Two variations of the Public Good Index for games with a priori unions

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Abstract

This paper discusses two variations of the Public Good Index for games with a priori unions. The first variation stresses the public good property which suggests that all members of a winning coalition derive equal power. The second one follows earlier work on the integration of a priori unions (see Owen 1977, 1982) and refers to essential subsets of a union when allocating power shares. Theoretical reasoning and numerical examples demonstrate that the numerical values that result from the two alternative measures may differ substantially.

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1 Introduction

Power is an important concept to study the social, political and economic relationships represented as simple games. Power indices are quantitative measures to express power in these games. The literature offers a series of alternative measures: the Shapley-Shubik index (Shapley and Shubik, 1954), the Banzhaf-Coleman index (Banzhaf, 1965; Coleman, 1971), the Deegan-Packel index (Deegan and Packel, 1978), and the Johnston power index (Johnston, 1978). In this paper we will focus on the Public Good Index (PGI). This measure was first applied in Holler (1978), then explicitly proposed in Holler (1982) and axiomatized in Holler and Packel (1983).

The results of the listed power indices depend on the characteristic function of the corresponding games. They do not take into account available information on coalition formation and, more specifically, the existence of a priori unions, i.e., an a priori coalition structure. However, the so-called Owen value, proposed and characterized in Owen (1977), is a two-step extension of the Shapley value that takes a priori unions into consideration. In the first step, i.e., in the induced game played by the a priori unions (quotient game), this measure distributes the total value among the unions in accordance with the Shapley value. In the second step, once again applying the Shapley value, the total reward of a union is allocated among its members taking into account the possibility of their joining other unions.

The Owen value is a coalitional value of the Shapley value: it coincides with the Shapley value if each a priori union contains one element only. The Owen value applies the Shapley value in both steps. One of its most appealing properties is the quotient game property: the assigned power to the players of an a priori union equals the power of this union in the game played by the unions. Another interesting property of the Owen value is the property of symmetry in the quotient game: given two unions which play symmetric roles in the quotient game, they are awarded with the same apportionment of the total payoff.

Using similar reasoning, Owen (1982) proposed an application of the Banzhaf value to the framework of TU games with a priori unions. This measure is referred to as Banzhaf-Owen value. Here the assessments in each step are given by the Banzhaf value. However, neither the property of symmetry in the quotient game nor the quotient game property are satisfied. This value has been axiomatically characterized by Albizuri (2001), Amer et al. (2002), and Alonso-Meijide et al. (2007).

Alonso-Meijide and Fiestras-Janeiro (2002) propose a value for TU games with a priori unions which extends the Banzhaf value, satisfies the quotient game property and the symmetry in the quotient game property, and assures balanced contributions for the unions. This value is the symmetric coalition Banzhaf value. It reflects the result of a bargaining procedure by which, in the quotient game, each a priori union receives a payoff determined by the Banzhaf value; and, within each union, the members share this payoff in accordance with the Shapley value.

In a recent paper, Alonso-Meijide et al. (2008b) extended the Deegan-Packel index to games with a priori unions. The proposed extension satisfies the quotient game property, symmetry inside unions, symmetry among unions, DP-mergeability in the quotient game, and DP-mergeability inside unions. Two similar versions of mergeability (PGI-mergeability) are quintessential to the PGI extensions to a priori unions which will be analyzed in the sequel.

In this paper, we will discuss two extensions of the PGI for games with a priori unions. The first one stresses the public good property which suggests that all members of a winning coalition derive equal power, irrespective of their possibility to form alternative coalitions. In games with a priori unions it seems "natural" to apply the notion of decisiveness and the concept of minimal winning coalition to the quotient game only. Partners in an a priori union cannot be excluded from enjoying the coalition value, but, as well, partners cannot absent themselves from the costs implied by an a priori union. The second extension follows earlier work on the integration of a priori unions (see Owen 1977, 1982). It refers to essential subsets of a union when allocating power shares, taking the outside options of the coalition members into consideration.

Theoretical reasoning and numerical examples demonstrate that the numerical values that result from the two alternative measures may differ substantially. Obviously, the two versions constitute different solution concepts. The discussion will demonstrate that "different solution concepts can... be thought of as results of choosing not only which properties one likes, but also which examples one wishes to avoid" (Aumann, 1977, p.471).

The paper is organized as follows. In Section 2, we introduce the analytical tools and recall some basic definitions. In Section 3, using the principle of solidarity inside unions, we define and characterize a first extension of the Public Good Index. In Section 4, we define and characterize a second extension of the Public Good Index, following a similar procedure to that of Owen. Finally, we illustrate and compare these extensions with a real–world example.

2 Preliminaries

2.1 Simple Games

A **simple-game** is a pair (N, W) where N is a coalition and W is a set of subsets of N satisfying:

- $N \in W, \emptyset \notin W$ and
- the monotonicity property, *i.e.*,

$$S \subseteq T \subseteq N \text{ and } S \in W \text{ implies } T \in W.$$

This representation of simple games follows the approach by Felsenthal and Machover (1998) and by Peleg and Sudhölter (2003). Intuitively, N is the set of members of a committee and W is the set of winning coalitions. For example, parliaments, town councils, and the UN Security Council are committees.

In a simple game (N, W), a coalition $S \subseteq N$ is **winning** if $S \in W$ and is **losing** if $S \notin W$. We denote by $SI(N)$ the set of simple games with player set N.

A winning coalition $S \in W$ is a **minimal winning coalition** (MWC) if every proper subset of S is a losing coalition, that is, S is a MWC in (N, W) if $S \in W$ and $T \notin W$ for any $T \subset S$. We denote by M^W the set of MWC of the simple game (N, W). Given a player $i \in N$ we denote by M_i^W the set of MWC such that i belongs to, that is, $M_i^W = \left\{S \in M^W / i \in S\right\}$.

A **null player** in a simple game (N, W) is a player i such that $i \notin S$ for all $S \in M^W$. Two players $i, j \in N$ are **symmetric** in a simple game (N, W)

if $S \cup i \in W$[1] if and only if $S \cup j \in W$ for all $S \subseteq N \backslash \{i, j\}$ such that $S \notin W$. Given a coalition $T \subseteq N$, the **unanimity game** of T, (N, W_T), is the simple game with $M^{W_T} = \{T\}$.

A **power index** is a function f which assigns an n-dimensional real vector $f(N, W)$ to a simple game (N, W), where the i-th component of this vector, $f_i(N, W)$, is the power of player i in the game (N, W) according to f. Here we recall three appealing properties of power indices.

A power index f satisfies *efficiency* if and only if for every simple game (N, W), $\sum_{i \in N} f_i(N, W) = 1$.

A power index f satisfies the *null player* property if and only if for every simple game (N, W) and $i \in N$ a null player, then $f_i(N, W) = 0$.

A power index f satisfies *symmetry* if and only if for every simple game (N, W), and $i, j \in N$ symmetric players in the game, $f_i(N, W) = f_j(N, W)$.

2.2 The Public Good Index

Holler (1982) proposed a power index, the Public Good Index (PGI). In the computation of the PGI, the MWC are the only relevant coalitions. It is assumed that coalitions that are not MWC do not matter, and thus should not be taken into consideration, when it comes to measuring power. That is, although only MWC are taking into account for the calculation of the PGI, it is not said that no other coalitions will be formed.

Given a simple game (N, W), the PGI assigns to each player $i \in N$ the real number:

$$\delta_i(N, W) = \frac{\left|M_i^W\right|}{\sum_{j \in N} \left|M_j^W\right|}^2. \tag{1}$$

That is, the PGI of a player i is equal to the total number of MWC containing player i, normalized by the sum of these numbers for all players.

An axiomatic characterization of this index can be found in Holler and Packel (1983). The characterization used in that paper applies the properties of symmetry, efficiency, null player, and PGI- mergeability. To specify the latter property, we introduce the definition of mergeable games.

Mergeable games. Given two simple games $(N, W), (N, V)$, the simple game $(N, W \vee V)$ is defined in such a way that a coalition $S \in W \vee V$ if $S \in W$ or $S \in V$. Two simple games (N, W) and (N, V) are mergeable if for any $S \in M^W$ and for any $T \in M^V$, $S \nsubseteq T$ and $T \nsubseteq S$.

The mergeability condition guarantees that the set of MWC of the game $(N, W \vee V)$ is the union of the MWC sets of the games (N, W) and (N, V) (when (N, W) and (N, V) are two mergeable games). Then, $\left|M^{W \vee V}\right| = \left|M^W\right| + \left|M^V\right|$.

PGI-mergeability. A power index f satisfies PGI-mergeability if for any

[1] We will use shorthand notation and write $S \cup i$ for the set $S \cup \{i\}$ and $S \backslash i$ for the set $S \backslash \{i\}$.

[2] We denote by $|S|$ the cardinality of a set S.

pair of mergeable games $(N, W), (N, V)$, it holds that

$$f(N, W \vee V) = \frac{\sum_{j \in N} \left| M_j^W \right|}{\sum_{j \in N} \left| M_j^{W \vee V} \right|} f(N, W) + \frac{\sum_{j \in N} \left| M_j^V \right|}{\sum_{j \in N} \left| M_j^{W \vee V} \right|} f(N, V). \qquad (2)$$

That is, power in a merged game is a weigthed mean of the power of the components games, with the sum of the number of MWC containing each player providing the weights.

Theorem 1 *(Holler and Packel, 1983) The unique power index f defined on $SI(N)$ satisfying PGI-mergeability, null player, symmetry, and efficiency is the PGI.*

Alternatively, Alonso-Meijide et al. (2008a) characterized the PGI replacing the property of PGI-mergeability with the property of PGI-minimal monotonicity. It takes into account a relation between two simple games (N, W) and (N, V), that is, given in terms of the cardinality of the sets of MWC.

PGI-minimal monotonicity. A power index f satisfies PGI-minimal monotonicity if for any pair of simple games (N, W), (N, V), it holds that for all player $i \in N$ such that $M_i^W \subseteq M_i^V$,

$$f_i(N, V) \sum_{j \in N} \left| M_j^V \right| \geq f_i(N, W) \sum_{j \in N} \left| M_j^W \right|.$$

That is, if the set of MWC containing a player $i \in N$ in game (N, W) is a subset of MWC containing this player in game (N, V), then the power of player i in game (N, V) is not less than power of player i in game (N, W) (first, this power must be normalized by the sum for all players of the total number of MWC containing each player in games (N, W) and (N, V)).

Theorem 2 *(Alonso-Meijide et al. 2008a) The unique power index f satisfying PGI-minimal monotonicity, null player, symmetry, and efficiency, is the PGI.*

2.3 Games with a priori unions

Given N, we will denote by $P(N)$ the set of all partitions of N. An element $P \in P(N)$ is called a coalition structure: it describes the a priori unions on N. A simple game with partition of players is a triple (N, W, P), where (N, W) is a simple game and $P \in P(N)$. We denote by SIU the set of simple games with a priori unions and by $SIU(N)$ the set of simple games with a priori unions and player set N.

Given $(N, W, P) \in SIU(N)$, with $P = \{P_1, P_2, \ldots, P_u\}$, the **quotient game** is the simple game $\left(U, \overline{W}\right)$, where the set of players $U = \{1, 2, \ldots, u\}$ are the unions. A set $R \subseteq U$ is a winning coalition in $\left(U, \overline{W}\right)$ if $\bigcup_{k \in R} P_k$ is a winning coalition in (N, W).

Two unions $P_k, P_s \in P$ are symmetric if k and s are symmetric players in $\left(U, \overline{W}\right)$.

Taken a simple game with a priori unions (N, W, P), where

$$M^W = \{S_1, S_2, \ldots, S_l\}, \ P = \{P_1, P_2, \ldots, P_u\}, \text{ and } U = \{1, 2, \ldots, u\}:$$

- Two trivial partitions of players are given by $P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and $P^N = \{N\}$.
- The representatives of a coalition $S \subseteq N$ in the quotient game $\left(U, \overline{W}\right)$ form a coalition $u(S) \subseteq U$, where $j \in u(S)$ if and only if there exists a player i of P_j such that $i \in S$. In this way, $u(S)$ is defined as a set of index numbers that characterize the a priori unions involved in the forming of S, that is,
$$u(S) = \{j \in U / P_j \cap S \neq \emptyset\}.$$
- We will denote by $M^{\overline{W}}$ the **set of MWC in the quotient game**, that is,
$$M^{\overline{W}} = \left\{R \subseteq U / R \in \overline{W} \text{ and } R' \notin \overline{W} \text{ for all } R' \subset R\right\}.$$
- Given an a priori union $k \in U$ we will denote by $M_k^{\overline{W}}$ the set of MWC in the quotient game such that a priori union P_k belongs to them, that is,
$$M_k^{\overline{W}} = \left\{R \in M^{\overline{W}} / k \in R\right\}.$$

Given a simple game with a priori unions (N, W, P), we will say that a coalition $S \in M^W$ is **irrelevant** if $u(S) \notin M^{\overline{W}}$. That is, an irrelevant coalition is a MWC in game (N, W) such that its representatives in the quotient game do not constitute a MWC in $\left(U, \overline{W}\right)$.

We say that two coalitions S and S' are **equivalent**, if $u(S) = u(S')$, i.e., if their representatives in the quotient game, i.e., the set of a priori unions, are the same.

An important assumption in Owen (1977) is that every coalition $S \subseteq P_k$ has the possibility of forming a winning coalition joining with one or more of the remaining unions different from P_k. The Owen value does not consider the possibility of a coalition among subset S and proper subsets of some of the others unions. Taking this limitation into account with respect to the formation of winning coalitions, we introduce the concept of essential subset of a union.

Given a simple game with a priori unions (N, W, P), we will say that a coalition $\emptyset \neq S \subseteq P_k$ is an **essential subset of a union** P_k with respect to R if and only if $R \in M^{\overline{W}}$, $k \in R$, $S \cup \left(\cup_{l \in R \setminus k} P_l\right) \in W$, and $T \cup \left(\cup_{l \in R \setminus k} P_l\right) \notin W$ for every $T \subset S$. $E^{k,R}(N, W, P)$ denotes the set of essential subsets of a union P_k of the game (N, W, P) with respect to R. $E_i^{k,R}(N, W, P)$ denotes the subset of $E^{k,R}(N, W, P)$ formed by coalitions S such that $i \in S$. Finally, $E(N, W, P)$ denotes the set of coalitions S such that there exist a union P_k and $R \in M^{\overline{W}}$ and $S \in E^{k,R}(N, W, P)$. In order to illustrate the concepts of irrelevant coalition and essential subset of a union, we reproduce here an example similar to *Example 1*, included in Alonso-Meijide et al. (2008b).

Example 3 *Take a simple game with a priori unions* (N,W,P) *with* $N = \{a,b,c,d,e,f\}$, $P = \{P_1,P_2,P_3\}$, *and thus* $U = \{1,2,3\}$, *where* $P_1 = \{a\}$, $P_2 = \{b,c,d\}$ *and* $P_3 = \{e,f\}$ *and* $M^W = \{S_1,S_2,S_3,S_4,S_5\}$ *where* $S_1 = \{a,b\}$, $S_2 = \{a,c,d\}$, $S_3 = \{a,e,f\}$, $S_4 = \{a,c,e\}$, *and* $S_5 = \{b,c,d,e,f\}$. *The minimal winning coalitions in the quotient game* $\left(U,\overline{W}\right)$ *are:*

$$M^{\overline{W}} = \{\{1,2\},\{1,3\},\{2,3\}\}.$$

In this game, $S_4 = \{a,c,e\}$ *is a minimal winning coaliton. However, its coalition of representatives* $u(S_4) = \{1,2,3\}$ *is not minimal in the quotient game. Hence,* S_4 *is irrelevant.*

An essential subset of the union P_1 *with respect to* $R = \{1,2\}$ *is given by* $\{a\}$, *and two essential subsets of the union* P_2 *with respect to* $R = \{1,2\}$ *are given by* $\{b\}$ *and* $\{c,d\}$.

In the context of simple games with a priori unions, a **coalitional power index** is a function f which assigns an n-dimensional real vector $f(N,W,P)$ to a simple game with a priori unions (N,W,P), where the i-th component of vector $f_i(N,W,P)$ is the power of player i in the game (N,W,P) according to f.

3 The solidarity Public Good Index

The primary application of the PGI was to analyze situations in which a public good is considered. In this section, we consider simple games with a priori unions. We assume that a coalitional power index satisfies certain conditions, taking into account the existence of unions. Then, the definition of this variation of the PGI is focused on the quotient game, and after the allocation is divided among the unions, it is assumed that the index is a solidarity one. The solidarity condition establishes that players in the same a priori union have the same power.

We define a new power index that we will name the solidarity Public Good Index.

Definition 4 *Given* $(N,W,P) \in SIU(N)$, *the solidarity Public Good Index of a player* $i \in P_k$ *is:*

$$\Theta_i(N,W,P) = \frac{\left|M_k^{\overline{W}}\right|}{\sum_{l\in U}\left|M_l^{\overline{W}}\right|}\frac{1}{|P_k|} = \delta_k\left(U,\overline{W}\right)\frac{1}{|P_k|}. \tag{3}$$

The index Θ is consistent with the previous conditions. Only MWC in the original game that give rise to a MWC in the quotient game have influence. The first term coincides with the Public Good Index of the union P_k in the quotient game. Finally, the term $1/|P_k|$ assures that the payoff for player i is the same as for the other $|P_k|-1$ players of the union P_k (solidarity inside unions). As we can see in the previous definition, the amount given by this index to a player $i \in P_k$ is independent of the player, i.e., it is the same for all players in a union.

Next, we provide a characterization of this index. First, we recall the well-known properties of efficiency, symmetry among unions, and quotient game property of a coalitional power index.

Efficiency. A coalitional power index f satisfies efficiency if and only if for every $(N, W, P) \in SIU(N)$, $\sum_{i \in N} f_i(N, W, P) = 1$.

Symmetry among unions. A coalitional power index f satisfies symmetry among unions if and only if for every $(N, W, P) \in SIU(N)$, and $k, l \in U$ symmetric players in the quotient game, then

$$\sum_{i \in P_k} f_i(N, W, P) = \sum_{i \in P_l} f_i(N, W, P).$$

Quotient game property. A coalitional power index f satisfies the quotient game property if and only if for every $(N, W, P) \in SIU(N)$, and $k \in U$, then

$$f_k(U, \overline{W}, P^u) = \sum_{i \in P_k} f_i(N, W, P).$$

The crucial property of the coalitional solidarity Public Good Index is the property of solidarity. This property says that players in the same union are awarded in the same way.

Solidarity. A coalitional power index f satisfies solidarity if and only if for every $(N, W, P) \in SIU(N)$, and $i, j \in P_k$, then $f_i(N, W, P) = f_j(N, W, P)$.

The property of null union says that players belonging to a null union in the quotient game have no power.

Null union. A coalitional power index f satisfies null union if and only if for every $(N, W, P) \in SIU(N)$, and $i \in P_k$ $f_i(N, W, P) = 0$, if the union k is a null player in the quotient game $\left(U, \overline{W}\right)$.

The following property is an adaptation of the property of mergeability. It is similar to that used to characterize the coalitional Deegan-Packel index in Alonso-Meijide et al. (2008b). We say that two games (N, W, P) and (N, V, P) are mergeable in the quotient game if the corresponding quotient games are mergeable. For mergeable games, see above.

If two games (N, W, P) and (N, V, P) are mergeable in the quotient game[3], the mergeability condition guarantees that

$$\sum_{k \in U} \left| M_k^{\overline{W \vee V}} \right| = \sum_{k \in U} \left| M_k^{\overline{W}} \right| + \sum_{k \in U} \left| M_k^{\overline{V}} \right|.$$

The property of PGI-mergeability in the quotient game states that **power** in a merged game is a weighted mean of power of the two component games, with the sum of the number of MWC of each union in the quotient game of each component game providing the weights.

[3]We point out that the two games assure the same partition of players.

PGI-Mergeability in the quotient game. A coalitional power index f satisfies PGI-mergeability in the quotient game if for any pair $(N, W, P), (N, V, P)$, such that $(U, \overline{W})$ and $(U, \overline{V})$ are mergeable, it holds that

$$f(N, W \vee V, P) = \frac{\sum_{k \in U} \left| M_k^{\overline{W}} \right|}{\sum_{k \in U} \left| M_k^{\overline{W \vee V}} \right|} f(N, W, P) + \frac{\sum_{k \in U} \left| M_k^{\overline{V}} \right|}{\sum_{k \in U} \left| M_k^{\overline{W \vee V}} \right|} f(N, V, P).$$

Independence of superfluous coalitions says that elimination of a MWC S of the game such that (a) it is irrelevant or (b) there is a coalition S' such that $u(S) = u(S')$, will not change the power of the players.

Independence of superfluous coalitions. A coalitional power index f satisfies independence of superfluous coalitions if for any (N, W, P), and $S \in M^W$, $f(N, W, P) = f(N, W', P)$ where $M^{W'} = M^W \setminus S$ if $\overline{W'} = \overline{W}$ or if $M^{\overline{W}} = M^{\overline{W'}}$.

Theorem 5 *The solidarity Public Good Index is the unique coalitional power index which satisfies the properties of null union, solidarity, symmetry among unions, efficiency, PGI-mergeability in the quotient game, and independence of superfluous coalitions.*

Proof.

Existence. Let (N, W, P) be a simple game with a priori unions where $P = \{P_1, \cdots, P_u\}$ and $U = \{1, \cdots, u\}$. We prove that the solidarity Public Good Index Θ satisfies the previous properties.

Null Union. If $k \in U$ is a null player in the quotient game, then, by the property of null player of the Public Good Index, $\delta_k\left(U, \overline{W}\right) = 0$, and then $\Theta_i(N, W, P) = 0$, for all $i \in P_k$.

Solidarity. The expression for the solidarity Public Good Index is the same for every player in the same union, then if $i, j \in P_k$, then:

$$\Theta_i(N, W, P) = \Theta_j(N, W, P).$$

Symmetry among unions. Given a union $P_k \in P$, it holds that

$$\sum_{i \in P_k} \Theta_i(N, W, P) = \delta_k\left(U, \overline{W}\right).$$

Thus, the amount assigned by the solution Θ to a union $P_k \in P$ coincides with the PGI of the player $k \in U$ in the quotient game. As the PGI is symmetric, if two unions $P_k, P_j \in P$ are symmetric in the quotient game, it holds that

$$\sum_{i \in P_k} \Theta_i(N, W, P) = \sum_{i \in P_j} \Theta_i(N, W, P).$$

Efficiency. From above, the solidarity Public Good Index satisfies the quotient game property and we know that the PGI is efficient. Then, we get that the solidarity Public Good Index is efficient.

PGI-mergeability in the quotient game. If two games $(N, W, P), (N, V, P) \in SIU(N)$ are mergeable in the quotient game, taking into account that the PGI satisfies the property of mergeability, it holds that for any player $i \in P_k$,

$$\Theta_i\left(N, W \vee V, P\right) = \delta_k\left(U, \overline{W \vee V}\right) \frac{1}{|P_k|}$$

$$= \frac{\sum_{k\in U}\left|M_k^{\overline{W}}\right|}{\sum_{k\in U}\left|M_k^{\overline{W\vee V}}\right|}\delta_k\left(U, \overline{W}\right)\frac{1}{|P_k|} + \frac{\sum_{k\in U}\left|M_k^{\overline{V}}\right|}{\sum_{k\in U}\left|M_k^{\overline{W\vee V}}\right|}\delta_k\left(U, \overline{V}\right)\frac{1}{|P_k|}$$

$$= \frac{\sum_{k\in U}\left|M_k^{\overline{W}}\right|}{\sum_{k\in U}\left|M_k^{\overline{W\vee V}}\right|}\Theta_i\left(N, W, P\right) + \frac{\sum_{k\in U}\left|M_k^{\overline{V}}\right|}{\sum_{k\in U}\left|M_k^{\overline{W\vee V}}\right|}\Theta_i\left(N, V, P\right).$$

Independence of superfluous coalitions. Suppose two games (N, W, P), $(N, W', P) \in SIU(N)$ satisfying the conditions of the property, then, the MWC of the corresponding quotient games $\left(U, \overline{W}\right)$ and $\left(U, \overline{W'}\right)$ are the same. Then, the PGI of the quotient game allocates the same quantity to each union. As the solidarity Public Good Index satisfies the properties of quotient game and solidarity, this implies that $\Theta_i(N, W, P) = \Theta_i(N, W', P)$ holds.

Unicity. Let us take a coalitional power index f which satisfies all the above properties. Let us take $(N, W, P) \in SIU(N)$ with $M^W = \{S_1, \ldots, S_l\}$. Since the power index f satisfies independence of superfluous coalitions we can assume that $u(S) \in M^{\overline{W}}$, for every $S \in M^W$ and for every $S, T \in M^W$, it holds that $u(S) \neq u\left(T\right).$

First, we assume that $l = 1$. In that case (N, W) is a unanimity game, for instance, (N, W_S) with $S \subset N$. Since solution f satisfies efficiency, solidarity, null union, and symmetry among unions, f assigns to a player $i \in P_k$

$$f_i\left(N, W_S, P\right) = \begin{cases} \frac{1}{|u(S)|}\frac{1}{|P_k|} & \text{if } S \cap P_k \neq \emptyset \\ 0 & \text{if } S \cap P_k = \emptyset \end{cases},$$

and then, f_i coincides with Θ_i for every $i \in N$, for every unanimity game.

Let us assume that $l > 1$. We know that $u\left(S_j\right) \neq u\left(S_p\right)$ if $j, p = 1, 2, \ldots, l$, $\left(j \neq p\right).$

Notice that the unanimity games $\left(N, W_{S_j}, P\right)$ and $\left(N, W_{S_p}, P\right)$ for $j, p = 1, 2, \ldots, l$, $(j \neq p)$ are mergeable in the quotient game. Then, by the property of mergeability in the quotient game, it holds that:

$$f_i\left(N, W, P\right) = \frac{\sum_{j=1}^{l}\sum_{k\in U}\left|M_k^{\overline{W}_{S_j}}\right|}{\sum_{k\in U}\left|M_k^{\overline{W}}\right|} f_i\left(N, W_{S_j}, P\right) = \Theta_i\left(N, W, P\right).$$

This finishes the proof. $\square$

4 The Owen-extended Public Good Index

In this section, we characterize an extension of the PGI similar to Owen´s elaboration of the Shapley value (Owen, 1977), and the Banzhaf value (Owen, 1982) and to Alonso-Meijide et al. (2008b) modification of the Deegan-Packel index.

We consider two levels of negotiation, (a) among unions, and (b) inside unions. In the process, a player $i \in P_k$ can collaborate with some players $S \subseteq P_k$ and/or with complete unions different of P_k. The potential of a player joining other unions is taking into account when we define this index.

Definition 6 *Given $(N, W, P) \in SIU(N)$, the Owen-extended Public Good Index of a player $i \in P_k$ is:*

$$\Gamma_i (N, W, P) = \frac{1}{\sum_{l \in U} \left| M_l^{\overline{W}} \right|} \sum_{R \in M_k^{\overline{W}}} \frac{\left| E_i^{k,R} (N, W, P) \right|}{\sum_{j \in P_k} \left| E_j^{k,R} (N, W, P) \right|}. \tag{4}$$

Next, we provide a characterization of this value. First, we recall the well-known properties of null player and symmetry inside unions of a coalitional power index.

Null Player. A coalitional power index f satisfies null player if and only if for every $(N, W, P) \in SIU(N)$ and $i \in N$ a null player in the game (N, W), then $f_i(N, W, P) = 0$.

Symmetry inside unions. A coalitional power index f satisfies symmetry inside unions if and only if for every $(N, W, P) \in SIU(N)$, and $i, j \in P_k$ symmetric players in the game (N, W), $f_i(N, W, P) = f_j(N, W, P)$.

The following property is another adaptation of the property of mergeability. In the characterization of the coalitional Deegan-Packel index in Alonso-Meijide et al. (2008b) a property with a similar flavour is used. First, we introduce the concept of mergeable game inside unions.

Mergeable games inside unions. Two simple games with a priori unions (N, V, P) and (N, W, P) are mergeable inside unions if:

- (N, W) and (N, V) are mergeable,
- there exists $k \in U$ such that for every $S \in M^W \cup M^V$ it holds $S \subset P_k$.

Notice that in such a case there is only one minimal winning coalition in the quotient game. The property of mergeability inside unions states that power in a merged game is a weighted mean of power of the two component games, with the sum of the number of MWC for every player of each component game providing the weights, when all the minimal winning coalitions of the merged games are included in the same a priori union.

PGI-Mergeability inside unions. A coalitional power index f satisfies mergeability inside unions if for any pair $(N, W, P), (N, V, P)$, of mergeable games inside unions, then

$$f(N, W \vee V, P) = \frac{\sum_{j\in N} \left|M_j^W\right|}{\sum_{j\in N} \left|M_j^{W\vee V}\right|} f(N, W, P) + \frac{\sum_{j\in N} \left|M_j^V\right|}{\sum_{j\in N} \left|M_j^{W\vee V}\right|} f(N, V, P).$$

The next two properties are also defined in Alonso-Meijide et al. (2008b).

Invariance with respect to essential subsets of a union says that in two simple games with a priori unions in which identical a priori unions are represented in all the MWC, the power of players does not vary if the sets of minimal winning correspond to an identical set of essential subsets of a union.

Invariance with respect to essential subsets of a union. A coalitional power index f satisfies invariance with respect to essential subsets of a union if for any pair $(N, W, P), (N, V, P)$ such that $E(N, W, P) = E(N, V, P)$, $u(S) = R$, for every $S \in M^W \cup M^V$, then $f(N, W, P) = f(N, V, P)$.

Independence of irrelevant coalitions says that elimination of irrelevant coalitions of the game, as defined above, will not change the power of the players.

Independence of irrelevant coalitions. A coalitional power index f satisfies independence of irrelevant coalitions if for any (N, W, P), given $S \in M^W$ such that $u(S) \notin M^{\overline{W}}$, then $f(N, W, P) = f(N, W', P)$ where $M^{W'} = M^W \setminus S$.

Without proof, we present a characterization for the Owen-extended PGI. The proof is very similar to that of the Theorem 1 in Alonso-Meijide et al. (2008b).

Theorem 7 *The Owen-extended PGI is the unique coalitional power index which satisfies the properties of efficiency, null player, symmetry inside unions, symmetry among unions, PGI-mergeability in the quotient game, PGI-mergeability inside unions, invariance with respect to essential subsets of a union, and independence of irrelevant coalitions.*

5 An Example

We compute the two coalitional versions of the PGI to analyze the Parliament of Catalonia which has been arisen from the elections held on November 1^{st}, 2006. This Parliament has also been studied in Carreras et al (2007). They used binomial semivalues to explain the behavior of one of the parties (ERC).

The Parliament of Catalonia consists of 135 members. Following these elections, the Parliament was composed of:

1. 48 members of CIU, *Convergéncia i Unió*, a Catalan nationalist middle-of-the-road party,

2. 37 members of PSC, *Partido de los Socialistas de Cataluña*, a moderate left-wing socialist party federated to the *Partido Socialista Obrero Español*,

3. 21 members of ERC, *Esquerra Republicana de Cataluña*, a radical Catalan nationalist left-wing party,

4. 14 members of PPC, *Partido Popular de Cataluña*, a conservative party which is a Catalan delegation of the *Partido Popular*,

5. 12 members of ICV, *Iniciativa por Cataluña-Los Verdes-Izquierda Alternativa*, a coalition of ecologist groups and Catalan eurocommunist parties federated to *Izquierda Unida*, and

6. 3 members of $C's$, *Ciudadanos-Partidos de la Ciudadanía*, a non-Catalanist party.

This Parliament can be represented as the following weighted majority game

$$v = [68; 48, 37, 21, 14, 12, 3].$$

For the sake of clarity we identify CIU as player 1, PSC as player 2, ERC as player 3, PPC as player 4, ICV as player 5 and $C's$ as player 6. Then, taking $N = \{1, 2, 3, 4, 5, 6\}$, the corresponding minimal winning coalitions are:

$$M^W = \{\{1, 2\}, \{1, 3\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}.$$

We see that members of the $C's$ party are null players. Two main aspects characterized politics in Catalonia: the Spanish centralism to Catalanism axis and left to right axis. Taking into account this fact we consider two possible partitions of players:

$$P^1 = \{\{1\}, \{2\}, \{3, 5\}, \{4\}, \{6\}\},$$

$$P^2 = \{\{1\}, \{2, 3, 5\}, \{4\}, \{6\}\}.$$

In Table 1, we present the PGI and the two variations of the coalitional PGI for the two partitions of players.

Party	Shares of seats	δ	$\Theta\left(P^1\right)$	$\Theta\left(P^2\right)$	$\Gamma\left(P^1\right)$	$\Gamma\left(P^2\right)$
CIU	0.3556	0.2308	0.3333	0	0.3333	0
PSC	0.2741	0.2308	0.3333	0.3333	0.3333	0.3333
ERC	0.1037	0.2308	0.1667	0.3333	0.25	0.3333
PPC	0.1156	0.1539	0	0	0	0
ICV	0.0889	0.1539	0.1667	0.3333	0.0834	0.3333
C's	0.0222	0	0	0	0	0

Table 1: Some power indices in the Catalonian Parliament November 2006.

P^1 represents the axis of Spanish centralism vs. Catalanism while P^2 represents the a priori unions that correspond to the left-right axis. The power values indicate that the alternative interpretation of a priori unions, as captured by Θ and Γ, matters. Moreover the focus on the Spanish centralism vs. Catalanism axis produces a larger diversity of power then the left-right axis. Perhaps this is the reason why this dimension is so prominent in the political discussion. Note also that the strongest party, CIU, has no power if the focus is on the left-right axis, irrespective of whether we apply Θ and Γ. This could be an argument why the axis of Spanish centralism vs. Catalanism is so popular. A comparison of δ with the Θ and Γ values shows that a priori union makes.

6 Remarks and conclusions

In Alonso-Meijide and Bowles (2005), new procedures based on generating functions are described to compute power indices for weighted majority games with a priori unions. Owen (1972) proposed the multilinear extension of games as a tool to compute the Shapley value. The multilinear extension has been used to compute the Banzhaf-Owen coalition value (Carreras and Magaña, 1994), the Owen coalition value (Owen and Winter, 1992) and the symmetric coalition Banzhaf value (Alonso-Meijide et al., 2005). These methods could be used to compute the two versions of the Public Good Index with a priori unions considered in this paper.

The Owen-extended Public Good Index coincides with the original PGI when the system of unions are the trivial ones, P^n and P^N. The solidarity Public Good Index coincides with the original PGI if $P = P^n$. However, if $P = P^N$, the solidarity Public Good Index coincides with the egalitarian solution $f_i(N, W, P) = 1/n$ for each $i \in N$.

This paper is part of ongoing research program that analyzes the properties of alternative power measures in order to give substantial characterizations of the measures and prepare for their applications. The underlying perspective is that there is no ‘right’ or ‘wrong’ measure: they are indicators, not predictors and as such they might be adequate or inadequate. The authors of this paper share Robert Aumann’s view that, in game theory, “different solution concepts are like different indicators of an economy; different methods for calculating a price index; different maps (road, topo, political, geologic, etc., not to speak of scale, projection, etc.); different stock indices (Dow Jones, ...). They depict or illuminate the situation from different angles; each one stresses certain aspects at the expense of others” (Aumann, 1977, p.464). However, to interpret the indicators and to apply them adequately, one has to know their properties. This, of course, is a major task, given the multitude of power measures, so far developed, and the large variation in the situations to which these measures are, or should be, applied. Moreover this program risks, like all successful research programs, the fate that no foreseeable end exists.

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