AXIOMATIC CHARACTERIZATIONS OF THE SYMMETRIC COALITIONAL BINOMIAL SEMIVALUES *

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Abstract

The symmetric coalitional binomial semivalues extend the notion of binomial semivalue to games with a coalition structure, in such a way that they generalize the symmetric coalitional Banzhaf value. By considering the property of balanced contributions within unions, two axiomatic characterizations for each one of these values are provided.

Keywords: cooperative game, coalition structure, binomial semivalue.
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1 Introduction

Games with a *coalition structure* were first considered by Aumann and Drèze [2], who extended the Shapley value to this new framework in such a manner that the game really splits into subgames played by the unions isolatedly from each other, and every player receives the payoff allocated to him by the Shapley value in the subgame he is playing within his union. A second approach was used by Owen [7], when introducing and axiomatically characterizing his coalitional value (*Owen value*). In this case, the unions play a *quotient game* among themselves, and each one receives a payoff which, in turn, is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by the Shapley value.

By applying a similar procedure to the Banzhaf value, Owen [8] got a second coalitional value, the modified Banzhaf value for games with a coalition structure or

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Owen–Banzhaf value. In this case the payoffs at both levels, that of the unions in the quotient game and that of the players within each union, are given by the Banzhaf value.

Alonso and Fiestras [1] realized that the Owen–Banzhaf value fails to satisfy two interesting properties of the Owen value: symmetry in the quotient game and the quotient game property. Then they suggested to modify the two–step allocation scheme and use the Banzhaf value for sharing in the quotient game and the Shapley value within unions. This gave rise to the *symmetric coalitional Banzhaf value*.

The notion of p–binomial semivalue was first given by Puente [9]. Carreras and Puente [3] extended this notion to games with a coalition structure and obtained at the same time a wide generalization of the Alonso and Fiestras value (essentially: $p \in [0, 1]$ instead of $p = 1/2$), the family of *symmetric coalitional p–binomial semivalues*.

Our aim here is to provide two axiomatic characterizations for each symmetric coalitional binomial semivalue, both based on the interesting property of *balanced contributions within unions*. First we use it jointly with additivity, the dummy player property, symmetry in the quotient game and the coalitional p–binomial total power property. Next, we prove that the symmetric coalitional p–binomial semivalue is the unique coalitional value of the p–binomial semivalue that satisfies balanced contributions within unions and the quotient game property.

The organization of the paper is as follows. In Section 2, a minimum of preliminaries is provided. In Section 3 we recall the definition of the symmetric coalitional binomial semivalues, introduce the property of balanced contributions within unions and state and prove the characterization theorems.

2 Preliminaries

2.1 Games and semivalues

Let N be a finite set of *players* and 2^N be the set of its *coalitions* (subsets of N). A *cooperative game* on N is a function $v : 2^N \to \mathbb{R}$, that assigns a real number $v(S)$ to each coalition $S \subseteq N$ with $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. A player $i \in N$ is a *dummy* in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$. Two players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

Endowed with the natural operations for real–valued functions, the set of all cooperative games on N is a vector space $\mathcal{G}_N$. For every nonempty coalition $T \subseteq N$, the *unanimity game* u_T is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. Finally, every permutation θ of N induces a linear automorphism of $\mathcal{G}_N$ given by $(\theta v)(S) = v(\theta^{-1} S)$ for all $S \subseteq N$ and all v.

By a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \to \mathbb{R}^N$, that assigns to every game v a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

Following Weber's [12] axiomatic description, $\psi : \mathcal{G}_N \to \mathbb{R}^N$ is a *semivalue* iff it satisfies the following properties:

(i) *linearity*: $\psi[v+v'] = \psi[v]+\psi[v']$ (*additivity*) and $\psi[\lambda v] = \lambda\psi[v]$ for all $v, v' \in \mathcal{G}_N$ and $\lambda \in \mathbb{R}$;

(ii) *anonymity*: $\psi_{\theta i}[\theta v] = \psi_i[v]$ for all θ on N, $i \in N$, and $v \in \mathcal{G}_N$;

(iii) *positivity*: if v is monotonic, then $\psi[v] \geq 0$;

(iv) *dummy player property*: if $i \in N$ is a dummy in game v, then $\psi_i[v] = v(\{i\})$.

There is an interesting characterization of semivalues, by means of *weighting coefficients*, due to Dubey, Neyman and Weber [4]. Set $n = |N|$. Then: (a) for every *weighting vector* $\{p_k\}_{k=0}^{n-1}$ such that $\sum_{k=0}^{n-1} p_k\binom{n-1}{k} = 1$ and $p_k \geq 0$ for all k, the expression

$$\psi_i[v] = \sum_{S\subseteq N\setminus\{i\}} p_s[v(S\cup\{i\}) - v(S)] \quad \text{for all } i \in N \text{ and all } v \in \mathcal{G}_N,$$

where $s = |S|$, defines a semivalue ψ; (b) conversely, every semivalue can be obtained in this way; (c) the correspondence given by $\{p_k\}_{k=0}^{n-1} \mapsto \psi$ is bijective.

Well known examples of semivalues are the *Shapley value* φ (Shapley [10]), for which $p_k = 1/n\binom{n-1}{k}$, and the *Banzhaf value* β (Owen [6]), for which $p_k = 2^{1-n}$. The Shapley value φ is the only *efficient* semivalue, in the sense that $\sum_{i\in N} \varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$.

Notice that these two classical values are defined for each N. The same happens with the *binomial semivalues*, introduced by Puente [9] as follows. Let $p \in [0,1]$ and $p_k = p^k(1-p)^{n-k-1}$ for $k = 0, 1, \ldots, n-1$. Then $\{p_k\}_{k=0}^{n-1}$ is a weighting vector and defines a semivalue that will be denoted as ψ^p and called the *p–binomial semivalue*. Of course, $\psi^{1/2} = \beta$.

2.2 Games with a coalition structure

Let us consider a finite set, say, $N = \{1, 2, \ldots, n\}$. We will denote by $P(N)$ the set of all partitions of N. Each $P \in P(N)$ is called a *coalition structure* or *system of unions* on N. The so–called *trivial coalition structures* are $P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and $P^N = \{N\}$. A *cooperative game with a coalition structure* is a pair $[v; P]$, where $v \in \mathcal{G}_N$ and $P \in P(N)$ for a given N. We denote by $\mathcal{G}_N^{cs}$ the set of all cooperative games with a coalition structure and player set N.

If $[v; P] \in \mathcal{G}_N^{cs}$ and $P = \{P_1, P_2, \ldots, P_m\}$, the *quotient game* v^P is the cooperative game played by the unions, or, rather, by the set $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^P(R) = v(\bigcup_{r\in R} P_r) \quad \text{for all } R \subseteq M.$$

Unions P_r, P_s are said to be *symmetric* in $[v; P]$ if r, s are symmetric players in v^P.

By a *coalitional value* on $\mathcal{G}_N^{cs}$ we will mean a map $g : \mathcal{G}_N^{cs} \to \mathbb{R}^N$, which assigns to every pair $[v; P]$ a vector $g[v; P]$ with components $g_i[v; P]$ for each $i \in N$.

Given a value f on $\mathcal{G}_N$, a *coalitional value of* f is a coalitional value g on $\mathcal{G}_N^{cs}$ such that $g[v; P^n] = f[v]$ for all $v \in \mathcal{G}_N$.

3 The symmetric coalitional p–binomial semivalues

The symmetric coalitional p–binomial semivalue represents a two–step bargaining procedure where, first, the unions are allocated in the quotient game the payoff given by the p–binomial semivalue ψ^p and, then, this payoff is efficiently shared within each union according to the Shapley value φ.

Definition 3.1 (Carreras and Puente [3]) Let $p \in [0,1]$. For any fixed player set N, the *symmetric coalitional p–binomial semivalue* is the coalitional value Ω^p defined on $\mathcal{G}_N^{cs}$ by

$$\Omega_i^p[v;P] = \sum_{R \subseteq M\backslash\{k\}} \sum_{T \subseteq P_k\backslash\{i\}} p^r(1-p)^{m-r-1} \frac{1}{p_k\binom{p_k-1}{t}}[v(Q \cup T \cup \{i\}) - v(Q \cup T)]$$

for all $i \in N$ and $[v;P] \in \mathcal{G}_N^{cs}$, where $P_k \in P$ is the union such that $i \in P_k$ and $Q = \bigcup_{r \in R} P_r$. In case $p = 1/2$, we get $\Omega^{1/2} = \Pi$, the *symmetric coalitional Banzhaf value* introduced by Alonso and Fiestras [1].

Definition 3.2 A coalitional value g on $\mathcal{G}_N^{cs}$ satisfies the property of *balanced contributions within unions* if, for all $[v;P] \in \mathcal{G}_N^{cs}$, all $P_k \in P$ and all $i, j \in P_k$,

$$g_i[v;P] - g_i[v;P_{-j}] = g_j[v;P] - g_j[v;P_{-i}],$$

where P_{-i} is the coalition structure that results when player i leaves the union he belongs to, i.e.,

$$P_{-i} = \{P_1, \ldots, P_{k-1}, P_k\backslash\{i\}, P_{k+1}, \ldots, P_m, \{i\}\},$$

and P_{-j} is defined analogously. Notice that in P_{-i} player i does not leave the game, but only union P_k.

This property states that the loss (or gain) of a player $i \in P_k$ when a player $j \in P_k$ decides to leave the union and remain alone is the same as the loss (or gain) of player j when player i decides to leave the union. It is reminiscent of Myerson's [5] fairness concept.

Let us consider the following properties for a coalitional value g on $\mathcal{G}_N^{cs}$:

- *additivity*: $g[v+v';P] = g[v;P] + g[v';P]$ for all v, v' and P
- *dummy player property*: if i is a dummy in v, then $g_i[v;P] = v(\{i\})$ for all P
- *coalitional p–binomial total power property*: for all $[v;P] \in \mathcal{G}_N^{cs}$,

$$\sum_{i \in N} g_i[v;P] = \sum_{k \in M} \sum_{R \subseteq M\backslash\{k\}} p^r(1-p)^{m-r-1}[v^P(R \cup \{k\}) - v^P(R)]$$

- *symmetry in the quotient game*: if $P_r, P_s \in P$ are symmetric in $[v;P]$ then

$$\sum_{i \in P_r} g_i[v;P] = \sum_{j \in P_s} g_j[v;P]$$

- *quotient game property*: for all $[v;P] \in \mathcal{G}_N^{cs}$,

$$\sum_{i \in P_k} g_i[v;P] = g_k[v^P;P^m] \quad \text{for all } P_k \in P$$

(this property makes sense only for coalitional values defined for any N; here and in the sequel we abuse the notation and use a unique symbol g on both $\mathcal{G}_N^{cs}$ and $\mathcal{G}_M^{cs}$).

In the next theorem we give a first characterization of each symmetric coalitional p–binomial semivalue. We will need a lemma whose proof is straightforward.

Lemma 3.3 *Let $p \in [0,1]$, $\emptyset \neq S \subseteq N$, $s = |S|$ and $i \in N$. Then $\psi_i^p[u_S] = p^{s-1}$ if $i \in S$, and $\psi_i^p[u_S] = 0$ otherwise.* $\square$

Theorem 3.4 *(First axiomatic characterization) Let $p \in [0,1]$. For any N there is a unique coalitional value on $\mathcal{G}_N^{cs}$ that satisfies additivity, the dummy player property, balanced contributions within unions, the coalitional p–binomial total power property and symmetry in the quotient game. It is the symmetric coalitional p–binomial semivalue Ω^p.*

Moreover, Ω^p satisfies the quotient game property, is a coalitional value of the p–binomial semivalue ψ^p, and yields $\Omega^p[v;P^N] = \varphi[v]$ for all $v \in \mathcal{G}_N$.

Proof: (a) (Existence) It suffices to show that the coalitional value Ω^p satisfies the five properties enumerated in the statement.

1. Additivity. It merely follows from the expression of $\Omega_i^p[v;P]$.

2. Dummy player property. Let $i \in N$ be a dummy player in game v and P be any coalition structure. Assume $i \in P_k$. Then $v(Q \cup T \cup \{i\}) - v(Q \cup T) = v(\{i\})$ for all R and T. As, moreover,

$$\sum_{R \subseteq M \setminus \{k\}} p^r (1-p)^{m-r-1} = 1 \quad \text{and} \quad \sum_{T \subseteq P_k \setminus \{i\}} \frac{1}{p_k \binom{p_k - 1}{t}} = 1,$$

we conclude that $\Omega_i^p[v;P] = v(\{i\})$.

3. Balanced contributions within unions. Let us take $[v;P] \in \mathcal{G}_N^{cs}$, with $P = \{P_1, P_2, \ldots, P_m\}$ and $M = \{1, 2, \ldots, m\}$. Let $P_k \in P$ and $i, j \in P_k$. Then we have

$$\Omega_i^p[v;P] =$$

$$\sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i,j\}} p^r (1-p)^{m-r-1} \frac{(p_k - t - 1)! t!}{p_k!} [v(Q \cup T \cup \{i\}) - v(Q \cup T)] +$$

$$\sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i\} : j \in T} p^r (1-p)^{m-r-1} \frac{(p_k - t - 1)! t!}{p_k!} [v(Q \cup T \cup \{i\}) - v(Q \cup T)] =$$

$$\sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i,j\}} p^r (1-p)^{m-r-1} \frac{(p_k - t - 1)! t!}{p_k!} [v(Q \cup T \cup \{i\}) - v(Q \cup T)] +$$

$$\sum_{R\subseteq M\setminus\{k\}} \sum_{T\subseteq P_k\setminus\{i,j\}} p^r(1-p)^{m-r-1}\frac{(p_k-t-2)!(t+1)!}{p_k!}[v(Q\cup T\cup\{j\}\cup\{i\})-v(Q\cup T\cup\{j\})],$$

where $Q = \bigcup_{r\in R} P_r$. We now consider

$$P_{-j} = \{P'_1, P'_2, \ldots, P'_{m+1}\},$$

where $P'_h = P_h$ for $h \in \{1,\ldots,k-1,k+1,\ldots,m\}$, $P'_k = P_k\setminus\{j\}$, $P'_{m+1} = \{j\}$ and $M' = \{1,2,\ldots,m+1\}$, and get, in a similar way,

$$\Omega^p_i[v;P_{-j}] =$$

$$\sum_{R\subseteq M'\setminus\{k\}} \sum_{T\subseteq P'_k\setminus\{i\}} p^r(1-p)^{m-r}\frac{(p_k-t-2)!t!}{(p_k-1)!}[v(Q\cup T\cup\{i\})-v(Q\cup T)] =$$

$$\sum_{R\subseteq M\setminus\{k\}} \sum_{T\subseteq P_k\setminus\{i,j\}} p^r(1-p)^{m-r}\frac{(p_k-t-2)!t!}{(p_k-1)!}[v(Q\cup T\cup\{i\})-v(Q\cup T)]+$$

$$\sum_{R\subseteq M\setminus\{k\}} \sum_{T\subseteq P_k\setminus\{i,j\}} p^{r+1}(1-p)^{m-r-1}\frac{(p_k-t-2)!t!}{(p_k-1)!}[v(Q\cup T\cup\{j\}\cup\{i\})-v(Q\cup T\cup\{j\})].$$

Thus

$$\Omega^p_i[v;P] - \Omega^p_i[v;P_{-j}] =$$

$$\sum_{R\subseteq M\setminus\{k\}} \sum_{T\subseteq P_k\setminus\{i,j\}} A_1[v(Q\cup T\cup\{i\})-v(Q\cup T)]+A_2[v(Q\cup T\cup\{j\}\cup\{i\})-v(Q\cup T\cup\{j\})],$$

where

$$A_1 = p^r(1-p)^{m-r-1}\frac{(p_k-t-1)!t!}{p_k!} - p^r(1-p)^{m-r}\frac{(p_k-t-2)!t!}{(p_k-1)!}$$

and

$$A_2 = p^r(1-p)^{m-r-1}\frac{(p_k-t-2)!(t+1)!}{p_k!} - p^{r+1}(1-p)^{m-r-1}\frac{(p_k-t-2)!t!}{(p_k-1)!}.$$

It is easy to check that $A_2 = -A_1$, so that

$$\Omega^p_i[v;P] - \Omega^p_i[v;P_{-j}] =$$

$$\sum_{R\subseteq M\setminus\{k\}} \sum_{T\subseteq P_k\setminus\{i,j\}} A_1[v(Q\cup T\cup\{i\})+v(Q\cup T\cup\{j\})-v(Q\cup T)-v(Q\cup T\cup\{i\}\cup\{j\})].$$

Since this expression depends on i in the same way as it depends on j,

$$\Omega^p_i[v;P] - \Omega^p_i[v;P_{-j}] = \Omega^p_j[v;P] - \Omega^p_j[v;P_{-i}].$$

4. Coalitional p–binomial total power property. Let $[v;P] \in \mathcal{G}_N^{cs}$. Fixing $k \in M$, for every $R \subseteq M \backslash \{k\}$ we consider the game $v_R \in \mathcal{G}_{P_k}$ defined by

$$v_R(T) = v(Q \cup T) - v(Q) \quad \text{for all } T \subseteq P_k.$$

The Shapley value gives, for each $i \in P_k$,

$$\varphi_i[v_R] = \sum_{T \subseteq P_k \backslash \{i\}} \frac{1}{p_k \binom{p_k - 1}{t}} [v(Q \cup T \cup \{i\}) - v(Q \cup T)].$$

Using the efficiency of φ, we get

$$\sum_{i \in P_k} \varphi_i[v_R] = v_R(P_k) = v(Q \cup P_k) - v(Q) = v^P(R \cup \{k\}) - v^P(R).$$

Hence

$$\sum_{i \in P_k} \Omega_i^p[v;P] = \sum_{R \subseteq M \backslash \{k\}} p^r (1-p)^{m-r-1} [v^P(R \cup \{k\}) - v^P(R)] = \psi_k^p[v^P]$$

and, finally,

$$\sum_{i \in N} \Omega_i^p[v;P] = \sum_{k \in M} \sum_{R \subseteq M \backslash \{k\}} p^r (1-p)^{m-r-1} [v^P(R \cup \{k\}) - v^P(R)].$$

5. Symmetry in the quotient game. It readily follows from the relationship

$$\sum_{i \in P_k} \Omega_i^p[v;P] = \psi_k^p[v^P],$$

stated in the previous point, and the anonymity of the p–binomial semivalue ψ^p.

(b) (Uniqueness) Let g be a coalitional value that satisfies the above five properties. We will see that g is uniquely determined, so that $g = \Omega^p$.

Using additivity and the fact that the unanimity games form a basis of $\mathcal{G}_N$, it suffices to see that g is uniquely determined on each pair of the form $[\lambda u_T; P]$. So let $\lambda \in \mathbb{R}$, $\emptyset \neq T \subseteq N$ and $P \in P(N)$. Let $R = \{k \in M : T \cap P_k \neq \emptyset\}$ and $R_k = T \cap P_k$ for each $k \in R$.

Using the dummy player property it follows that $g_i[\lambda u_T; P] = 0$ if $i \notin T$. Now we apply the coalitional p–binomial total power property:

$$\sum_{i \in N} g_i[\lambda u_T; P] = \sum_{k \in M} \sum_{S \subseteq M \backslash \{k\}} p^s (1-p)^{m-s-1} [(\lambda u_T)^P (S \cup \{k\}) - (\lambda u_T)^P (S)].$$

It is easy to see that $(\lambda u_T)^P = \lambda u_T^P$. Then, by the definition of the p–binomial semivalue and its linearity, we have

$$\sum_{i \in N} g_i[\lambda u_T; P] = \sum_{k \in M} \psi_k^p[\lambda u_T^P] = \lambda \sum_{k \in M} \psi_k^p[u_T^P].$$

As, moreover, $u_T^P = u_R$, using Lemma 3.3 yields

$$\sum_{i \in N} g_i[\lambda u_T; P] = \lambda \sum_{k \in M} \psi_k^p[u_R] = \lambda \sum_{k \in R} p^{r-1} = \lambda r p^{r-1}.$$

Let $k \in R$. From the dummy player property and symmetry in the quotient game we get

$$\sum_{i \in R_k} g_i[\lambda u_T; P] = \sum_{i \in P_k} g_i[\lambda u_T; P] = \lambda p^{r-1}.$$

It remains to see that $g_i[\lambda u_T; P] = \dfrac{\lambda p^{r-1}}{r_k}$ for all $i \in R_k$. To this end, we use induction on $r_k = |R_k|$.

If $r_k = 1$ it is obvious because $R_k = \{i\}$. So, let $r_k > 1$. If $i, j \in R_k$, from balanced contributions within unions it follows that

$$g_i[\lambda u_T; P] - g_i[\lambda u_T; P_{-j}] = g_j[\lambda u_T; P] - g_j[\lambda u_T; P_{-i}].$$

Now, the cardinality of the corresponding subsets $(R_{-i})_k$ and $(R_{-j})_k$, for both P_{-i} and P_{-j}, is $r_k - 1$, whereas $|R_{-i}| = |R_{-j}| = r+1$, so that, by the inductive hypothesis,

$$g_i[\lambda u_T; P_{-j}] = \frac{\lambda p^r}{r_k - 1} = g_j[\lambda u_T; P_{-i}]$$

and hence

$$g_i[\lambda u_T; P] = \frac{\lambda p^{r-1}}{r_k} = g_j[\lambda u_T; P].$$

This completes the uniqueness proof.

(c) First, if $P = P^N$ then

$$\Omega_i^p[v; P^N] = \sum_{T \subseteq N \setminus \{i\}} \frac{1}{n\binom{n-1}{t}} [v(T \cup \{i\}) - v(T)] = \varphi_i[v]$$

for all $i \in N$ and all $v \in \mathcal{G}_N$. Analogously, Ω^p is a coalitional value of the p–binomial semivalue ψ^p. Indeed, for $P = P^n$

$$\Omega_i^p[v; P^n] = \sum_{R \subseteq N \setminus \{i\}} p^r (1-p)^{m-r-1} [v(R \cup \{i\}) - v(R)] = \psi_i^p[v].$$

Finally, the quotient game property: as we have seen when showing the symmetry in the quotient game in part (a) of this proof, and using the preceding property for $\mathcal{G}_M^{cs}$,

$$\sum_{i \in P_k} \Omega_i^p[v; P] = \psi_k^p[v^P] = \Omega_k^p[v^P; P^m]. \quad \square$$

In Vázquez et al. [11], it was shown that the Owen value is the unique coalitional value of the Shapley value that satisfies the properties of quotient game and balanced contributions within unions. Analogously, Alonso and Fiestras [1] proved that the symmetric coalitional Banzhaf value Π is the unique coalitional value of the Banzhaf value that satisfies these two properties. In the next theorem we generalize this result.

Theorem 3.5 *(Second axiomatic characterization) Let $p \in [0,1]$. The symmetric coalitional p–binomial semivalue Ω^p is the unique coalitional value of the p–binomial semivalue ψ^p defined for any N that satisfies balanced contributions within unions and the quotient game property.*

Proof: (a) (Existence) It follows from Theorem 3.4.

(b) (Uniqueness) Assume that $g^1 \neq g^2$ are coalitional values of the p–binomial semivalue ψ^p defined for any N and satisfying the above two properties. Let N be such that $g^1 \neq g^2$ on $\mathcal{G}_N^{cs}$ and take, among those $[v;P] \in \mathcal{G}_N^{cs}$ such that $g^1[v;P] \neq g^2[v;P]$, a pair $[v;P]$ with the maximum number of unions m.

As g^1 and g^2 satisfy the quotient game property, for all $k \in M$ we have

$$\sum_{i \in P_k} g_i^h[v;P] = g_k^h[v^P;P^m] \quad \text{for } h = 1,2,$$

and, from both being coalitional values of ψ^p (also on M, of course),

$$\sum_{i \in P_k} g_i^1[v;P] = \psi_k^p[v^P] = \sum_{i \in P_k} g_i^2[v;P] \quad \text{for all } k \in M,$$

so that g^1 and g^2 coincide (say, additively) on each union P_k. If $P_k = \{i\}$ then $g_i^1[v;P] = g_i^2[v;P]$. If $p_k > 1$, let $i,j \in P_k$ be distinct. Using the property of balanced contributions within unions,

$$g_i^h([v;P] - g_j^h[v;P] = g_i^h[v;P_{-j}] - g_j^h[v;P_{-i}] \quad \text{for } h = 1,2.$$

By the maximality of m, it follows that

$$g_i^1[v;P_{-j}] - g_j^1[v;P_{-i}] = g_i^2[v;P_{-j}] - g_j^2[v;P_{-i}]$$

and hence

$$g_i^1[v;P] - g_j^1[v;P] = g_i^2[v;P] - g_j^2[v;P],$$

that is,

$$g_i^1[v;P] - g_i^2[v;P] = c_k \;\; \text{(a constant)} \quad \text{for all } i \in P_k.$$

However

$$0 = \sum_{i \in P_k} g_i^1[v;P] - \sum_{i \in P_k} g_i^2[v;P] = p_k c_k,$$

so that $c_k = 0$ and therefore g^1 and g^2 coincide on each player of P_k; thus, $g^1 = g^2$ on N, a contradiction. $\square$

Remark 3.6 *(A third axiomatic characterization)* A further axiomatic characterization of each symmetric coalitional p–binomial semivalue Ω^p was carried out in Carreras and Puente [3] by just replacing the property of balanced contributions within unions with

- *symmetry within unions*: if $i,j \in P_k$ are symmetric in v then $g_i[v;P] = g_j[v;P]$.

Remark 3.7 *(Restriction to simple games)* Axiomatic characterizations, analogous to those of Theorems 3.4 and 3.5 (in case of Remark 3.6, see Carreras and Puente [3]), can be established for the restriction of each symmetric coalitional p–binomial semivalue Ω^p to the class of (monotonic) simple games by just replacing additivity with

- *transfer property*: $g[v \vee v'; P] = g[v; P] + g[v'; P] - g[v \wedge v'; P]$.

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