A COMPARATIVE AXIOMATIC CHARACTERIZATION OF THE BANZHAF–OWEN COALITIONAL VALUE

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Abstract

A compact axiomatic characterization of the modified Banzhaf value for games with a coalition structure (Banzhaf–Owen value, for short) is provided. The axiomatic system used here can be compared with parallel axiomatizations of other coalitional values such as the Owen value or the Alonso–Fiestras value, thus giving arguments to defend the use of one of them that will depend on the context where they are to be applied.

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1 Introduction

The assessment of the strategic position of each player in any game is a main objective of the cooperative game theory, as it can be applied to e.g. sharing costs or profits in economic problems or measuring the power of each agent in a collective decision–making system. The *Shapley value* φ is the best known concept in this respect, and its axiomatic presentation (Shapley [38], also in Roth [37]) introduced a new, elegant style in game theory and opened a fruitful research line.

As a sort of reaction to the application of φ to simple games as a power index, suggested by Shapley and Shubik [39], and following a more classical procedure, Banzhaf [12] introduced a different index of power (essentially equivalent to those

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proposed by Penrose [36] and Coleman [17]) that gave rise to a *Banzhaf value* β on all cooperative games first defined by Owen [30]. Many axiomatic characterizations of one, the other or both values may be found in the literature (see, e.g., Owen [32], Dubey and Shapley [18], Young [45], Lehrer [26], Straffin [40], Amer and Carreras [7], Nowak [28] or Laruelle and Valenciano [24]). A most interesting one was stated by Feltkamp [20], who gave *parallel* characterizations of the Shapley and Banzhaf values that enhance the similarities and differences between them. Indeed, only one property distinguishes these values: efficiency for the Shapley value versus total power for the Banzhaf value.

Forming coalitions is a most natural behavior in cooperative games, and the evaluation of the consequences that derive from this action is also of great interest to game theorists. *Games with a coalition structure* were first considered by Aumann and Drèze [11], who extended the Shapley value to this new framework in such a manner that the game really splits into subgames played by the unions isolatedly from each other, and every player receives the payoff allocated to him by the Shapley value in the subgame he is playing within his union. A second approach was used by Owen [31] (also in Owen [34]), when introducing and axiomatically characterizing his coalitional value Φ (*Owen value*). In this case, the unions play a *quotient game* among themselves, and each one receives a payoff which, in turn, is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by the Shapley value. In addition to the initial one, many other axiomatic characterizations of Φ can be found in the literature (Hart and Kurz [23], Winter [44], Amer and Carreras [8] and [9], Vázquez–Brage et al. [41], Hamiache [22] or Albizuri [2] among others).

By applying a similar procedure to the Banzhaf value, Owen [33] obtained a second coalitional value, the modified Banzhaf value Ψ for games with a coalition structure or *Banzhaf–Owen value*. Here, the payoffs at both levels, that of the unions in the quotient game and that of the players within each union, are given by the Banzhaf value. In this case, no axiomatization was initially provided. A first axiomatic characterization was reached by Albizuri [1], but only on the restricted domain of (monotonic) simple games. Amer et al. [10] were the first to establish a characterization of Ψ on the full domain of all cooperative games. However, as they said in Remark 3.3(b) under a suggestion of a referee of their article, their characterization is far from giving rise to an almost common axiomatization of both Φ and Ψ similar to Feltkamp's one for φ and β. (For a wide generalization of Owen's procedure to coalitional semivalues, which encompasses the four coalitional values that will be considered here, the interested reader is referred to Albizuri and Zarzuelo [3]. For a way of extending to games with coalition structure the notion of sharing function, please see van den Brink and van der Laan [13].)

Our aim here is to provide a new axiomatic characterization for the Banzhaf–Owen value Ψ that is able to be compared with some of the existing ones for Φ and, more precisely, with the characterization reached by Vázquez–Brage et al. [41].

The organization of the paper is then as follows. In Section 2, a minimum of preliminaries is provided. In Section 3 we give the axiomatic characterization of the Banzhaf–Owen value. Section 4 is devoted to comparing it with parallel axiomatizations of Φ, of Alonso and Fiestras' symmetric coalitional Banzhaf value π and even of a sort of "counterpart" value μ introduced by Amer et al. [10] (both to be defined below in due time) and to discussing by the way our results.

2 Preliminaries

Although the reader is assumed to be generally familiar with the cooperative game theory, we recall here some basic notions.

2.1 Games and values

A finite transferable utility cooperative game (from now on, simply a *game*) is a pair (N, v) defined by a finite set of *players* N, usually $N = \{1, 2, \ldots, n\}$, and a function $v : 2^N \to \mathbb{R}$, that assigns to each *coalition* $S \subseteq N$ a real number $v(S)$ and satisfies $v(\emptyset) = 0$. In the sequel, $\mathcal{G}_N$ will denote the family of all games on a given N and $\mathcal{G}$ the family of all games.

A player $i \in N$ is a *dummy* in game (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N\backslash\{i\}$, that is, if all his marginal contributions equal $v(\{i\})$. Two players $i, j \in N$ are *symmetric* in game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N\backslash\{i, j\}$, i.e., if their marginal contributions to each coalition coincide.

By a *value* we will mean a map f that assigns to every game $(N, v) \in \mathcal{G}$ a vector $f(N, v) \in \mathbb{R}^N$ with components $f_i(N, v)$ for all $i \in N$.

Definition 2.1 (Owen [30]) The *Banzhaf value* β is the value defined by

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N\backslash\{i\}} \Big[v(S \cup \{i\}) - v(S)\Big] \quad \text{for any } i \in N \text{ and any } (N, v) \in \mathcal{G}. \quad (1)$$

2.2 Games with a coalition structure

Let us consider a finite set, say, $N = \{1, 2, \ldots, n\}$. We will denote by $P(N)$ the set of all partitions of N. Each $P \in P(N)$, of the form $P = \{P_1, P_2, \ldots, P_m\}$, is called a *coalition structure* or *system of unions* on N. The so–called *trivial coalition structures* are $P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$, where each union is a singleton, and $P^N = \{N\}$, where the *grand coalition* forms. Given $i \in N$, $P(i)$ will denote the subfamily of coalition structures $P \in P(N)$ such that $\{i\} \in P$. If $i \in P_k \in P$, P_{-i} will denote the partition obtained from P when player i leaves union P_k and becomes isolated, i.e.,

$$P_{-i} = \{P_h \in P : h \neq k\} \cup \{P_k\backslash\{i\}, \{i\}\}.$$

A *cooperative game with a coalition structure* is a triple (N, v, P) where $(N, v) \in \mathcal{G}$ and $P \in P(N)$. The set of all cooperative games with a coalition structure will be denoted by $\mathcal{G}^{cs}$, and by $\mathcal{G}^{cs}_N$ the subset where N is the player set.

If $(N, v, P) \in \mathcal{G}^{cs}$ and $P = \{P_1, P_2, \ldots, P_m\}$, the *quotient game* (M, v^P) is the cooperative game played by the unions, or, rather, by the set $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^P(R) = v(\bigcup_{r \in R} P_r) \quad \text{for all } R \subseteq M. \tag{2}$$

Notice that (M, v^P) is nothing but (N, v) whenever $P = P^n$. A not completely trivial case is the following.

Example 2.2 Let $(N, v) \in \mathcal{G}$ be given and $i, j \in N$ be distinct players. Let $P^{i,j}$ be the coalition structure of N where just i and j form a union while the remaining players stay all isolated. In this case, we can slightly alter the notations, take ij as a new player representing both i and j and each other player k as representing himself, thus considering $N^{i,j} = \{ij, 1, 2, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\}$ (where $\hat{i}$ and $\hat{j}$ mean that i and j have been removed) as quotient player set and, as quotient game, the game $v^{i,j}$ defined by

$$v^{i,j}(S) = v(S) \quad \text{and} \quad v^{i,j}(S \cup \{ij\}) = v(S \cup \{i, j\}) \quad \text{for any } S \subseteq N \backslash \{i, j\}.$$

As we see, this game, considered by Lehrer [26] and Nowak [28] in their axiomatizations of the Banzhaf value and called "reduced game" or "amalgamation of i and j", is nothing but the easiest non–trivial case of quotient game. Its generalization to the case where, instead of $\{i, j\}$, some coalition $S \subseteq N$ with $|S| \geq 2$ forms is straightforward.

By a *coalitional value* we will mean a map g that assigns to every game with a coalition structure (N, v, P) a vector $g(N, v, P) \in \mathbb{R}^N$ with components $g_i(N, v, P)$ for each $i \in N$.

Definition 2.3 (Owen [33]) The *Banzhaf–Owen value* Ψ is the coalitional value defined by

$$\Psi_i(N, v, P) = \sum_{R \subseteq M \backslash \{k\}} \sum_{T \subseteq P_k \backslash \{i\}} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \Big[v(Q \cup T \cup \{i\}) - v(Q \cup T) \Big] \tag{3}$$

for all $i \in N$ and all $(N, v, P) \in \mathcal{G}^{cs}$, where $P_k \in P$ is the union such that $i \in P_k$, $m = |M|$, $p_k = |P_k|$ and $Q = \bigcup_{r \in R} P_r$.

Definition 2.4 Given a value f on $\mathcal{G}$, a coalitional value g on $\mathcal{G}^{cs}$ is a *coalitional f–value* if

$$g(N, v, P^n) = f(N, v) \quad \text{for all} \quad (N, v) \in \mathcal{G}. \tag{4}$$

3 An axiomatic approach

We shall consider the following properties for a coalitional value g on $\mathcal{G}^{cs}$.

A1. (*2–Efficiency*) For all $(N, v) \in \mathcal{G}$, and any pair of distinct players $i, j \in N$,

$$g_i(N, v, P^n) + g_j(N, v, P^n) = g_{ij}(N^{i,j}, v^{i,j}, P^{n-1}).$$

A2. (*Dummy player*) If $i \in N$ is a dummy in (N, v) then $g_i(N, v, P^n) = v(\{i\})$.

A3. (*Symmetry*) If $i, j \in N$ are symmetric players in (N, v) then $g_i(N, v, P^n) = g_j(N, v, P^n)$.

A4. (*Equal marginal contributions*) If (N, v) and (N, w) are games with a common player set N, and some player $i \in N$ satisfies $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \backslash \{i\}$, then $g_i(N, v, P^n) = g_i(N, w, P^n)$.

A5. (*Indifference within unions*) If $(N, v, P) \in \mathcal{G}^{cs}$, $P_k \in P$, and $i, j \in P_k$ are distinct players, then

$$g_i(N, v, P) = g_i(N, v, P_{-j}).$$

A6. (*Quotient game property for one–player unions*) If $(N, v, P) \in \mathcal{G}^{cs}$ and $P \in P(i)$ for some $i \in N$, then

$$g_i(N, v, P) = g_k(M, v^P, P^m),$$

where $P_k = \{i\}$.

Axioms A2 and A3 (also called *equal treatment property*) are standard in the literature. Axiom A1 was introduced by Lehrer [26] in a slightly different form (as an inequality), although it was soon discovered (see, e.g., Carreras and Magaña [14]) that the equality holds, as reported also by Nowak [28], while A4 was introduced by Young [45]. We refer to these sources, and also to Haller [21] and Malawski [27], for discussions about the meaning and scope of these properties. The discussion on axioms A5 and A6 will be done in the next section.

In the sequel, A0 will mean the conjunction of A1–A4. This will make our statements simpler. We will first establish a close relationship between the coalitional values satisfying A0 and the Banzhaf value. More precisely:

Proposition 3.1 *A coalitional value g satisfies A0 if, and only if, it is a coalitional Banzhaf value, i.e.*

$$g_i(N, v, P^n) = \beta_i(N, v) \quad \textit{for all} \quad i \in N \quad \textit{and all} \quad (N, v) \in \mathcal{G}. \tag{5}$$

Proof. The proof follows the same guidelines as Nowak's [28] (non–trivial) proof. The only difference between Nowak's statement and ours is that he is talking about values, whereas we are referring to coalitional values, although the connection is given by the appearance of the trivial coalition structure P^n in our axiom set A1–A4. □

Remark 3.2 The interest of this first result lies on the existence of an analogous result for the Shapley value obtained by Young [45] and also cited by Nowak [28] (the difference will be explained in the next section). This is a sort of starting point for our aim to "put in parallel" the Owen value Φ (as a coalitional Shapley value) and the Banzhaf–Owen value Ψ (a coalitional Banzhaf value in the above sense).

Now, we are ready to state and prove our main result.

Theorem 3.3 *(Existence and uniqueness) A coalitional value g satisfies A0, A5 and A6 if, and only if, it is the Banzhaf–Owen coalitional value Ψ. In other words, Ψ is the unique coalitional Banzhaf value that satisfies A5 and A6.*

Proof. (a) (Existence) 1. The Banzhaf–Owen coalitional value Ψ satisfies A0, that is, A1–A4. According to Proposition 3.1, it suffices to check equation (5). Let $(N, v) \in \mathcal{G}$ and $i \in N$. As we will deal with $P = P^n$, we have $M = N$, $P_k = \{i\}$ so $k = i$ and $|p_k| = 1$, $T = \emptyset$ and $Q = R$ when applying formula (3). Thus

$$\Psi_i(N, v, P) = \sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i\}} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \Big[v(Q \cup T \cup \{i\}) - v(Q \cup T) \Big]$$

reduces to

$$\Psi_i(N, v, P^n) = \sum_{R \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}} \Big[v(R \cup \{i\}) - v(R) \Big] = \beta_i(N, v).$$

2. The Banzhaf–Owen value Ψ satisfies A5, the property of indifference whitin unions. Let $(N, v, P) \in \mathcal{G}^{cs}$, $P_k \in P$, and $i, j \in P_k$ be distinct players. Let

$$P_{-j} = \{P'_1, P'_2, \ldots, P'_{m+1}\},$$

where $P'_h = P_h$ for every $h \in M \setminus \{k\}$, $P'_k = P_k \setminus \{j\}$ and $P'_{m+1} = \{j\}$, and let $M' = \{1, 2, \ldots, m, m+1\}$. Then, $m' = m + 1$ and $p'_k = p_k - 1$ so that

$$\Psi_i(N, v, P_{-j}) = \sum_{R \subseteq M' \setminus \{k\}} \sum_{T \subseteq P'_k \setminus \{i\}} \frac{1}{2^{m'-1}} \frac{1}{2^{p'_k - 1}} \Big[v(Q \cup T \cup \{i\}) - v(Q \cup T) \Big]$$

reduces, by separating the cases $m + 1 \in R$ and $m + 1 \notin R$ and grouping terms again, to

$$\Psi_i(N, v, P_{-j}) = \sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i\}} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \Big[v(Q \cup T \cup \{i\}) - v(Q \cup T) \Big] = \Psi_i(N, v, P).$$

3. The Banzhaf–Owen value Ψ satisfies A6, the quotient game property for one–player unions. Let $(N, v, P) \in \mathcal{G}^{cs}$ be such that $P \in P(i)$, and let $P_k = \{i\}$. Then $|p_k| = 1$, so that $T = \emptyset$ and therefore

$$\Psi_i(N, v, P) = \sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i\}} \frac{1}{2^{m-1}} \frac{1}{2^{p_k - 1}} \Big[v(Q \cup T \cup \{i\}) - v(Q \cup T) \Big]$$

reduces to

$$\Psi_i(N,v,P) = \sum_{R\subseteq M\setminus\{k\}} \frac{1}{2^{m-1}}\Big[v(Q\cup\{i\}) - v(Q)\Big]$$

$$= \sum_{R\subseteq M\setminus\{k\}} \frac{1}{2^{m-1}}\Big[v^P(R\cup\{k\}) - v^P(R)\Big] = \beta_k(M,v^P) = \Psi_i(M,v^P,P^m).$$

(b) (Uniqueness) Let us assume for a while that two coalitional Banzhaf values g^1 and g^2 satisfy indifference within unions (A5) and the quotient game property for one–player unions (A6). Then we can find a game (N,v) and a coalition structure P on N with the maximum number of unions such that $g^1(N,v,P) \neq g^2(N,v,P)$, i.e., $g_i^1(N,v,P) \neq g_i^2(N,v,P)$ for some $i \in N$.

As g^1 and g^2 are coalitional Banzhaf values, it follows that $m < n$. Let us take $P_k \in P$ such that $i \in P_k$. Two possible cases arise:

- $|P_k| = 1$. Then, $P_k = \{i\}$. By A6 we have
$$g_i^1(N,v,P) = g_k^1(M,v^P,P^m) \quad \text{and} \quad g_i^2(N,v,P) = g_k^2(M,v^P,P^m).$$
Since g^1 and g^2 are coalitional Banzhaf values
$$g_k^1(M,v^P,P^m) = \beta_k(M,v^P) = g_k^2(M,v^P,P^m).$$
Therefore, $g_i^1(N,v,P) = g_i^2(N,v,P)$, a contradiction.

- $|P_k| > 1$. Then, there is some $j \in P_k$ such that $j \neq i$. By A5,
$$g_i^1(N,v,P) = g_i^1(N,v,P_{-j}) \quad \text{and} \quad g_i^2(N,v,P) = g_i^2(N,v,P_{-j}).$$
By the maximality of partition P it follows that
$$g_i^1(N,v,P_{-j}) = g_i^2(N,v,P_{-j}),$$
and this leads to $g_i^1(N,v,P) = g_i^2(N,v,P)$, a contradiction again. □

Remark 3.4 (*Independence of the axiomatic system*) By considering A0 (conjunction of A1–A4, that is, equivalent to being a coalitional Banzhaf value) as a sole property, the axiom system {A0,A5,A6} is independent. Indeed:

(i) The coalitional value g, given by $g(N,v,P) = \beta(N,v)$ for all $(N,v,P) \in \mathcal{G}^{cs}$, is a coalitional Banzhaf value (satisfies A0) that satisfies A5 but not A6.

(ii) The *symmetric coalitional Banzhaf value* π, introduced by Alonso–Meijide and Fiestras–Janeiro [4] and defined, with the same notation as in case of Ψ for equation (3) and adding $t = |T|$, by

$$\pi_i(N,v,P) = \sum_{R\subseteq M\setminus\{k\}} \sum_{T\subseteq P_k\setminus\{i\}} \frac{1}{2^{m-1}} \frac{t!(p_k-t-1)!}{p_k!}\Big[v(Q\cup T\cup\{i\}) - v(Q\cup T)\Big], \tag{6}$$

is a coalitional Banzhaf value that satisfies A6 but not A5 (for details, see Alonso–Meijide and Fiestras–Janeiro [4]). This coalitional value will be considered in detail in the next section.

(iii) Finally, given $a \in \mathbb{R}$, the coalitional value g, given by $g(N, v, P) = a$ for all $(N, v, P) \in \mathcal{G}^{cs}$, satisfies A5 and A6 but is not a coalitional Banzhaf value.

4 Discussion and conclusions

This section is devoted to the analysis and criticism of the results obtained in Section 3.

4.1 On the axioms

Some comments are in order concerning the properties we have used as axioms. First, it is worthy of mention that property A0 (i.e., axioms A1–A4), might be replaced with any other system of axioms that characterizes the Banzhaf value by adding the mention of the trivial coalition structure as we did before. In particular, A1 is equivalent, in presence of A2–A4, to the so–called *total power property* (with regard to P^n, of course), which states

$$\sum_{i \in N} g_i(N, v, P^n) = \frac{1}{2^{m-1}} \sum_{S \subseteq N} \sum_{i \notin S} \Big[v(S \cup \{i\}) - v(S) \Big] \quad \text{for all} \quad (N, v) \in \mathcal{G},$$

and can be traced back (omitting P^n) at least to Owen [32] and Dubey and Shapley [18]. Moreover, note that no use has been made of additivity and neither of strong monotonicity (Young [45]) in our system, although it should be noticed that A4 is what Young calls *independence*, a property weaker than strong monotonicity. Not only φ and β, but all *semivalues* (see Weber [42], Dubey et al. [19] or Weber [43] for this notion) satisfy A2–A4 (always omitting P^n) and an *ad hoc* modification of the total power property for each one of them.

A5 (indifference within unions) describes the invariance of the allocations given by a coalitional value to the players of any union in front of the existence of self–isolating players in that union. This property is stronger than the "balanced contributions property" that will be considered below.

Finally, A6 (quotient game property for one–player unions) states that, using the coalitional value in the original game with a coalition structure, any isolated player gets the same payoff as the union he forms if we use the same coalitional value in the quotient game with the trivial singleton structure. As we will see, this property, weaker than the "quotient game property" described below, could also be called *1–quotient game property*.

4.2 Other properties, other values

Let us now consider, for comparative purposes, a new series of properties for a coalitional value that have been used as axioms in the literature. We recall that $P_k, P_h \in P$

are *symmetric unions* in (N, v, P) if k and h are symmetric players in the quotient game (M, v^P).

B1. (*Efficiency*) For all $(N, v) \in \mathcal{G}$

$$\sum_{i \in N} g_i(N, v, P^n) = v(N).$$

B2. Equal to A2.

B3. Equal to A3.

B4. Equal to A4.

B0. Conjunction of B1–B4.

B5. (*Balanced contributions within unions*) For all $(N, v, P) \in \mathcal{G}^{cs}$ and $i, j \in P_k \in P$

$$g_i(N, v, P) - g_i(N, v, P_{-j}) = g_j(N, v, P) - g_j(N, v, P_{-i}).$$

B6. (*Quotient game property*) For all $(N, v, P) \in \mathcal{G}^{cs}$ and all $P_k \in P$

$$\sum_{i \in P_k} g_i(N, v, P) = g_k(M, v^P, P^m).$$

B7. (*Symmetry in the quotient game*) If P_k and P_h are symmetric unions in (N, v, P), then

$$\sum_{i \in P_k} g_i(N, v, P) = \sum_{j \in P_h} g_j(N, v, P).$$

As was said in Subsection 4.1, A5 is stronger than B5, whereas A6 is weaker than B6. Both relations are clear.

Now, we put together in Table 1 all these properties and their counterparts A0–A6, and specify which of them are satisfied by each one of the four coalitional values we are considering in this paper, namely, Φ (Owen value), Ψ (Banzhaf–Owen value), π (symmetric coalitional Banzhaf value or Alonso–Fiestras value, for short), and μ (a "counterpart" of π introduced as a counterexample by Amer et al. [10] which simply follows Owen's [31] and [33] two–step procedure but applies the Shapley value in the quotient game and the Banzhaf value within unions). Note therefore that these four values cover all the variations of Owen's scheme using the Shapley and Banzhaf values.

An OK (resp., empty) entry in the four final columns means that the corresponding coalitional value satisfies (resp., fails to satisfy) the corresponding property. The verification of the positive (OK) entries and suitable counterexamples in case of failure can be easily found in the literature concerning these four values already mentioned here. It is worthy of mention that property A1, 2–efficiency, is specific of Ψ. The total power property for this value differs from the corresponding one for π (see Alonso–Meijide and Fiestras–Janeiro [4]), although they share a common spirit.

The last row—not essential to our discussion, but nice enough to be included here—refers to the possibility to compute a value by using, as in the well known cases of the Shapley value (Owen [29]) and the Banzhaf value (Owen [30]), the multilinear extension of the game where it is applied. The references are Owen and Winter [35] for Φ, Carreras and Magaña [14] (see also Carreras and Magaña [15]) for Ψ, and Alonso–Meijide et al. [5] for π and μ. The multilinear extension, introduced by Owen [29], becomes therefore a very interesting tool, for both theory and practice, in the framework of coalitional values.

Table 1. Properties and coalitional values

symbol	**property**	Φ	Ψ	π	μ
A0	coalitional β–value (Banzhaf)		OK	OK	
B0	coalitional φ–value (Shapley)	OK			OK
A1	2–efficiency / total power		OK	OK	
B1	efficiency	OK			OK
A2 = B2	dummy player	OK	OK	OK	OK
A3 = B3	symmetry	OK	OK	OK	OK
A4 = B4	equal marginal contributions	OK	OK	OK	OK
A5	indifference within unions		OK		
B5	balanced contributions within unions	OK	OK	OK	OK
A6	1–quotient game	OK	OK	OK	OK
B6	quotient game	OK		OK	
B7	symmetry in the quotient game	OK		OK	
MLE	computation by multilinear extensions	OK	OK	OK	OK

For a moment, we will disregard value μ and focus on the remaining values Φ, Ψ and π. The important point is that we have, then, *parallel* (i.e., very close) axiomatic characterizations of these coalitional values. We state them.

Theorem 4.1 *(Vázquez–Brage et al. [41]) A coalitional value satisfies B0, B5 and B6 if, and only if, it is the Owen value* Φ. □

Theorem 4.2 *(Theorem 3.3 in this paper) A coalitional value satisfies A0, A5 and A6 if, and only if, it is the Banzhaf–Owen value* Ψ. □

Theorem 4.3 *(Alonso–Meijide and Fiestras–Janeiro [4]) A coalitional value satisfies A0, B5 and B6 if, and only if, it is the symmetric coalitional Banzhaf value* π. □

As is seen, the only basic difference between Φ and π lies in the fact that the former is a coalitional φ–value whereas the latter is a coalitional β–value. Instead, the differences between Φ and Ψ arise in axioms A0/B0, A5/B5 and A6/B6, the latter two pairs being linked by an implication relationship, while the differences between π and Ψ are limited to A5/B5 and A6/B6. We feel that this is a complete generalization of Feltkamp's [20] axiomatic characterizations of φ and β. It also enhances the role of π as an "intermediate" value between Φ and Ψ, and we wish to mention here that Theorem 4.3 extends in a natural way to all symmetric coalitional binomial semivalues π^p for $p \in [0,1]$ (introduced by Carreras and Puente [16]), as is shown in Alonso–Meijide et al. [6].

It has not been an easy task making changes on B5 and B6 in order to get, respectively, really simple and sharp axioms A5 and A6, those that mark the essential differences between Φ and Ψ. It has been necessary to "split hairs" accurately.

Which is the reason to have included the symmetric coalitional Banzhaf value π in our considerations? Well, Alonso–Meijide and Fiestras–Janeiro [4] realized that Ψ fails to satisfy two in principle interesting properties of Φ, namely B6 (quotient game property) and B7 (symmetry in the quotient game). Then they suggested to modify Owens' two–step allocation scheme (common to Φ and Ψ) and use β for sharing in the quotient game and φ to sharing within unions. This gave rise to π, that satisfies B6 and B7 but differs from the Owen value Φ in satisfying A0 instead of B0. In Subsection 4.3, we will look again at the meaning of π.

Now, we would like to refer to the work by Amer et al. [10]. Their first axiomatic characterization of the Banzhaf–Owen value on the domain of all cooperative games was reached by considering six properties that, for our purposes, do not need to be stated in detail:

C1. *Additivity.*

C2. *Dummy player property.*

C3. *Symmetry within unions.*

C4. *Many null players.*

C5. *Delegation neutrality.*

C6. *Delegation transfer.*

Properties C1–C3 are standard in the literature. C4 is perhaps the most striking one. C5 and C6 refer to the so–called "delegation game", close to Lehrer's [26] "reduced game" but avoiding the use of different player sets. In order to show—partially—the independence of this axiom system, it is introduced in Remark 2.1(b) the "fourth value" μ (a mixed coalitional value that can be now viewed as a "counterpart" of π), since it satisfies all properties but C4 (see Remark 3.3(a) in Amer et al. [10]), thus proving that this "rare" property does not follow from the remaining ones. The problem with this axiomatic characterization is that it is far from any of the existing ones for the Owen value Φ, as the authors recognize in their Remark 3.3(b) following a suggestion of a referee of their article, because Φ satisfies all but property C6 but it is hard to imagine which property—if any—would be able to replace C6 and complete a hypothetical parallel axiomatization of Φ. In our opinion, the relative "failure", only in this sense, of the work by Amer et al. [10] enforces still more our result (Theorem 3.3).

Finally, let us go back to value μ. We first introduce two more properties for a coalitional value g:

D1. (*Quotient game property for k–player unions if $k > 1$*) If $(N, v, P) \in \mathcal{G}^{cs}$ and $P_k \in P$ is such that $|P_k| > 1$ then

$$\sum_{i \in P_k} g_i(N, v, P) = g_k(M, v^P, P^m).$$

D2. (*2–Efficiency within unions*) For all $(N, v, P) \in \mathcal{G}^{cs}$, any $P_k \in P$ and any pair of distinct players $i, j \in P_k$,

$$g_i(N, v, P) + g_j(N, v, P) = g_{ij}(N^{i,j}, v^{i,j}, P^{ij}),$$

where P^{ij} is the partition that arises from P by simply collapsing i and j in a single player ij.

Note that property D1 could in fact be written as B6/A6, and B6 = (B6/A6) $\cap$ A6. Furthermore, it is not difficult to see that D2 coincides with C6 (the delegation transfer property introduced by Amer et al. [10]): the only difference is, roughly speaking, that, in the original delegation transfer property, the delegating player becomes a null player, whereas this player disappears in D2. Thus, B6/A6 and C6 are consistent notations for D1 and D2, respectively, so that we will use them in the sequel.

Then, we have a fourth result concerning μ and using C6 (Theorem 4.4), whose proof is omitted since it is similar to that of Theorem 3.3, and a new and interesting comparative table referred to the four values (Table 2) where the splitting of B6 into A6 and B6/A6 matters.

Theorem 4.4 *A coalitional value satisfies B0, B5, A6 and C6 if, and only if, it is the counterpart value μ.* □

Table 2. New properties and coalitional values

symbol	**property**	Φ	Ψ	π	μ
A0	coalitional β–value (Banzhaf)		OK	OK	
B0	coalitional φ–value (Shapley)	OK			OK
B5	balanced contributions within unions	OK	OK	OK	OK
A6	1–quotient game	OK	OK	OK	OK
B6/A6	k–quotient game ($k > 1$)	OK		OK	
C6	2–efficiency within unions / delegation transfer		OK		OK

Remark 4.5 Even a final axiomatic characterization of Ψ can be stated, once we have considered C6. In effect: a coalitional value satisfies A0, B5, A6 and C6 if, and only if, it is the Banzhaf–Owen value Ψ. The proof is analogous to that of 3.3.

4.3 On the philosophy behind axiomatics

To close this section, we would like to make some comments about what is behind this "logical game" of axiomatizations.

For any value, understood as a solution concept for cooperative conflicts, it is always interesting at most, in both theory and practice, to have an explicit formula and even an alternative computation procedure. This is the case of the four coalitional values mentioned here, Φ, Ψ, π and μ, all of which are obtained by combining the weighting coefficients of φ and β in similar formulas and also by applying the multilinear extension technique.

Also a list of properties of the value, as long as possible, is always desirable. Then, why axiomatic systems? There are some reasons for this interest of game theorists in getting them. First, for a mathematically elegant and pleasant spirit. Second, because a set of basic (and assumed independent and hence minimal) properties is a most convenient and economic tool to decide on the use of the value. Finally, such a set allows a researcher to compare a given value with others and select the most suitable one for the problem he or she is facing each time.

Why *parallel* axiomatic characterizations are especially interesting? Because they favour the easiness when comparing different options to be chosen as the preferred value.

Then, we feel that one should strongly prevent from being dogmatic at this point. Probably, there is no value able to cover all situations. For example, there is no unanimous criterion to choose among using either the Shapley value φ or the Banzhaf value β as power index *in all cases.* We contend that pure and applied game theorists should be flexible at most in this respect. On one hand, in both theory and practice, one has often to handle additional information not stored in either the characteristic function v of the game or the coalition structure when evaluating this couple. On the other hand, only a few properties found in the literature can really be considered absolutely compelling, i.e. almost no axiom is compelling *in vacuo,* but only inserted in the framework of a given, specific cooperative conflict. Even those that appear as the best placed in this sense might well be conditioned by the characteristics of the problem where we pretend to use the value they define. The conclusion is that all of us should look at axioms with an open mind and without a priori value judgements. The history of science is full of examples of theoretical models that only after a certain period of time have been proven to be useful in practice. Let us briefly illustrate these considerations by means of some simple instances.

Example 4.6 (a) Assume that N is a set of workers in a given production area and P reflects the classification of them into the firms they are working for. Assume, besides, that g is a coalitional value that allocates to each worker his salary and to each firm its (net) income (say, per year in both cases). In this context, axiom B5 (balanced contributions within unions) is too weak since here it seems more suitable to assume the stronger hypothesis that the salary of a worker will not change if a partner leaves the firm, and this is precisely axiom A5 (indifference within unions). Furthermore, axiom B6 (quotient game property) is neither compelling because the sum of salaries of the workers of a firm needs not coincide with the net income of the firm. This seems too strong. However, if a worker creates and holds his own firm alone, it is very reasonable that his salary coincides with the net income of his firm, and this is precisely the weaker axiom A6 (quotient game property for 1–player unions). Thus, we have in mind the Banzhaf–Owen value Ψ.

(b) Now, assume that political parties are the agents in a parliamentary context and the coalition structures reflect the coalition formation. Assume, moreover, that g is a coalitional value that measures, in some sense, the "power" of both parties and coalitions. In this case, A5 (indifference within unions) might not be a reasonable property, but not necessarily should be automatically replaced with B5 (balanced contributions within unions): maybe the effect on a party of the desertion of a coalition partner is not the same as the effect when the roles are interchanged. Also B6 (quotient game property) may be not completely convincing, since the power concept at the coalition level might well be different from power at the party level, and hence the sum of the power indices of the colligated parties might differ from the power of their union in the quotient game—at least, it is not completely clear why they *should* coincide. Instead, it seems much more reasonable that this coincidence holds in the case of a party that remains isolated, and this is property A6 (quotient game for 1–player unions).

(c) Still in the parliamentary framework, one can consider that parties in the original game, and unions in the quotient game, fight for something called "power". However, once each union got its fraction of power in the quotient game, it is often convenient to share this index among its members *efficiently.* For example, and especially, whenever the coalition is winning and gives rise to a coalition government, that will need to share cabinet and parliamentary positions as presidencies and ministries and budgets management among its members. Even if one prefers the Banzhaf value as power index, he/she will apply it in the quotient game, but will necessarily prefer the *efficient* Shapley value when sharing within the union. In other words, he/she will prefer the symmetric coalitional Banzhaf value π, because of the failure of Ψ as to B6 and B7.

These examples show the relativity of the term "compelling" and hence the convenience of looking at axioms and axiomatic characterizations with no constraints and to appreciate those axiomatizations that permit a comparison between different (coalitional, in this case) values. We hope that the reader will hold this view and agree, therefore, with our opinion so far expressed.

Remark 4.7 Finally, we would like to point out an additional criterion that supports the use of the Banzhaf–Owen value Ψ as power index and comes from a rather different, not axiomatic approach: the probabilistic one. We are referring to a nice paper by Laruelle and Valenciano [25] where three meanings of Ψ are provided in the voting context. By interpreting power as the ability—say, probability—to become decisive in a voting process, the authors state three interpretations of this coalitional value: (a) as a modified Banzhaf index of the given voting rule; (b) as the Banzhaf index of a modified voting rule; and (c) as an (extended) Banzhaf index of an (extended notion of) voting rule. Two conclusions of this article deserve also being mentioned: (1) the Banzhaf–Owen values of different agents can be compared *only in case of players of the same union*, a condition to be taken into account in the applications of this power index; (2) similar interpretations of other coalitional values in the voting context are problematic; in other words, the arguments given for Ψ do not adapt convincingly to them. Unfortunately, the authors leave to the reader the checking of this but give no hint.

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