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The truncated core for games with limited aspirations*

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Abstract

We define and study games with limited aspirations. In a game with limited aspirations there are upper bounds on the possible payoffs for some coalitions. These restrictions require adjustments in the definitions of solution concepts. In the current paper we study the effect of the restrictions on the core and define and study the so-called truncated core.

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1 Introduction

We define and study games with limited aspirations. In a game with limited aspirations there are upper bounds on the possible payoffs for some coali-

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tions. These restrictions require adjustments in the definitions of solution concepts. In the current paper we study the effect of the restrictions on the core and define and study the so-called truncated core.

Games with limited aspirations derive from our earlier work on interval games. In Carpente et al. (2005) we argue that to desribe players' opportunities in strategic-form games, it is natural to associate so-called coalitional interval games with strategic-form games. Coalitional interval games arise when one cannot provide a sharp worth for each coalition of players but instead associates with each coalition a lower bound and an upper bound for its worth – leading to the definition of an interval associated with each coalition. Coalitional interval games were first introduced by Branzei et al. (2004) in the context of bankruptcy problems. Within this context, Branzei et al. (2003) define two possible extensions of the Shapley value. Carpente et al. (2005) provide axiomatic characterizations of the method that associates with each strategic-form game a coalitional interval game. However, to the best of our knowledge, little else is known about interval games.

The current paper is inspired by our desire to develop a better understanding of coalitional interval games. Our research program focusses on defining and studying solution concepts for these games. In the current paper we concentrate on the core. We do this in a slightly more general setting than that of interval games. Whereas in an interval game there is an upper bound to the interval for each coalition, we allow for the possibility that such upper bounds may exist only for some coalitions. Hence, we arrive at the definition of games with limited aspirations.

The organization of the paper is as follows. In Section 2 we formally define games with limited aspirations. In this section we also provide two examples of situations where games with limited aspirations arise, namely minimum cost spanning tree problems and queueing problems, and we derive the associated games with limited aspirations. In Section 3 we define a core concept for games with limited aspirations which we call the truncated core. We derive conditions for non-emptiness of the truncated core of a game with limited aspirations which we call t-balancedness, we study the relation between cores and truncated cores, and we provide an axiomatic characterization of the truncated core as a solution for games with limited aspirations.

2 Games with limited aspirations

We are interested in the effect that upper limits on payoffs for coalitions can have on solution concepts for coalitional games. To study this issue, we define games with limited aspirations. A *game with limited aspirations* is a 4-tuple $(N, \mathcal{F}, v, a)$, where (N, v) is a coalitional game, $\mathcal{F}$ is a collection of coalitions, and a is a map from $\mathcal{F}$ to $\mathbb{R}$. The coalitional game (N, v) with set of players N provides for every coalition $S \subseteq \dot{N}$ the value or worth $v(S) \in \mathbb{R}$ that the members of the coaliton can jointly obtain if they coordinate their actions. We make the usual assumption that $v(\emptyset) = 0$. Unlike ordinary coalitional games, however, in a game with limited aspirations, for some coalitions there is an upper bound on how much the players in the coalition can jointly receive in payouts. $\mathcal{F} \subseteq 2^N$ denotes the set of such coalitions and for each $S \in \mathcal{F}$ the upper bound on joint payouts is $a(S)$. In line with the interpretations of v and a, we assume that $a(S) \geq v(S)$ for all $S \in \mathcal{F}$. Also, whenever convenient to minimize notation, we will use $a(\varnothing) = 0$.

We denote by $GA(N)$ the set of games with limited aspirations with player set N and by GA the set of all games with limited aspirations with a finite player set. As the player set N can be backed out of the characteristic function v and the collection of coalitions $\mathcal{F}$ can be backed out of the function a, we identify $(N, \mathcal{F}, v, a)$ with (v, a) whenever there is no need to stress the sets N and $\mathcal{F}$.

We define games with limited aspirations in a general manner, However, for most of the situations with which we associate games with limited aspirations, the associated games satisfy $\mathcal{F} = 2^N \setminus \emptyset$ (there is an upper bound on payouts for every non-empty coalition) and the corresponding bounds satisfy a subadditivity condition.[1]

[1]The subadditivity condition is $a(S) + a(T) \geq a(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$.

2.1 Examples of games with limited aspirations

Minimum cost spanning tree problems and queueing problems are two examples of situations that naturally give rise to games with limited aspirations.

2.1.1 Games with limited aspirations in minimum cost spanning tree problems

Consider a source node 0 and set of agents $N = \{1, \ldots, n\}$ who all need to get connected to the source node, directly or indirectly, to get access to its resources. An example is a trunk line for electricity to which agents need to connect to get power. In the basic model, we assume that there is no loss of the resource due to transportation and each agent i who is connected to the source node receives a benefit b_i. Denote the vector of benefits by $b = (b_i)_{i \in N}$. Of course, there are costs associated with the various possible connections. These costs are reflected in a cost matrix $C = (c_{ij})_{i,j \in N_0}$, where $N_0 = N \cup \{0\}$ and $c_{ij} \in \mathbb{R}$ represents the cost of a connection between i and j for all pairs $i, j \in N_0$. Connections are undirected and costly, so that $c_{ij} = c_{ji} > 0$ for each pair $i, j \in N_0$. It is assumed that the benefits of being connected are large compared to the costs so that each agent finds it worth while to pay the cost of getting connected.[2] The triple (N_0, b, C) is a *minimum cost spanning tree problem.* For brevity, we refer to such a problem as *mcstp.*

In a *mcstp*, when the agents build connections, a network g on N_0 results, where $g \subseteq \{\{i,j\} \mid i, j \in N_0, i \neq j\}$. The elements of g are called links or connections and the cost associated with a network g, $c(g)$, is the sum of the costs of its links:

$$c(g) = \sum_{\{i,j\} \in g} c_{ij}.$$

The objective of the agents is to build a network that connects them to the source in the least costly way possible. As there is no loss of the resource due

If $\mathcal{F} = 2^N \setminus \emptyset$ and we define $a(\emptyset) = 0$, then the subadditivity condition is equivalent to subadditivity of the coalitional game (N, a).

[2]For example, $b_k > \sum_{i,j \in N_0} c_{ij}$ for each $k \in N$ would be a bound that gives this result, although smaller benefits will be sufficient in most cases.

to transportation, agents can get the resource by connecting to other agents that are already connected to the source. In general, a network g on a set of nodes M is said to *connect* the nodes in M if for every two nodes $i, j \in M$ there exists a *path* in the network from node i to node j, i.e., a sequence of links $\{i_0, i_1\}, \ldots, \{i_{l-1}, i_l\}$ in g with $i_0 = i$ and $i_l = j$. The number of links in the path (l) is called the *length* of the path. A least costly network to connect a set of nodes will be a network that has no cycles – paths of length at least 3 that begin and end in the same node and do not use the same link more than once. Such a network is known as a *tree* on the set of nodes. In a tree there exists a unique path between any two of its nodes.

Consider a *mcstp* (N_0, b, C) and denote the set of networks that connect the nodes in N_0 by $\mathcal{G}(N)$. Hence, $\mathcal{G}(N)$ is the set of all networks $g \subseteq \{\{i, j\} : i, j \in N_0, i \neq j\}$ with the property that there exists a path in g between any two nodes in N_0. A minimum cost spanning tree (or *mcst*) for N is a network $g_N \in \mathcal{G}(N)$ with the property that

$$c(g_N) = min_{g \in \mathcal{G}(N)} c(g).^3$$

It is well-known that a (not necessarily unique) *mcst* exists.[4] We denote the cost of a *mcst* for N in the *mcstp* (N_0, b, C) by $m(N, C) = min_{g \in \mathcal{G}(N)} c(g)$.

Hence, the total cost that the agents in N have to incur to get connected to the source node is $m(N, C)$. This cost burden needs to be distributed among the individual agents in some way. An approach to this problem that is often encountered in the literature uses coalitional games. In a first step a coalitional game is associated with the *mcstp* and then a solution concept for coalitional games, e.g. the Shapley value, is applied to obtain a distribution of the costs.

Bird (1976) formulated a coalitional cost game c associated with a *mcstp* that assigns to each coalition its stand-alone cost. We will look at the situation from the perspective of revenues and assign to each coalition it's mem-

[3] It follows from earlier remarks that a minimum cost network that connects the players in N to the source is a tree.

[4] Prim (1957) introduced an algorithm for constructing a *mcst* for any set of nodes and corresponding costs for links.

bers benefits from being connected to the source minus the stand-alone costs. For each coalition $S \subseteq N$, consider the *mcstp* (S_0, b, C) on S induced by (N_0, b, C) and define the stand-alone cost of coalition S by $c(S) = m(S, C)$ and the coalition's revenue by

$$v(S) = \sum_{i \in S} b_i - c(S).$$

Hence, the cost of a coalition S is defined to be the minimal cost of connecting all agents in S to the source node, assuming that no indirect connections through agents not in S can be used. This is a pessimistic approach because agents in $N \setminus S$ want to be connected to the source node as well, so it would seem reasonable to assume those agents would not object if players in S want to connect through them as long as the agents in S bear the cost. This could enable the members of S to connect to the source node for lower costs and therefore raise the coalition's revenue.

Bergantiños and Vidal-Puga (2007) take an optimistic point of view by looking at coalitions' marginal costs. Let $S \subseteq N$ and suppose that a network $g_{N \setminus S} \in \mathcal{G}(N \setminus S)$ has already been formed that connects the nodes in $N \setminus S$ to the source node 0.[5] The marginal cost $\widetilde{c}(S)$ of the agents in S equals

$$\widetilde{c}(S) = min_{g \in \mathcal{G}(N): g_{N \setminus S} \subseteq g} \left(c(g) - c(g_{N \setminus S}) \right)$$

and the corresponding revenue of coalition S is

$$\widetilde{v}(S) = \sum_{i \in S} b_i - \widetilde{c}(S).$$

The drawback of this optimistic approach is that the total cost of connecting all agents to the source node may exceed the sum of their marginal costs, as would be the case if all links between an agent node and the source node are much more expensive than links between agent nodes. However, it seems reasonable to impose that each coalition of agents pays at least its marginal cost, so that a coalition S is not subsidized by the other agents.

[5]Note that to determine the marginal cost of coalition S it does not matter which network $g_{N \setminus S}$ exists that connects the players in $N \setminus S$ to the source node.

Games with limited aspirations give us the possibility to take these considerations on board. We associate with a *mcstp* (N_0, b, C) a game with limited aspirations (v, a), where v is the revenue game based on Bird's (1976) pessimistic cost game, and a gives upper bounds on revenues as implied by the optimistic cost $\widetilde{c}\,(S)$, i.e., $a\,(S) = \widetilde{v}\,(S)$ for each coalition $S \subseteq N$. Note that this satisfies the requirement $a(S) \geq v(S)$ for all S as a coalition's marginal cost can never exceed its stand-alone cost.

2.1.2 Games with limited aspirations in queueing problems

Consider a set of agents $N = \{1, \ldots, n\}$ who all need to be served by a server that processes agents' jobs in succession. Examples of such situations are abundant, such as people queueing to obtain new driver's licences, queues at immigration offices, queues to purchase tickets to concerts, etc. Each agent $i \in N$ has a disutility of waiting equal to $\theta_i \geq 0$ per unit of time spent waiting. The waiting time of an agent depends on his position in the queue and the time it takes to serve those in front of him. In the basic model, we assume that each agent takes exactly one period of time to get served and that an agent i receives a benefit b_i from being served that is high enough such that each agent finds it worth waiting as long as is necessary to get served[6] (queueing in immigration offices seems a good example here, as I am sure those of you who have ever tried to obtain a visa would agree). Therefore, if agents' positions in the queue are given by the permutation $\sigma : N \longrightarrow N$, then agent $i \in N$ has a position $\sigma(i)$ in the queue and his disutility of waiting equals $(\sigma(i) - 1)\,\theta_i$ as he needs to wait for the $\sigma(i) - 1$ agents ahead of him in the queue to be served. The triple (N, b, θ), where N is the set of agents in the queue, $\theta = (\theta_i)_{i \in N}$ is the vector of disutilities or costs, and $b = (b_i)_{i \in N}$ is the vector of benefits, is a *queueing problem.*

The disutility of waiting is a cost and agents can be compensated for waiting by payments. If an agent i receives a transfer t_i and is $\sigma(i)^{th}$ to be served, then his total benefit equals $b_i - (\sigma(i) - 1)\,\theta_i + t_i$. If agents differ in their disutility of waiting, it is beneficial for an agent with a higher disutility

[6] Technically, this condition is $b_i > \sum_{j \in N \setminus \{i\}} \theta_j$ for each $i \in N$.

of waiting to pay an agent with a lower disutility of waiting who is ahead of him in the queue to switch positions. If $\theta_i > \theta_j$ and $\sigma(i) = \sigma(j) + 1$, then both agents will be better off if agents i and j switch positions and agent i pays agent j a transfer $t_i \in (\theta_j, \theta_i)$.

Therefore, it is optimal for the agents to arrange themselves into a queue where an agent with a higher disutility of waiting is in front of one with a lower disutility of waiting. Denoting the set of all permutations of N by $\mathcal{P}(N)$ and the total cost associated with a permutation σ by

$$c(N, \sigma) = \sum_{i \in N} (\sigma(i) - 1)\, \theta_i,$$ [7]

we say that a permutation is *efficient* for queueing problem (N, b, θ) if its total cost is lowest among that of all possible permutations. Hence, the total cost (i.e. disutility of waiting) that the agents in N have to incur to all get served is $m(N, \theta) = min_{\sigma \in \mathcal{P}(N)} c(N, \sigma)$. However, the waiting time will be distributed unevenly among the agents; the first agent in line has no waiting time and the last agent in line has the maximum possible waiting time. Hence, in order to get all agents to agree on forming an efficient queue, the agents need to determine appropriate transfers between themselves that compensate the agents that need to wait longer. Coalitional games have been used in the literature to approach this question.

Chun (2004) takes a pessimistic point of view of a queueing problem (N, b, θ) by considering a coalition's cost if they have to let the other agents go first without being compensated for waiting for them to be served. Let $S \subseteq N$, denote the number of agents in S by s, and suppose that the agents in $N \backslash S$ are arranged in some order $\sigma_{N \backslash S} : N \backslash S \longrightarrow \{1, \ldots, n - s\}$.[8] The cost $c(S)$ of the players in S is defined by[9]

$$c\,(S) = min_{\sigma \in \mathcal{P}(N) : \sigma_{N \backslash S} \subseteq \sigma} \sum_{i \in S} (\sigma(i) - 1)\, \theta_i$$

[7] Note that transfers between agents do not play a role in determining the total cost as they are budget neutral.

[8] Technically, $\sigma_{N \backslash S}$ is a bijection between $N \backslash S$ and $\{1, \ldots, n - s\}$.

[9] Formally, $\sigma_{N \backslash S} \subseteq \sigma$ can be described as $\sigma(i) = \sigma_{N \backslash S}(i)$ for each $i \in N \backslash S$.

and the revenue of the coalition is

$$v(S) = \sum_{i \in S} b_i - c\,(S)\,.$$

Maniquet (2003) formulated a coalitional cost game $\widetilde{c}$ associated with a queueing problem (N, b, θ) that assigns to each coalition its stand-alone cost – the cost it would incur if its members are the only agents to be served. For each coalition $S \subseteq N$, consider the queuing problem (S, b, θ) on S induced by (N, b, θ) and define $\widetilde{c}(S) = m(S, \theta)$ and the corresponding revenues by

$$\widetilde{v}(S) = \sum_{i \in S} b_i - \widetilde{c}(S).$$

Hence, the cost of a coalition S is defined to be the minimal cost of the agents in S being served without them waiting for or compensating the agents in $N \backslash S$. This is, of course, a very optimistic estimate of the cost to the agents in S. The drawback of this optimistic approach is that the cost of any coalition of agents that consists of more than one agent exceeds the sum of the costs of its individual members. Hence, the costs according to $\widetilde{c}$ do not go far toward finding a reasonable solution to the queueing problem but they do provide a lower bound.

Games with limited aspirations give us the possibility to take these considerations on board. We associate with a queueing problem (N, b, θ) a game with limited aspirations (v, a) where v is the revenue game based on Chun's (2004) pessimistic cost game as defined above and a gives the upper bounds on revenues as implied by Maniquet's (2003) optimistic cost game, i.e. $a\,(S) = \widetilde{v}\,(S)$ for each coalition $S \subseteq N$. Note that this satisfies the requirement $a(S) \geq v(S)$ for all S.

3 The truncated core

In this section, we extend the core concept to games with limited aspirations and define a truncated core that takes the upper bounds on payoffs into account. We define conditions on games with limited aspirations that are necessary for the existence of allocations in the truncated core. We also investigate when the core and the truncated core coincide.

The *core* of a game with limited aspirations $(N, \mathcal{F}, v, a)$ is $C(v) = \{x \in \mathbb{R}^N \mid x(S) \geq v(S)$ for all $S \subseteq N$ and $x(N) = v(N)\}$.[10] Note that the aspirations do not play a role in the determination of the core of a game. When taking the aspirations into account, we arrive at the defintion of the truncated core. The *truncated core* of the game with limited aspirations is

$$C^T(v,a) = \{x \in \mathbb{R}^N \mid x(S) \geq v(S) \text{ for all } S \subseteq N,$$

$$x(S) \leq a(S) \text{ for all } S \in \mathcal{F}, \text{ and } x(N) = v(N)\}.$$

The truncated core can also be expressed as

$$C^T(v,a) = \{x \in C(v) \mid x(S) \leq a(S) \text{ for all } S \in \mathcal{F}\}.$$

Obviously, $C^T(v,a) \subseteq C(v)$ for all games with limited aspirations $(N, \mathcal{F}, v, a)$.

3.1 t-Balancedness

In this subsection we present a necessary and sufficient condition for the non-emptiness of the truncated core of a game with limited aspirations.

Let $(N, \mathcal{F}, v, a) \in GA(N)$ be a game with limited aspirations. A family of coalitions $\mathcal{G} \subseteq 2^N$ is said to be *t-balanced* if there exist non-negative coefficients $\{y_S \mid S \in \mathcal{G}\}$ and $\{w_S \mid S \in \mathcal{G} \cap \mathcal{F}\}$, satisfying the following requirements

1. $y_S > 0$ for all $S \in \mathcal{G} \setminus \mathcal{F}$
2. $\min\{y_S, w_S\} = 0$ and $\max\{y_S, w_S\} > 0$ for all $S \in \mathcal{G} \cap \mathcal{F}$
3. $\sum_{S \in \mathcal{G}, i \in S} y_S - \sum_{S \in \mathcal{G} \cap \mathcal{F}, i \in S} w_S = 1$ for all $i \in N$.

If $\mathcal{G}$ is a *t-balanced* collection of coalitions then coefficients $\{y_S \mid S \in \mathcal{G}\}$ and $\{w_S \mid S \in \mathcal{G} \cap \mathcal{F}\}$ satisfying the above requirements are called *t-balancing coefficients*. Note that condition 2 means that for every $S \in \mathcal{G} \cap \mathcal{F}$ either $y_S > 0$ or $w_S > 0$ but not both.

[10] Throughout the paper, we use $x(S) = \sum_{i \in S} x_i$.

A game with limited aspirations $(N, \mathcal{F}, v, a)$ is *t-balanced* if for every t-balanced family of coalitions $\mathcal{G}$ with t-balancing coefficients $\{y_S \mid S \in \mathcal{G}\}$ and $\{w_S \mid S \in \mathcal{G} \cap \mathcal{F}\}$ it holds that

$$\sum_{S \in \mathcal{G}} y_S v(S) - \sum_{S \in \mathcal{G} \cap \mathcal{F}} w_S a(S) \leq v(N).$$

In this inequality, due to condition 2 for t-balancing coefficients, for each $S \in \mathcal{G} \cap \mathcal{F}$ either $y_S > 0$ and $v(S)$ appears with a positive weight, or $w_S > 0$ and $a(S)$ appears with a negative weight, but not both.

t-Balancedness is a necessary and sufficient condition for non-emptiness of the truncated core. To prove this result, we first prove a lemma stating a relation between t-balancedness and a linear program. Let $(N, \mathcal{F}, v, a) \in GA$ be a game with limited aspirations. Define the associated linear program $P(v, a)$ by

$$\begin{array}{ll} \text{maximize} & \sum_{S \subseteq N} y_S v(S) - \sum_{S \in \mathcal{F}} w_S a(S) \\ \text{subject to} & \sum_{S \subseteq N, i \in S} y_S - \sum_{S \in \mathcal{F}, i \in S} w_S = 1,\ i \in N, \\ & y_S \geq 0,\ S \subseteq N, \\ & w_S \geq 0,\ S \in \mathcal{F}. \end{array}$$

The game $(N, \mathcal{F}, v, a)$ is t-balanced if ond only if the value $v_{P(v,a)}$ of the linear program $P(v, a)$ is equal to $v(N)$.

Proof. It is obvious that the game $(N, \mathcal{F}, v, a) \in GA$ is t-balanced if $v_{P(v,a)} = v(N)$.

Now, suppose that the game $(N, \mathcal{F}, v, a) \in GA$ is t-balanced. Choosing $y_N = 1$ and $y_S = 0$ for all $S \subset N$, $S \neq N$, and $w_S = 0$ for all $S \in \mathcal{F}$, it is clear that $v_{P(v,a)} \geq v(N)$. To show that $v_{P(v,a)} \leq v(N)$, choose y_S and w_S that satisfy the constraints of the linear program $P(v, a)$, i.e., $y_S \geq 0$, $S \subseteq N$, and $w_S \geq 0$, $S \in \mathcal{F}$, and $\sum_{S \subseteq N, i \in S} y_S - \sum_{S \in \mathcal{F}, i \in S} w_S = 1$ for all $i \in N$. Define $\mathcal{G} = \{S \subseteq N \mid S \notin \mathcal{F} \text{ and } y_S \neq 0\} \cup \{S \subseteq N \mid S \in \mathcal{F} \text{ and } y_S \neq w_S\}$. If $S \in \mathcal{G} \cap \mathcal{F}$, we define $\widetilde{y}_S = y_S - \min\{y_S, w_S\}$ and $\widetilde{w}_S = w_S - \min\{y_S, w_S\}$. This assures that $\min\{\widetilde{y}_S, \widetilde{w}_S\} = 0$ and $\max\{\widetilde{y}_S, \widetilde{w}_S\} > 0$ for all $S \in \mathcal{G} \cap \mathcal{F}$. To simplify notation, we define $\widetilde{y}_S = y_S$ for all $S \in \mathcal{G} \setminus \mathcal{F}$. Then, clearly,

$\widetilde{y}_S > 0$ for all $S \in \mathcal{G} \setminus \mathcal{F}$. Also, for each $i \in N$, it holds that

$$\sum_{S\in\mathcal{G},i\in S} \widetilde{y}_S - \sum_{S\in\mathcal{G}\cap\mathcal{F},i\in S} \widetilde{w}_S$$

$$= \sum_{S\in\mathcal{G}\setminus\mathcal{F},i\in S} \widetilde{y}_S + \sum_{S\in\mathcal{G}\cap\mathcal{F},i\in S} (\widetilde{y}_S - \widetilde{w}_S)$$

$$= \sum_{S\in\mathcal{G}\setminus\mathcal{F},i\in S} y_S + \sum_{S\in\mathcal{G}\cap\mathcal{F},i\in S} (y_S - w_S)$$

$$= \sum_{S\subseteq N,S\notin\mathcal{F},i\in S} y_S + \sum_{S\subseteq N,S\in\mathcal{F},i\in S} (y_S - w_S)$$

$$= \sum_{S\subseteq N,i\in S} y_S - \sum_{S\in\mathcal{F},i\in S} w_S = 1.$$

This shows that the family of coalitions $\mathcal{G} \subseteq 2^N$ is t-balanced with t-balancing coefficients $\{\widetilde{y}_S \mid S \in \mathcal{G}\}$ and $\{\widetilde{w}_S \mid S \in \mathcal{G} \cap \mathcal{F}\}$. Because $(N, \mathcal{F}, v, a)$ is t-balanced (by assumption), it holds that $\sum_{S\in\mathcal{G}} \widetilde{y}_S v(S) - \sum_{S\in\mathcal{G}\cap\mathcal{F}} \widetilde{w}_S a(S) \leq v(N)$. From this we derive

$$\sum_{S\subseteq N} y_S v(S) - \sum_{S\in\mathcal{F}} w_S a(S)$$

$$= \sum_{S\subseteq N,S\notin\mathcal{F}} y_S v(S) + \sum_{S\in\mathcal{G}\cap\mathcal{F}} (y_S v(S) - w_S a(S)) + \sum_{S\in\mathcal{F}\setminus\mathcal{G}} (y_S v(S) - w_S a(S))$$

$$\leq \sum_{S\subseteq N,S\notin\mathcal{F}} y_S v(S) + \sum_{S\in\mathcal{G}\cap\mathcal{F}} (y_S v(S) - w_S a(S))$$

$$\leq \sum_{S\subseteq N,S\notin\mathcal{F}} y_S v(S) + \sum_{S\in\mathcal{G}\cap\mathcal{F}} (y_S v(S) - w_S a(S))$$

$$- \sum_{S\in\mathcal{G}\cap\mathcal{F}} \min\{y_S, w_S\} \left(v(S) - a(S)\right)$$

$$= \sum_{S\in\mathcal{G}\setminus\mathcal{F}} \widetilde{y}_S v(S) + \sum_{S\in\mathcal{G}\cap\mathcal{F}} (\widetilde{y}_S v(S) - \widetilde{w}_S a(S))$$

$$= \sum_{S\in\mathcal{G}} \widetilde{y}_S v(S) - \sum_{S\in\mathcal{G}\cap\mathcal{F}} \widetilde{w}_S a(S) \leq v(N),$$

where the first inequality follows from the fact that $y_S = w_S$ and $v(S) \leq a(S)$ for all $S \in \mathcal{F} \setminus \mathcal{G}$, the second inequality follows from the fact that $\min\{y_S, w_S\} \geq 0$ and $v(S) \leq a(S)$ for all $S \in \mathcal{G} \cap \mathcal{F}$, and the second equality uses that $y_S = 0$ if $S \notin \mathcal{F}$ and $S \notin \mathcal{G}$, while $\widetilde{y}_S = y_S$ if $S \in \mathcal{G} \setminus \mathcal{F}$.

This shows that $v_{P(v,a)} \leq v(N)$. □

A game with limited aspirations $(N, \mathcal{F}, v, a)$ is t-balanced if and only if its truncated core $C^T(v, a)$ is non-empty.

Proof. *Necessity* Let $(N, \mathcal{F}, v, a) \in GA$ such that $C^T(v, a) \neq \varnothing$ and take $x \in C^T(v, a)$. Let $\mathcal{G}$ be a t-balanced family of coalitions with t-balancing coefficients $\{y_S \mid S \in \mathcal{G}\}$ and $\{w_S \mid S \in \mathcal{G} \cap \mathcal{F}\}$. Then

$$\sum_{S \in \mathcal{G}} y_S v(S) - \sum_{S \in \mathcal{G} \cap \mathcal{F}} w_S a(S) \leq \sum_{S \in \mathcal{G}} y_S x(S) - \sum_{S \in \mathcal{G} \cap \mathcal{F}} w_S x(S)$$

$$= \sum_{i \in N} \left(x_i \sum_{S \in \mathcal{G}, i \in S} y_S \right) - \sum_{i \in N} \left(x_i \sum_{S \in \mathcal{G} \cap \mathcal{F}, i \in S} w_S \right) = \sum_{i \in N} x_i = v(N),$$

where the inequality and the last equality follow from the conditions for the truncated core, and the second equality follows from the third condition for the t-balancing coefficients. This shows that $(N, \mathcal{F}, v, a)$ is t-balanced.

Sufficiency Let $(N, \mathcal{F}, v, a) \in GA$ be t-balanced. By Lemma 3.1, this implies that $v_{P(v,a)} = v(N)$. This, in turn, is equivalent to the statement that the value of the dual program of $P(v, a)$ equals $v(N)$. This dual program $DP\,(v, a)$ is given by

$$\begin{array}{ll} \text{minimize} & \sum_{i \in N} x_i \\ \text{subject to} & \sum_{i \in S} x_i \geq v(S),\ S \subseteq N, \\ & \sum_{i \in S} x_i \leq a(S),\ S \in \mathcal{F}. \end{array}$$

Therefore, the statement $v_{DP(v,a)} = v(N)$ implies that there exists a $x \in \mathbb{R}^N$ such that $x(S) \geq v(S)$ for all $S \subseteq N$, $x(S) \leq a(S)$ for all $S \in \mathcal{F}$, and $x(N) = v(N)$. Hence, $C^T(v, a) \neq \varnothing$. □

3.2 Cores and truncated cores

In this subsection we explore conditions under which the truncated core of a game with limited aspirations is equal to its core. As $C^T\,(v, a) \subseteq C(v)$

for all games with limited aspirations $(N, \mathcal{F}, v, a)$, it follows trivially that $C^T(v, a) = C(v)$ when $C(v) = \varnothing$. Hence, we can restrict our attention to considering games for which the core is not empty and in the remainder of this subsection we assume that the game with limited aspirations is balanced, i.e. the game (N, v) is balanced.

3.2.1 A sufficient condition

A sufficient condition for the truncated core and the core of a game to coincide is $a(S) \geq v(N) - v(N \backslash S)$ for all $S \in \mathcal{F}$.

Let $(N, \mathcal{F}, v, a)$ be a game with limited aspirations with the property that $a(S) \geq v(N) - v(N \backslash S)$ for all $S \in \mathcal{F}$. Then $C^T(v, a) = C(v)$.

Proof. In light of $C^T(v, a) = \{x \in C(v) \mid x(S) \leq a(S) \text{ for all } S \in \mathcal{F}\}$, we only need to show that for every $x \in C(v)$ it holds that $x(S) \leq a(S)$ for all $S \in \mathcal{F}$. Let $x \in C(v)$ and $S \in \mathcal{F}$. Then $x(S) = x(N) - x(N \backslash S) = v(N) - x(N \backslash S) \leq v(N) - v(N \backslash S) \leq a(S)$, where the first inequality follows from the fact that $x \in C(v)$ and the last one follows from the condition in the statement of the theorem. $\square$

The condition $a(S) \geq v(N) - v(N \backslash S)$ for all $S \in \mathcal{F}$ that appears in Theorem 3.2.1 is not necessary for $C^T(v, a) = C(v)$. This is illustrated in the following example.

Consider a 3-player game with limited aspirations. $N = \{1, 2, 3\}$. Revenues are given by $v(S) = 1$ if and only if both of the conditions $S \cap \{2, 3\} \neq \varnothing$ and $1 \in S$ hold, and $v(S) = 0$ for other $S \subseteq N$. Aspirations are given by $a(S) = 1$ if $1 \in S$ and $a(S) = 0$ if $1 \notin S$. For this game, the core has exactly one element, namely $x = (1, 0, 0)$, where player 1 (who is needed to get any positive revenue at all) gets the revenue of the grand coalition. Note that $x \in C^T(v, a)$ as well, because $x(S) > 0$ only if $1 \in S$. This shows that $C(v) = C^T(v, a)$. However, $a(2, 3) = 0 < 1 = v(N) - v(1)$.

3.2.2 Necessary and sufficient conditions

In this subsection, we provide necessary and sufficient conditions for the truncated core and the core of a game to coincide.

First we show an intermediate result. Let (N, v) be a balanced game. Define a game (N, w^v) by $w^v(S) = \min_{x \in C(v)} x(S)$ for each $S \subseteq N$. Then the core of the game (N, w^v) coincides with that of the game (N, v).

Let $(N, \mathcal{F}, v, a)$ be a balanced game with limited aspirations. Then $C(v) = C(w^v)$.

Proof. First we observe that $w^v(N) = \min_{x \in C(v)} x(N) = v(N)$, because $x(N) = v(N)$ for all $x \in C(v)$.

Now, let $x \in C(v)$. Then $x(N) = v(N) = w^v(N)$ and for all $S \subseteq N$ it holds that $x(S) \geq \min_{y \in C(v)} y(S) = w^v(S)$. We conclude that $x \in C(w^v)$. This proves that $C(v) \subseteq C(w^v)$.

To prove that $C(w^v) \subseteq C(v)$, let $x \in C(w^v)$. Then $x(N) = w^v(N) = v(N)$. Also, for each $S \subseteq N$ it holds that $x(S) \geq w^v(S) = \min_{y \in C(v)} y(S) \geq v(S)$, where the last inequality follows from the fact that $y(S) \geq v(S)$ for all $y \in C(v)$. We conclude that $x \in C(v)$. □

It is clear that we cannot increase the revenue of any coalition S above $w^v(S)$ without losing the property reflected in the previous theorem. In this sense, then, (N, w^v) is the largest game that has the same core as the game (N, v). The idea behind this result is that if we increase the revenues of coalitions just up to the level where the core constraint for the coalition would become binding for at least one core element, then we would not affect the core.

This result inspires a constraint on aspirations that guarantees equality between the core and the truncated core of a game.

Let $(N, \mathcal{F}, v, a)$ be a balanced game with limited aspirations. Then $C^T(v, a) = C(v)$ if and only if the aspirations satisfy the property that $a(S) \geq \max_{x \in C(v)} x(S)$ for each $S \in \mathcal{F}$.

Proof. We first prove the 'if' part. Suppose that $a(S) \geq \max_{x \in C(v)} x(S)$ for each $S \in \mathcal{F}$. In light of $C^T(v, a) = \{x \in C(v) \mid x(S) \leq a(S) \text{ for all } S \in \mathcal{F}\}$, we only need to show that for every $x \in C(v)$ it holds that $x(S) \leq a(S)$ for all $S \in \mathcal{F}$. This, however, follows directly from $a(S) \geq \max_{x \in C(v)} x(S)$ for each $S \in \mathcal{F}$.

We proceed with the proof of the 'only if' part. Suppose that $C^T(v, a) =$

$C(v)$. Let $S \in \mathcal{F}$. Then for all $x \in C(v)$ it holds that $x \in C^T(v, a)$ and $x(S) \leq a(S)$. Hence, $a(S) \geq \max_{x \in C(v)} x(S)$. □

We can use the bounds on aspirations in the previous theorem to obtain a result similar to that which we obtained in Theorem 3.2.2.

Take a balanced game with limited aspirations $(N, \mathcal{F}, v, a)$. Define a game $(N, \mathcal{F}, v, a^v)$ by changing the aspirations to $a^v(S) = \max_{x \in C(v)} x(S)$ for each $S \in \mathcal{F}$.

Let $(N, \mathcal{F}, v, a)$ be a balanced game with limited aspirations. Then $C^T(v, a^v) = C(v)$.

Proof. The result follows immediately by applying Theorem 3.2.2. □

It follows from Theorem 3.2.2 that if $C^T(v, a) = C(v)$, then

$$a(S) \geq \max_{x \in C(v)} x(S)$$

for each $S \in \mathcal{F}$. Therefore, for a balanced game with limited aspirations such that $C^T(v, a) = C(v)$, changing the aspirations from a to a^v represents a (weak) lowering of the aspirations. It is clear that we cannot decrease the aspiration of any coalition $S \in \mathcal{F}$ below $a^v(S)$ without losing the property reflected in the previous theorem. Example 3.2.1 demonstrates this, as $C^T(v, a)$ would be empty in that example if we chose $a(1) < 1$. In this sense, then, the aspirations a^v are the lowest aspirations that have no impact on the core of the game (N, v). The idea behind this result is that if we add aspirations of coalitions that are just low enough so that they are binding for at least one core element, then we would not affect the core.

Stringing previous results together, we derive that $C^T(v, a^v) = C(v) \supseteq C^T(v, a)$. It is possible that $C^T(v, a^v) \neq C^T(v, a)$ when $C(v) \neq C^T(v, a)$ which can only happen for a balanced game with limited aspirations $(N, \mathcal{F}, v, a)$ for which the aspiration of one or more $S \in \mathcal{F}$ is too low, specifically $a(S) < \max_{x \in C(v)} x(S)$.

The lowest aspirations that do not affect the truncated core are given by $b^{v,a}(S) = \min\{a(S), \max_{x \in C(v)} x(S)\} = \min\{a(S), a^v(S)\}$ for each $S \in \mathcal{F}$.

Let $(N, \mathcal{F}, v, a)$ be a balanced game with limited aspirations. Then $C^T(v, b^{v,a}) = C^T(v, a)$.

Proof. Note that $C^T(v,a) = \{x \in C(v) \mid x(S) \leq a(S) \text{ for all } S \in \mathcal{F}\}$ and $C^T(v, b^{v,a}) = \{x \in C(v) \mid x(S) \leq b^{v,a}(S) \text{ for all } S \in \mathcal{F}\}$. Hence, we only need to show that for every $x \in C(v)$ it holds that $x(S) \leq a(S)$ for all $S \in \mathcal{F}$ if and only if $x(S) \leq b^{v,a}(S)$ for all $S \in \mathcal{F}$.

The implication one way, if $x(S) \leq b^{v,a}(S)$ then $x(S) \leq a(S)$, follows immediately from $b^{v,a}(S) \leq a(S)$ for each $S \in \mathcal{F}$.

To prove the other implication, let $x \in C(v)$ and $S \in \mathcal{F}$ and assume that $x(S) \leq a(S)$. Then $x(S) \leq \max_{x \in C(v)} x(S)$ and combining this with $x(S) \leq a(S)$ leads to $x(S) \leq b^{v,a}(S)$. □

Note that for a balanced game with limited aspirations changing the aspirations from a to $b^{v,a}$ represents a (weak) lowering of the aspirations. It is clear that we cannot decrease the aspiration of any coalition $S \in \mathcal{F}$ below $b^{v,a}(S)$ without losing the equality $C^T(v, b^{v,a}) = C^T(v,a)$.

3.3 Axiomatic characterization

In this subsection we provide an axiomatic characterization of the truncated core. To this end, we introduce several properties of solutions on the class of games with limited aspirations.

Formally, a *solution* on a set A of games with limited aspirations is a correspondence f defined on $A \subset GA$ that assigns to each game with limited aspirations $(N, \mathcal{F}, v, a) \in A$ a set of payoff vectors $f(v,a) \subseteq \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq v(N)\}$.[11] The following properties of solutions for games with limited aspirations are straightforward extensions of similar properties of solutions for coalitional games. Let $A \subset GA$ be a set of games with limited aspirations and f a solution on A.

Non-Emptiness. For all $(N, \mathcal{F}, v, a) \in A$ it holds that $f(v,a) \neq \varnothing$.

Individual Rationality. For all $(N, \mathcal{F}, v, a) \in A$ it holds that $f(v,a) \subseteq \{x \in \mathbb{R}^N \mid x_i \geq v(i) \text{ for all } i \in N \text{ and } x_i \leq a(i) \text{ for all } \{i\} \in \mathcal{F}\}$.[12]

Superadditivity. For all $(N, \mathcal{F}, v_1, a_1) \in A$ and $(N, \mathcal{F}, v_2, a_2) \in A$ such

[11] When we want to stress the player set, we will use the notation $f(N, \mathcal{F}, v, a)$.

[12] We make the usual simplification to notation of denoting $v(i)$ and $a(i)$ instead of $v(\{i\})$ and $a(\{i\})$.

that $(N,\mathcal{F},v_1+v_2,a_1+a_2)\in A$, it holds that $f(v_1,a_1)+f(v_2,a_2)\subseteq f(v_1+v_2,a_1+a_2)$.[13]

We are also going to use consistency properties. To define those, we need to define reduced games with limited aspirations. Let $(N,\mathcal{F},v,a)$ be a game with limited aspirations, let $S\in 2^N\backslash\{\emptyset,N\}$, and let $x\in\mathbb{R}^N$ be such that $x(N)\leq v(N)$. The reduced game with limited aspirations with respect to S and x is the game $(S,\mathcal{F}_S,v_{S,x},a_S)$ where $\mathcal{F}_S=\{T\subseteq S\mid T\in\mathcal{F}\}$, $a_S(T)=a(T)$ for all $T\in\mathcal{F}_S$, and

$$v_{S,x}(T)=\begin{cases}0 & \text{if } T=\emptyset,\\ v(N)-x(N\backslash S) & \text{if } T=S,\\ \max_{Q\subseteq N\backslash S}\{v(T\cup Q)-x(Q)\} & \text{if } T\in 2^S\backslash\{\emptyset,S\}.\end{cases}$$

Consistency. For all $(N,\mathcal{F},v,a)\in A$, $S\in 2^N\backslash\{\emptyset,N\}$, and $x\in f(N,\mathcal{F},v,a)$, it holds that $(S,\mathcal{F}_S,v_{S,x},a_S)\in A$ and $x_S\in f(S,\mathcal{F}_S,v_{S,x},a_S)$.[14]

Converse Consistency. Let $(N,\mathcal{F},v,a)\in A$ and $x\in\mathbb{R}^N$ such that $x(N)=v(N)$. If for all $S\in 2^N\backslash\{\emptyset,N\}$ it holds that $(S,\mathcal{F}_S,v_{S,x},a_S)\in A$ and $x_S\in f(S,\mathcal{F}_S,v_{S,x},a_S)$, then $x\in f(N,\mathcal{F},v,a)$.

Let $GA_c=\{(N,\mathcal{F},v,a)\in GA \mid C^T(v,a)\neq\emptyset\}$ be the set of t-balanced games with limited aspirations.

The truncated core on GA_c satisfies consistency.

Proof. Let $(N,\mathcal{F},v,a)\in GA_c$, $S\in 2^N\backslash\{\emptyset,N\}$, and $x\in C^T(v,a)$. As the core satisifes consistency with respect to the reduced game $v_{S,x}$,[15] we know that $x_S\in C(S,v_{S,x})$. Therefore, to prove that $x_S\in C^T(v_{S,x},a_S)$ and, consequently, $(S,\mathcal{F}_S,v_{S,x},a_S)\in GA_c$, it suffices to prove that $\sum_{i\in T}x_i\leq a_S(T)$ for all $T\in\mathcal{F}_S$. This follows straightforwardly from $x\in C^T(v,a)$ and $a_S(T)=a(T)$. $\square$

The truncated core on GA_c satisfies converse consistency.

Proof. Let $(N,\mathcal{F},v,a)\in GA_c$ and $x\in\mathbb{R}^N$ with $x(N)=v(N)$ such that $(S,\mathcal{F}_S,v_{S,x},a_S)\in GA_c$ and $x_S\in C^T(S,\mathcal{F}_S,v_{S,x},a_S)$ for all $S\in 2^N\backslash\{\emptyset,N\}$. As the core satisifes converse consistency with respect to the reduced games

[13] The sum of games is defined, as usual, on a per-coalition basis. The sum of aspirations is defined likewise.

[14] As usual, x_S denotes the projection of x on $\mathbb{R}^S$.

[15] See Lemma 2.11 in Peleg (1986).

$v_{S,x}$,[16] we know that $x \in C(N, v)$. Therefore, to prove that $x \in C^T(N, \mathcal{F}, v, a)$, it suffices to prove that $x(S) \leq a(S)$ for all $S \in \mathcal{F}$. If $S \in \mathcal{F}$, $S \neq N$, then this follows from $x_S \in C^T(S, \mathcal{F}_S, v_{S,x}, a_S)$ and, consequently, $x(S) \leq a_S(S) = a(S)$. If $N \in \mathcal{F}$ and $S = N$, then this follows from $x(N) = v(N) \leq a(N)$. $\Box$

Peleg (1986) uses bilateral converse consistency in an axiomatic characterization of the core. Bilateral converse consistency is stronger than converse consistency as it stipulates that $x_S \in f(S, \mathcal{F}_S, v_{S,x}, a_S)$ only for coalitions S with two players is sufficient to conclude that $x \in f(N, \mathcal{F}, v, a)$. Such a property of bilateral converse consisteny is not satisfied by the truncated core of games with limited aspirations. For example, consider the game $(N, \mathcal{F}, v, a) \in GA_c$ with $N = \{1, 2, 3, 4\}$, $\mathcal{F} = \{\{1, 2, 3\}\}$, $v(S) = 1$ if $\mid S \mid= 1$, $v(S) = 3$ if $\mid S \mid= 2$, $v(S) = 5$ if $\mid S \mid= 3$, and $v(N) = 12$, and $a(\{1, 2, 3\}) = 8$. Taking $x = (3, 3, 3, 3)$, it holds that $x \in C(v)$ and $x_S \in C^T(v_{S,x}, a_S)$ for all $S \subset N$ with $\mid S \mid= 2$. Nevertheless $x \not\in C^T(v, a)$.

Consistency can be used to provide an axiomatic characterization of the truncated core.

The truncated core is the unique solution on GA_c satisfying non-emptiness, individual rationality, superadditivity, and consistency.

The proof of this theorem follows along the lines of the proof of Theorem 5.4 in Peleg (1986). We provide the proof of Theorem 3.3 in the appendix.

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[16]This follows from Lemma 3.2 in Peleg (1986).

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Appendix

Theorem 3.3: The truncated core is the unique solution on GA_c satisfying non-emptiness, individual rationality, superadditivity, and consistency.

Proof of Theorem 3.3. It is straightforward to prove that the truncated core satisfies individual rationality, superadditivity, and non-emptiness on GA_c, in addition to consistency. To show uniqueness, let f be a solution on GA_c that satisfies non-emptiness, individual rationality, superadditivity, and consistency. The proof proceeds in two parts.

Part 1. We first show that $f(v,a) \subseteq C^T(v,a)$ for all $(N, \mathcal{F}, v, a) \in GA_c$. We do this by induction on the number of players.

If $|N| = 1$, then $f(v,a) \subseteq C^T(v,a)$ by individual rationality of f.

Suppose $|N| = 2$. Let $x \in f(v,a)$ and $i \in N$. By consistency it follows that $x_i \in f(\{i\}, \mathcal{F}_{\{i\}}, v_{\{i\},x}, a_{\{i\}})$. Together with individual rationality, this implies that $x_i \geq v_{\{i\},x}(i) = v(N) - x(N\backslash\{i\}) \geq x_i$, where the last inequality follows from $x(N) \leq v(N)$ (which holds because f is a solution). Hence, we conclude that $x(N) = v(N)$ has to hold. This and individual rationality give us that $x_j \geq v(j)$ for all $j \in N$, $x_j \leq a(j)$ for all $\{j\} \in \mathcal{F}$, and $x(N) = v(N)$, which is to say that $x \in C^T(v,a)$.

Now suppose that $|N| > 2$ and that $f(v,a) \subseteq C^T(v,a)$ for all games with fewer players. If $x \in f(N, \mathcal{F}, v, a)$, then by consistency $x_S \in f(S, \mathcal{F}_S, v_{S,x}, a_S)$ for all $S \in 2^N\backslash\{\emptyset, N\}$. By the induction hypothesis, we then know that $x_S \in C^T(v_{S,x}, a_S)$ for all $S \in 2^N\backslash\{\emptyset, N\}$. As the truncated core on GA_c satisfies converse consistency (see Lemma 3.3), we can conclude that $x \in C^T(N, \mathcal{F}, v, a)$.

Part 2. We now show that $C^T(v,a) \subseteq f(v,a)$ for all $(N, \mathcal{F}, v, a) \in GA_c$. Let $(N, \mathcal{F}, v, a) \in GA_c$. We distinguish three cases.

Case 1. $|N| = 1$. Suppose $N = \{i\}$. As $f(N, \mathcal{F}, v, a) \subseteq C^T(N, \mathcal{F}, v, a) = \{v(i)\}$, non-emptiness of f implies that $v(i) \in f(N, \mathcal{F}, v, a)$.

Case 2. $|N| \geq 3$. Let $x \in C^T(v,a)$. We define a game $(N, \mathcal{F}, w, b)$ by $w(\{i\}) = v(\{i\})$ for all $i \in N$, $w(S) = x(S)$ if $|S| > 1$, and $b(S) = a(S)$ for all $S \in \mathcal{F}$. Then, obvioulsy, $C^T(w,b) = \{x\}$. By non-emptiness of f and Part 1 of this proof, it then follows that $f(w,b) = \{x\}$. We define a game $(N, \mathcal{F}, u, c)$ by $u(S) = v(S) - w(S)$ for all $S \in 2^N$ and $c(S) = 0$ for all $S \in \mathcal{F}$. This implies that $u(\{i\}) = 0$ for all $i \in N$, $u(S) \leq 0$ for all $S \in 2^N$, and $u(N) = 0$. Then $C^T(u,c) = \{0\}$ and by non-emptiness of f and Part 1 of this proof it follows that $f(u,c) = \{0\}$. Superadditivity of f now gives us $\{x\} = f(w,b) + f(u,c) \subseteq f(w+u, b+c) = f(v,a)$. This proves that $C^T(v,a) \subseteq f(v,a)$.

Case 3. $|N| = 2$. Suppose $N = \{i,j\}$. Let $k \notin N$ and define $M = \{i,j,k\}$. Define the game $(M, \mathcal{G}, u, b)$ as follows: $u(S) = v(S \cap N)$ if $S \in 2^M\backslash\{M\}$ and $u(M) = v(N)$, $\mathcal{G} = \mathcal{F} \cup \{\{k\}, \{i,k\}, \{j,k\}, M\}$, $b(S) = a(S)$ for all $S \in \mathcal{F}$, $b(\{k\}) = 0$, $b(\{i,k\}) = a(i)$, $b(\{j,k\}) = a(j)$, and $b(M) = v(N)$. Let $x = (x_i, x_j) \in C^T(N, \mathcal{F}, v, a)$. Then $y = (x_i, x_j, 0) \in C^T(M, \mathcal{G}, u, b)$. As

$|M| = 3$, we can apply the results we found in Case 2 and conclude that $y \in f(M, \mathcal{G}, u, b)$. Also, $(N, \mathcal{G}_N, u_{N,y}, b_N) = (N, \mathcal{F}, v, a)$. Consistency of f then allows us to conclude that $x = y_N \in f(N, \mathcal{G}_N, u_{N,y}, b_N) = f(N, \mathcal{F}, v, a)$. This proves that $C^T(v, a) \subseteq f(v, a)$. $\square$