V_s\big(c_k - \sum_{j=1}^{k-1} x_j, \dots, c_k - \sum_{j=1}^{k-1} x_j\big) dx_{k-1} \dots dx_1 \\
&= \int_0^{c_1} \dots \int_0^{c_{k-1}-\sum_{j=1}^{k-2} x_j} \frac{1}{s!}\big(c_k - \sum_{j=1}^{k-1} x_j\big)^s dx_{k-1} \dots dx_1
\end{aligned}$$

where the last equality is obtained applying Lemma 3. Now, we expand the integrand by the binomial theorem, setting $X_k = c_k - c_{k-1}$ and $Y_{k-1} = c_{k-1} - \sum_{j=1}^{k-1} x_j$. Then,

$$\frac{1}{s!}\big(c_k - \sum_{j=1}^{k-1} x_j\big)^s = \frac{1}{s!}(X_k + Y_{k-1})^s = \frac{1}{s!}\sum_{i=0}^{s}\binom{s}{i} X_k^i Y_{k-1}^{s-i} = \sum_{i=0}^{s} \frac{1}{i!(s-i)!} X_k^i Y_{k-1}^{s-i}.$$

Therefore,

$$A^p_{k,s} = \sum_{i=0}^{s} \frac{1}{i!}\Big(\int_0^{c_1} \dots \int_0^{c_{k-1}-\sum_{j=1}^{k-2} x_j} \frac{1}{(s-i)!} Y_{k-1}^{s-i} dx_{k-1} \dots dx_1\Big) X_k^i.$$

Again, by Lemma 3, $\frac{1}{(s-i)!} Y_{k-1}^{s-i} = V_{s-i}(Y_{k-1}, \dots, Y_{k-1})$ and, consequently,

$$\begin{aligned}
\int_0^{c_1} \dots \int_0^{c_{k-1}-\sum_{j=1}^{k-2} x_j} \frac{1}{(s-i)!} Y_{k-1}^{s-i} dx_{k-1} \dots dx_1 &= \int_0^{c_1} \dots \int_0^{c_{k-1}-\sum_{j=1}^{k-2} x_j} V_{s-i}(Y_{k-1}, \dots, Y_{k-1}) dx_{k-1} \dots dx_1 \\
= \int_0^{c_1} \dots \int_0^{c_{k-2}-\sum_{j=1}^{k-3} x_j} V_{s+1-i}\Big(c_{k-1} - \sum_{j=1}^{k-2} x_j, \dots, c_{k-1} - \sum_{j=1}^{k-2} x_j\Big) & dx_{k-2} \dots dx_1 = A^p_{k-1,s+1-i}.
\end{aligned}$$

The above equality leads to $A^p_{k,s} = \sum_{i=0}^{s} \frac{1}{i!} A^p_{k-1,s+1-i}\ X_k^i$.

The case $k = p$ and the second part of the proof can be easily adapted from the previous one. □

Lemma 8. *Given $p, k, s \in \mathbb{N}$ such that $k \geq p > 1$ and $0 < c_1 \leq \dots \leq c_p \leq \dots \leq c_k$, let $X_k = c_k - c_{k-1}$. Then,*

$$\Delta^p_k(s, s+1) = \sum_{i=0}^{2s}\Big(\sum_{r=0}^{r_i} B(i,r)\Delta^{\delta(p,k)}_{k-1}(t(i,r), q(i,r))\Big) X_k^i.$$

where, $r_i \in \mathbb{N}$ for all $i \in \{0, \dots, 2s\}$. Besides, for all $r \in \{0, \dots, r_i\}$, it holds that $B(i,r) \geq 0$ and $t(i,r) < q(i,r)$.

Proof. Using the decomposition of Proposition 24, one can derive that

$$\begin{aligned}
\Delta^p_k(s, s+1) = &\sum_{i=0}^{s}\Big(\sum_{r=0}^{i} \frac{1}{r!(i-r)!} \Delta^{\delta(p,k)}_{k-1}(t_1(i,r), q_1(i,r))\Big) X_k^i \\
&+ \sum_{i=s+1}^{2s+1}\Big(\sum_{r=0}^{2s+1-i} \frac{1}{(s+1-r)!(i-(s+1-r))!} \Delta^{\delta(p,k)}_{k-1}(t_2(i,r), q_2(i,r))\Big) X_k^i.
\end{aligned}$$

where, $t_1(i,r) = s+1-r$, $q_1(i,r) = s+2-(i-r)$, $t_2(i,r) = 2s+2-i-r$ and $q_2(i,r) = r+1$.

First, we examine the coefficients of the powers X_k^i, $i = 0,\dots,s$, in the sum above. Observe that the coefficient of X_k^0, that corresponds to $i=0$, $r=0$, is just $\Delta_{k-1}^{\delta(p,k)}(s+1,s+2)$. Next, fix $i \in \{1,\dots,s\}$ and denote $r_i^* = \frac{i-1}{2}$. Clearly, if i is an odd number, $r_i^* \in \mathbb{N}$ and the term corresponding to the index $r = r_i^*$ is zero, because $t_1(i,r_i^*) = q_1(i,r_i^*)$. As a consequence, the coefficient of X_k^i has an odd number of addends, in fact, i when i is odd and $i+1$ when i is even. In any case, the term corresponding to the index $r = i$ (the last one) is, $B(i,i)\Delta_{k-1}^{\delta(p,k)}(t(i,i),q(i,i))$ where $B(i,i) = \frac{1}{i!} \geq 0$, $t(i,i) = s+1-i$ and $q(i,i) = s+2$. Since $i \in \{0,\dots,s\}$, $t(i,i) < q(i,i)$. Therefore, we are left with an even number of remaining terms.

Now, consider the terms corresponding to indices $r_1, r_2 \in \{0,\dots,i-1\}$ such that $r_1 < r_2$ and $r_1 + r_2 = i-1$. We have that, $r_1 < r_i^* < r_2$, $t_1(i,r_1) = q_1(i,r_2)$ and $q_1(i,r_1) = t_1(i,r_2)$. Subsequently, $\Delta_{k-1}^{\delta(p,k)}(t_1(i,r_1),q_1(i,r_1)) = -\Delta_{k-1}^{\delta(p,k)}(t_1(i,r_2),q_1(i,r_2))$, so we can add up both terms and write

$$\frac{1}{r_1!(i-r_1)!}\Delta_{k-1}^{\delta(p,k)}(t_1(i,r_1),q_1(i,r_1)) + \frac{1}{r_2!(i-r_2)!}\Delta_{k-1}^{\delta(p,k)}(t_1(i,r_2),q_1(i,r_2))$$
$$= \Big(\frac{1}{r_2!(i-r_2)!} - \frac{1}{r_1!(i-r_1)!}\Big)\Delta_{k-1}^{\delta(p,k)}(t_1(i,r_2),q_1(i,r_2)).$$

Therefore each pair of indices r_1, r_2, with the properties listed above, produces a single term of the form $B(i,r)\Delta_{k-1}^{\delta(p,k)}(t(i,r),q(i,r))$ satisfying:

1. $B(i,r) \geq 0$.
   Indeed, $\frac{1}{r_2!(i-r_2)!} - \frac{1}{r_1!(i-r_1)!} \geq 0$ if and only if $\frac{(i-r_1)!}{r_2!} \geq \frac{(i-r_2)!}{r_1!}$. But, $r_1 + r_2 = i-1$ implies that $i - r_1 = r_2 + 1$, $i - r_2 = r_1 + 1$. Then, $\frac{(i-r_1)!}{r_2!} = r_2 + 1 > \frac{(i-r_2)!}{r_1!} = r_1 + 1$.

2. $t(i,r) < q(i,r)$.
   Certainly, $t(i,r) = t_1(i,r_2) = s+1-r_2 < q(i,r) = q_1(i,r_2) = s+2-(i-r_2)$ if and only if $r_2 > \frac{i-1}{2} = r_i^*$.

A similar analysis can be done for the coefficients of the powers X_k^i, $i = s+1,\dots,2s+1$. □

Lemma 9. *For all $p,s \in \mathbb{N}$, $\mu_p(c_1,\dots,c_p,\overset{s}{\dots},c_p) \geq \mu_p(c_1,\dots,c_p,\overset{s+1}{\dots},c_p)$.*

Proof. We proceed by induction on p. The case $p=1$, that is, $\mu_1(c_1,\overset{s}{\dots},c_1) \geq \mu_1(c_1,\overset{s+1}{\dots},c_1)$ for all $s \in \mathbb{N}$, is a simple consequence of the fact that $\mu_1(c_1,\overset{s}{\dots},c_1) = \frac{c_1}{s}$. Next, assume that the result holds for $p-1$, that is, for all $s \in \mathbb{N}$,

$$\mu_{p-1}(c_1,\dots,c_{p-1},\overset{s}{\dots},c_{p-1}) \geq \mu_{p-1}(c_1,\dots,c_{p-1},\overset{s+1}{\dots},c_{p-1}).$$

Then, directly from Theorem 2, $\Delta_{p-1}^{p-1}(s,s+1) \geq 0$ for all $s \in \mathbb{N}$, or equivalently, $\Delta_{p-1}^{p-1}(t,q) \geq 0$ whenever $t < q$. We have to prove that the result holds for p. But, again, that is equivalent to prove that $\Delta_p^p(s,s+1) \geq 0$, for all $s \in \mathbb{N}$, which is a direct consequence of Lemma 8 and the induction hypothesis. □

Lemma 10. *For all $p,k,s \in \mathbb{N}$ such that $k \geq p$,*

$$\mu_p(c_1,\dots,c_p,\dots,c_k,\overset{s}{\dots},c_k) \geq \mu_p(c_1,\dots,c_p,\dots,c_k,\overset{s+1}{\dots},c_k).$$

Proof. We proceed by induction on k. The case $k=p$ was proven in Lemma 9. Next, assume that the result holds for $k-1 \geq p$, that is, for all $s \in \mathbb{N}$,

$$\mu_p(c_1,\dots,c_p,\dots,c_{k-1},\overset{s}{\dots},c_{k-1}) \geq \mu_p(c_1,\dots,c_p,\dots,c_{k-1},\overset{s+1}{\dots},c_{k-1}).$$

Then, $\Delta_{k-1}^p(s,s+1) \geq 0$ for all $s \in \mathbb{N}$, or equivalently, $\Delta_{k-1}^p(t,q) \geq 0$ whenever $t < q$. We have to prove that the result holds for $k > p$. But, again, that is equivalent to prove that $\Delta_k^p(s,s+1) \geq 0$, for all $s \in \mathbb{N}$. But this inequality follows immediately from Lemma 8 and the induction hypothesis. □

Theorem 5. *Given $p, s \in \mathbb{N}$ and costs $0 < c_1 \leq \cdots \leq c_p \leq c_{p+1} \leq \cdots \leq c_{p+s}$, we have that*

$$\mu_p(c_1, \ldots, c_p) \geq \mu_p(c_1, \ldots, c_p, c_{p+1}) \geq \cdots \geq \mu_p(c_1, \ldots, c_p, c_{p+1}, \ldots, c_{p+s})$$

Proof. Observe that $\mu_p(c_1, \ldots, c_p) = (c_p - c_{p-1}) + \hat{\mu}_{p-1}(c_1, \ldots, c_{p-1})$ and $\mu_p(c_1, \ldots, c_p, c_{p+1}) = \hat{\mu}_p(c_1, \ldots, c_p)$. Then, $\mu_p(c_1, \ldots, c_p) \geq \mu_p(c_1, \ldots, c_p, c_{p+1})$ if and only if $\Delta_p = (c_p - c_{p-1})A^{p-1}_{p-1,1}A^p_{p,1} + A^{p-1}_{p-1,2}A^p_{p,1} - A^p_{p,2}A^{p-1}_{p-1,1} \geq 0$. Note that Δ_p does not depend on the cost c_{p+1}, therefore, using Lemma 9, it is easy to see that the first inequality of the chain is satisfied.

Now, whenever $k > p$, $\mu_p(c_1, \ldots, c_k) \geq \mu_p(c_1, \ldots, c_k, c_{k+1})$ if and only if $\hat{\mu}_p(c_1, \ldots, c_{k-1}) \geq \hat{\mu}_p(c_1, \ldots, c_k)$. Using Theorem 2, the last inequality is equivalent to

$$\hat{A}^p_{k-1,1}A^p_{k,1} - \hat{A}^p_{k,1}A^p_{k-1,1} \geq 0.$$

Consider the left hand expression as a function of the cost c_k, that is, $f(c_k) = \hat{A}^p_{k-1,1}A^p_{k,1} - \hat{A}^p_{k,1}A^p_{k-1,1}$, $c_k \in [c_{k-1}, c_{k+1}]$. A straightforward computation shows that $f'(c_k) = 0$. Henceforth, f is constant in the interval $[c_{k-1}, c_{k+1}]$. Consequently, $f(c_k) \geq 0$ if and only if $f(c_{k-1}) \geq 0$. Since $f(c_{k-1}) = \Delta^p_{k-1}(1, 2)$ then $f(c_{k-1}) \geq 0$ if and only if $\hat{\mu}_p(c_1, \ldots, c_p, \ldots, c_{k-1}) \geq \hat{\mu}_p(c_1, \ldots, c_p, \ldots, c_{k-1}, c_{k-1})$. Finally, the last inequality has already been established in Lemma 10. □

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