Modification of the Banzhaf value for games with a coalition structure

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In this paper we introduce a new coalitional value in the context of TU games with an *a priori* system of unions, which it is called the symmetric coalitional Banzhaf value. This value satisfies the property of symmetry in the quotient game, the quotient game property, and it is a coalitional value of Banzhaf. Several characterizations are provided and two political examples illustrate the differences with respect to the Owen value and the Banzhaf-Owen value.

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1. Introduction

An n-person cooperative game with transferable utility (TU game) models conflict situations where the involved agents can achieve binding agreements and the joint utility can be split in any way among the players. One of the most interesting questions that appears is which partition of the total utility will be unanimously accepted.

The most important answer was provided by Shapley [13] and is known as the Shapley value. Another interesting value was introduced by Banzhaf [3], initially proposed in the context of voting games and later on this value was extended to general TU games by Owen [9]. Their relevance is seen in the set of plausible properties that characterizes each one. Both are probabilistic values (Weber [17]). Thus a player's assessment is the average of his marginal contribution to any coalition to which the player does not belong. But they differ in the associated weights to each coalition. The Banzhaf value arises from considering that every player is equally likely to enter any coalition. The Shapley value considers that every player is equally likely to join any coalition of the same size and that all coalitions of a given size are equally likely. Moreover, any pair of symmetric players gets the same value in each solution. In addition, the Shapley value is efficient, while the Banzhaf value satisfies the total power property. Thus, the Shapley value distributes the total utility among players while the total amount that players get from Banzhaf's allocation depends on the structure of the TU game.

For representing cooperative situations more accurately, more sophisticated models appear. One of them is a TU game with an *a priori* system of unions. A system of unions is a partition of the player set which describes an *a priori* coalition structure.

The Owen value, proposed and characterized in [10], is an extension of the Shapley value to this context. This value initially splits the total amount among the unions, according to the Shapley value, in the induced game played by the unions (quotient game) and then once again using the Shapley value within each union, its total reward is allocated among its members (quotient game property), taking into account the possibilities of their joining other unions. One of the most appealing properties that the Owen value satisfies is the property of symmetry in the quotient game: given two unions which play symmetric roles in the quotient game, they are awarded the same apportionment of the total payoff.

Later on, Owen [11] proposed an extension of the Banzhaf value to the framework of TU games with a system of unions, that we will call Banzhaf-Owen value. This solution follows the expressed ideas for the Owen value above, but here the assessments in each step, are given by the Banzhaf value. Nevertheless, neither the property of symmetry in the quotient game nor the quotient game property are satisfied as we will see with an example.

Our aim in this paper is to propose a new value for TU games with a system of unions which is *a coalitional value of Banzhaf*, symmetric in the quotient game and satisfies the property of quotient game but not efficiency. Consequently, the total amount obtained for the players according to this new value depends on the structure of the game, in a similar way to the Banzhaf value. We will name this value *the symmetric coalitional Banzhaf value*. It turns out that the earnings of a player can be computed in two steps: first we calculate the total payoff that a union can obtain by means of the Banzhaf value and then, for each union its total allocation is split among its members using the Shapley value.

The paper is organized as follows. In section 2 below we recall some preliminary definitions and characterizations of the Shapley, Banzhaf, and Owen values. In section 3 we define and characterize *the symmetric coalitional Banzhaf value* in the class of TU games with an *a priori* system of unions and in the subclass of simple games. Finally, in section 4 by means of two political examples we illustrate the differences among the value we propose, the Banzhaf-Owen value and the Owen value.

2. Preliminaries

2.1. TU games

A TU game is a pair (N,v) where $N=\{1,\ldots,n\}$ is the set of players and v, the characteristic function, is a real-valued function on the subsets of N with $v(\emptyset)=0$. A subset $S\subseteq N$ is called a coalition. Given a coalition S, $v(S)$ is the amount of utility that the members of S can obtain from the game, whatever the players are not in S may do. We will use shorthand notation and write $S\cup i$ for the set $S\cup\{i\}$, and $S\backslash i$ for the set $S\backslash\{i\}$. We denote by $G(N)$ the set of TU-games with player set N, and identify (N,v) with v when no confusion can arise. Given a finite set T, we denote by t the cardinality of T, i.e., $|T|=t$.

A TU game (N,v) is superadditive if for every pair of disjoint coalitions $S,T\subseteq N, v(S\cup T)\geq v(S)+v(T)$. It is monotone if $v(S)\leq v(T)$ for every $S\subseteq T$. A simple game is a monotonic game (N,v) where v is ranged to $\{0,1\}$. In a simple game, coalitions S with $v(S)=1$ are called winning coalitions and coalitions with $v(S)=0$ losing coalitions. A winning coalition whose all proper subsets are losing coalitions is called a minimal winning coalition.

A null player in a TU game (N,v) is a player i such that $v(S)=v(S\backslash i)$ for all $S\subseteq N$ containing i. A carrier of a TU game (N,v) is a coalition $T\subseteq N$ such that $v(S)=v(S\cap T)$ for all $S\subseteq N$. Two players $i,j\in N$ are symmetric in a TU game (N,v) if $v(S\cup i)=v(S\cup j)$ for all $S\subseteq N\backslash\{i,j\}$.

Given a family of games $H\subseteq G(N)$, a solution (or a value) on H is a function $f:H\longrightarrow\mathbb{R}^n$ which assigns to a TU game $v\in H$ a vector $(f_1(v),\ldots,f_n(v))$ where the real number $f_i(v)$ is the value of the player i in the game (N,v) according to f. It is useful to single out a list of desirable properties of solutions:

1. A solution f is additive (ADD) if $f(v+w)=f(v)+f(w)$ for every $v,w\in H$, where $(v+w)(S)=v(S)+w(S)$.
2. A solution f satisfies the null player property (NP) if $f_i(v)=0$ for every $v\in H$ and every null player $i\in N$ of v.
3. A solution f is symmetric (SYM) if $f_i(v)=f_j(v)$, for every $v\in H$ and for every pair of symmetric players $i,j\in N$ of v.
4. A solution f is efficient (EFF) if $\sum\limits_{i\in N} f_i(v)=v(N)$ for every $v\in H$.
5. A solution f satisfies the Banzhaf total power property (TP) if

$$\sum_{i\in N} f_i(v)=\frac{1}{2^{n-1}}\sum_{i=1}^{n}\sum_{\substack{S\subseteq N\\ i\in S}}[v(S)-v(S\backslash i)] \text{ for every } v\in H.$$

The property of Banzhaf total power establishes that the total payoff obtained for the players is the sum of all marginal contributions of every player normalized by 2^{n-1}. It is clear that if a value is efficient then this value cannot satisfy the Banzhaf total power.

The most well-known values for $G(N)$ are the Shapley value [13] and the Banzhaf value [3]. In [6] a characterization of these values is given using some of the previous properties. We recall them here:

- The unique solution f defined on $G(N)$ that satisfies ADD, NP, SYM and EFF is the Shapley value. Given a game (N,v), this solution assigns to each player $i \in N$ the real number:

$$\varphi_i(N,v) = \sum_{S \subseteq N \backslash i} \frac{s!\,(n-s-1)!}{n!} \left(v(S \cup i) - v(S)\right).$$

- The unique solution f defined on $G(N)$ that satisfies ADD, NP, SYM and TP is the Banzhaf value. Given a game (N,v), this solution assigns to each player $i \in N$ the real number:

$$\beta_i(N,v) = \sum_{S \subseteq N \backslash i} \frac{1}{2^{n-1}} \left(v(S \cup i) - v(S)\right).$$

Notice that only one property differentiates both characterizations: the Shapley value satisfies efficiency while the Banzhaf value satisfies total power. Other characterizations of the Shapley and Banzhaf values are given in [4], [7], [8] and [13].

2.2. Games with a system of unions

A system of unions (or a coalition structure) in an n-person TU game is a partition of the set of players. For any set T, we denote by $P(T)$ the set of partitions of T and by P^t the trivial system of unions for the set T defined by $P^t = \{\{x\} : x \in T\}$. A TU game with a system of unions is a triplet (N,v,P), where (N,v) is a TU game and $P \in P(N)$. The partition P gives the *a priori* structure of cooperation among players. We denote by $U(N)$ the set of TU games with a system of unions and player set N.

If $(N,v,P) \in U(N)$, with $P = \{P_i : i \in M,\ M = \{1,\ldots,m\}\}$, the quotient game v^P is the TU game played by the unions, i.e.

$$\left(M, v^P\right) \in G(M) \textit{ and } v^P(R) = v\left(\bigcup_{k \in R} P_k\right) \textit{ for all } R \subseteq M.$$

Two unions $P_k, P_s \in P$ are symmetric if k and s are symmetric players in $\left(M, v^P\right)$.

Given a family of games $H \subseteq U(N)$, a solution (or a value) on H is a function $f : H \longrightarrow \mathbb{R}^n$ which assigns to a game $(N,v,P) \in H$ a vector $(f_1(N,v,P), \ldots, f_n(N,v,P))$ where the real number $f_i(N,v,P)$ is the value of the player i in the game (N,v,P) according to f.

In this framework, two interesting properties for a solution are:

1. A solution f is symmetric in the unions (SU) if for every $(N, v, P) \in H$ it satisfies that $f_i(N, v, P) = f_j(N, v, P)$, for all pair of symmetric players $i, j \in P_k$.
2. A solution f is symmetric in the quotient (SQ) if for every $(N, v, P) \in H$ it satisfies that $\sum_{i \in P_k} f_i(N, v, P) = \sum_{j \in P_s} f_j(N, v, P)$, for all pair of symmetric unions $P_k, P_s \in P$.

Owen [10] proposed a modification of the Shapley value for $U(N)$, the Owen value. Among others, the previous properties are used in the following characterization:

- The unique solution f defined on $U(N)$ that satisfies ADD, NP, SU, SQ and EFF is the Owen value. Given a game $(N, v, P) \in U(N)$ this solution assigns to each player $i \in N$ the real number:

$$\Phi_i(N, v, P) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{mp_k} \frac{1}{\binom{m-1}{r}} \frac{1}{\binom{p_k-1}{t}} \left(v(Q \cup T \cup i) - v(Q \cup T)\right) \tag{2.1}$$

where $P_k \in P$ is the union such that $i \in P_k$ and $Q = \bigcup_{r \in R} P_r$.

Other characterizations of the Owen value can be found in [15], [16] and [18].

In [12] a modification (without characterization) of the Banzhaf value is proposed. Given a game $(N, v, P) \in U(N)$ this value, that we will call Banzhaf-Owen value, assigns to each player $i \in N$ the real number:

$$\Psi_i(N, v, P) = \sum_{R \subseteq M \setminus k} \sum_{T \subseteq P_k \setminus i} \frac{1}{2^{m-1}} \frac{1}{2^{p_k-1}} \left(v(Q \cup T \cup i) - v(Q \cup T)\right) \tag{2.2}$$

where $P_k \in P$ is the union such that $i \in P_k$ and $Q = \bigcup_{r \in R} P_r$.

Next example shows that the Banzhaf-Owen value does not satisfy SQ.

Example 1. Given a coalition $T \subseteq N$, the unanimity game of T, (N, u_T), is a simple game defined as:

$$u_T(S) = \begin{cases} 1 & \textit{for every } S \subseteq N \textit{ with } T \subseteq S \\ 0 & \textit{for every } S \subseteq N \textit{ with } T \nsubseteq S \end{cases}$$

Let us consider the five-person simple game (N, u_N), where $N = \{1, 2, 3, 4, 5\}$ and u_N is the unanimity game of the grand coalition. Let us take the coalition structure $P = \{P_1, P_2\}$, where $P_1 = \{1, 2, 3\}$ and $P_2 = \{4, 5\}$. As it is easy to verify, the Banzhaf-Owen value for this game is

$$\Psi(N, u_N, P) = (1/8, 1/8, 1/8, 1/4, 1/4)$$

The quotient game u_N^P is the unanimity game with player set $M = \{1, 2\}$. In this game, both players are symmetric, but $\sum_{i \in P_1} \Psi_i (N, u_N, P) \neq \sum_{i \in P_2} \Psi_i (N, u_N, P)$.

3. The symmetric coalitional Banzhaf value

In this section we suggest another modification of the Banzhaf value for games with an *a priori* system of unions for which we will provide two characterizations.

First of all, we will propose a modification of the total power property, taking into account that the existence of a coalition structure limits the cooperation among players. In the definition of this new property, we will use the property of quotient game (see [18]) and the concept of coalitional value.

A solution $f : U(N) \longrightarrow \mathbb{R}^n$ satisfies the quotient game property (QG) if for all $(N, v, P) \in U(N)$ and all $P_k \in P$

$$\sum_{i \in P_k} f_i (N, v, P) = f_k \left(M, v^P, P^m \right).$$

This property states that in a game with an *a priori* system of unions, the joint utility obtained by the players of a union coincides with the utility obtained by this union in the quotient game.

This property is satisfied by the Owen value but not by the Banzhaf-Owen value as you can easily see from example 1.

For any game $(N, v, P^n) \in U(N)$, the Owen value coincides with the Shapley value of (N, v). In this sense, the Owen value is a generalization of the Shapley value. To capture this idea, in [15] the notion of coalitional Shapley value is introduced. We generalize this idea to any solution by means of the notion of *coalitional value.*

Definition 1. *Given a value* $f : G(N) \longrightarrow \mathbb{R}^n$, *a coalitional value of* f *is a solution* g *defined in* $U(N)$ *which assigns to every game* $(N, v, P) \in U(N)$ *an element of* $\mathbb{R}^n$ *in such a way that* $g(N, v, P^n) = f(N, v)$.

Our interest is restricted to *coalitional values* of the Banzhaf value (*coalitional Banzhaf values*). This is equivalent to restricting attention to values which satisfy any set of properties that characterize the Banzhaf value. Besides, if this new value f satisfies QG, we obtain that, for any game $(N, v, P) \in U(N)$ with $P = \{P_i : i \in M,\ M = \{1, \ldots, m\}\}$,

$$\sum_{i \in N} f_i (N, v, P) = \sum_{k \in M} \sum_{i \in P_k} f_i (N, v, P) = \sum_{k \in M} f_k \left(M, v^P, P^m \right).$$

Moreover, *a coalitional Banzhaf value* satisfies:

$$\sum_{k\in M} f_k\left(M, v^P, P^m\right) = \sum_{k\in M} \beta_k\left(M, v^P\right) = \frac{1}{2^{m-1}} \sum_{k\in M} \sum_{T\subseteq M\backslash k} \left[v^P\left(T\cup k\right) - v^P\left(T\right)\right].$$

Definition 2. *A solution $f : U(N) \longrightarrow \mathbb{R}^n$ satisfies the coalitional total power property (CTP) if for all $(N, v, P) \in U(N)$,*

$$\sum_{i\in N} f_i\left(N, v, P\right) = \frac{1}{2^{m-1}} \sum_{k\in M} \sum_{T\subseteq M\backslash k} \left[v^P\left(T\cup k\right) - v^P\left(T\right)\right],$$

with $P = \{P_i : i \in M,\ M = \{1, \ldots, m\}\}$.

Now, we propose and characterize a new value in the next result:

Theorem 1. *There is a unique solution f defined on $U(N)$ that satisfies ADD, NP, SU, SQ and CTP. We will refer to this value as the symmetric coalitional Banzhaf value. Given a game $(N, v, P) \in U(N)$ this solution assigns to each player $i \in N$ the real number:*

$$\Pi_i\left(N, v, P\right) = \sum_{R\subseteq M\backslash k} \sum_{T\subseteq P_k\backslash i} \frac{1}{2^{m-1}} \frac{t!\left(p_k - t - 1\right)!}{p_k!} \left(v\left(Q\cup T\cup i\right) - v\left(Q\cup T\right)\right) \tag{3.1}$$

where $P_k \in P$ is the union such that $i \in P_k$ and $Q = \bigcup\limits_{r\in R} P_r$.

Proof.

(1) Existence. We show that Π satisfies the properties enumerated in the theorem. First of all, we prove that Π satisfies the property of coalitional total power. Given $(N, v, P) \in U(N)$

$$\sum_{i\in N} \Pi_i\left(N, v, P\right) = \sum_{k\in M} \sum_{i\in P_k} \Pi_i\left(N, v, P\right) =$$

$$\sum_{k\in M} \sum_{R\subseteq M\backslash k} \frac{1}{2^{m-1}} \sum_{i\in P_k} \sum_{T\subseteq P_k\backslash i} \frac{t!\left(p_k - t - 1\right)!}{p_k!} \left[v\left(Q\cup T\cup i\right) - v\left(Q\cup T\right)\right]. \tag{3.2}$$

For every $P_k \in P$, every $R \subseteq M\backslash k$, we consider $Q = \bigcup\limits_{r\in R} P_r$ and the TU game $\left(P_k, \tilde{v}^Q\right) \in G\left(P_k\right)$, with characteristic function $\tilde{v}^Q\left(T\right) = v\left(Q\cup T\right) - v\left(Q\right)$, for all $T \subseteq P_k$. Then, the Shapley value of a player $i \in P_k$ is equal to:

$$\varphi_i\left(P_k, \tilde{v}^Q\right) = \sum_{T\subseteq P_k\backslash i} \frac{t!\left(p_k - t - 1\right)!}{p_k!} \left[\tilde{v}^Q\left(T\cup i\right) - \tilde{v}^Q\left(T\right)\right] \tag{3.3}$$

$$= \sum_{T \subseteq P_k \backslash i} \frac{t!\,(p_k - t - 1)!}{p_k!} \left[v\left(Q \cup T \cup i\right) - v\left(Q \cup T\right)\right].$$

By the efficiency of the Shapley value, we obtain

$$\sum_{i \in P_k} \varphi_i\left(P_k, \tilde{v}^Q\right) = \tilde{v}^Q\left(P_k\right) = v\left(Q \cup P_k\right) - v\left(Q\right).$$

Inserting this result in (3.2), we have:

$$\sum_{i \in N} \Pi_i\left(N, v, P\right) = \sum_{k \in M} \sum_{R \subseteq M \backslash k} \frac{1}{2^{m-1}} \left[v\left(Q \cup P_k\right) - v\left(Q\right)\right]$$

$$= \frac{1}{2^{m-1}} \sum_{k \in M} \sum_{R \subseteq M \backslash k} \left[v^P\left(R \cup k\right) - v^P\left(R\right)\right].$$

From the expression (3.1) it is easy to prove that the previous value satisfies the properties of null player and symmetry in the unions.

We proved that the amount obtained by a union P_k is equal to

$$\sum_{i \in P_k} \Pi_i\left(N, v, P\right) = \frac{1}{2^{m-1}} \sum_{R \subseteq M \backslash k} \left[v^P\left(R \cup k\right) - v^P\left(R\right)\right] \tag{3.4}$$

Therefore, this value satisfies the property of symmetry in the quotient: given two symmetric unions $P_k, P_s \in P$ it holds that $v^P\left(R \cup k\right) = v^P\left(R \cup s\right)$, for all $R \subseteq M \backslash \{k, s\}$ and, $v^P\left(R \cup k\right) = v^P\left(T \cup s\right)$, where $T = \left(R \backslash s\right) \cup \{k\}$, for all $R \subseteq M \backslash k$ and $s \in R$. Thus, from (3.4), $\sum_{i \in P_k} \Pi_i\left(N, v, P\right) = \sum_{j \in P_s} \Pi_j\left(N, v, P\right)$. Finally, the previous expression is linear in v, and then, it is additive.

(2) Uniqueness. It is sufficient to prove that any solution f that satisfies the properties enumerated in the theorem coincides to Π for the unanimity games. The property of additivity and the fact that the family of unanimity games forms a basis, guarantees the uniqueness of this value for the family of TU games with a system of unions.

Let us consider a finite set N and the unanimity game u_T, with $T \subseteq N$. Given a partition $P = \{P_i : i \in M,\ M = \{1, \ldots, m\}\} \in P\left(N\right)$ we consider $T' = \{j \in M / P_j \cap T \neq \emptyset\}$ and $T'_j = T \cap P_j$. The quotient game $\left(M, u_T^P\right)$ is defined as

$$u_T^P\left(R\right) = \begin{cases} 1 & \textit{if } T' \subseteq R \\ 0 & \textit{if } T' \nsubseteq R \end{cases}$$

for all $R \subseteq M$.

Let f be a solution that satisfies the mentioned properties. Since f satisfies the null player property, $f_i\left(N, u_T, P\right) = 0$, for every $i \notin T$. Moreover, from the properties of coalitional total power, null player and symmetry in the quotient, it is derived that

$$\sum_{i\in P_k} f_i\left(N,u_T,P\right)=\begin{cases}0 & if\ k\notin T'\\ \frac{1}{2^{t'-1}} & if\ k\in T'\end{cases}$$

for all $k\in M$.

A solution f that satisfies the property of symmetry in the unions, gives to every player which belongs to T'_k the same value. Thus, we obtain:

$$f_i\left(N,u_T,P\right)=\begin{cases}0 & if\ i\notin T\\ \frac{1}{t'_k 2^{t'-1}} & if\ i\in T'_k\end{cases}$$

On the other hand, if $i\notin T$ it is easy to prove that $\Pi_i\left(N,u_T,P\right)=0$. When $i\in T$ then, $i\in T'_k=T\cap P_k$ for some $k\in M$, and $u_T\left(Q\cup S\cup i\right)-u_T\left(Q\cup S\right)=1$ if and only if $Q=\bigcup\limits_{r\in R} P_r$ where $T'\backslash k\subseteq R\subseteq M\backslash k$, and $T'_k\backslash i\subseteq S\subseteq P_k\backslash i$. Thus, if $i\in T'_k$

$$\Pi_i\left(N,u_T,P\right)=\sum_{T'\backslash k\subseteq R\subseteq M\backslash k}\ \sum_{T'_k\backslash i\subseteq S\subseteq P_k\backslash i}\frac{1}{2^{m-1}}\frac{s!\left(p_k-s-1\right)!}{p_k!}=\frac{1}{t'_k 2^{t'-1}}.$$

Then, f and Π coincides for any game (N,u_T,P) and the proof is finished. ■

This result reveals that the satisfied properties by the symmetric coalitional Banzhaf value and the Owen value are very similar. The unique difference lies in the former satisfies the property of coalitional total power and the latter the efficiency property. Notice that this fact is in correspondence with the distinction between the Banzhaf value and the Shapley value reflected previously.

It follows from expression (3.4) that the amount assigned to a union P_k by the value Π coincides to the Banzhaf value in the corresponding quotient game. Then, this quantity is split among the members of P_k according to the Shapley value (see expression (3.3)).

In [14] it is proved that the Owen value satisfies the property of balanced contributions for the unions, and in [15] Vázquez Brage *et al.* characterized the Owen value as the unique coalitional Shapley value which satisfies the properties of quotient game and balanced contributions for the unions. Next we will see that the symmetric coalitional Banzhaf value satisfies analogous properties.

Lemma 2. *The symmetric coalitional Banzhaf value is a coalitional Banzhaf value and satisfies the quotient game property.*

Proof. (1) Let us take a TU game (N,v). Let us consider the trivial system of unions P^n and the game (N,v,P^n). Since the cardinality of each union P_j is one, by substitution in (3.1), we get $\Pi_i\left(N,v,P^n\right)=\beta_i\left(N,v\right)$. This means that Π is a coalitional value of Banzhaf.

(2) From the expression (3.4), it immediately follows that the symmetric coalitional Banzhaf value satisfies the quotient game property. ■

The property of balanced contributions for the unions states that the loss (or gain) of a player $i \in P_k$ when a player $j \in P_k$ decides to leave the union and be alone is the same as the loss (or gain) of player j when player i decides to leave the union. This property reflects the idea that all players in a union should profit equally from joining the union.

A solution $f : U(N) \longrightarrow \mathbb{R}^n$ satisfies the property of balanced contributions for the unions (BCU) if for all $(N, v, P) \in U(N)$, all $P_k \in P$ and all $i, j \in P_k$.

$$f_i(N, v, P) - f_i(N, v, P_{-j}) = f_j(N, v, P) - f_j(N, v, P_{-i}),$$

where P_{-i} is the system of unions that results when player i separates from the union he belongs to, i.e., $P_{-i} = \{P_1, \ldots, P_{k-1}, P_k \backslash i, P_{k+1}, \ldots, P_m, \{i\}\}$, and P_{-j} is defined analogously.

Below, we prove that Π satisfies the previous property.

Lemma 3. *The symmetric coalitional Banzhaf value satisfies BCU.*

Proof. Let us take a game $(N, v, P) \in U(N)$, with $P = \{P_1, P_2, \ldots, P_m\}$ and $M = \{1, 2, \ldots, m\}$. Let $P_k \in P$ and $i, j \in P_k$. By (3.1),

$$\Pi_i(N, v, P) = \sum_{R \subseteq M \backslash k} \sum_{T \subseteq P_k \backslash \{i,j\}} \frac{1}{2^{m-1}} \frac{1}{p_k} \frac{1}{\binom{p_k - 1}{t}} [v(Q \cup T \cup i) - v(Q \cup T)]$$

$$+ \sum_{R \subseteq M \backslash k} \sum_{T \subseteq P_k \backslash \{i,j\}} \frac{1}{2^{m-1}} \frac{1}{p_k} \frac{1}{\binom{p_k - 1}{t+1}} [v(Q \cup T \cup j \cup i) - v(Q \cup T \cup j)],$$

where $Q = \bigcup_{r \in R} P_r$. We consider

$$P_{-j} = \{P'_1, P'_2, ..., P'_{m+1}\}$$

where $P'_l = P_l$, for all $l \in \{1, ..., k-1, k+1, ..., m\}$, $P'_k = P_k \backslash j$, $P'_{m+1} = \{j\}$ and $M' = \{1, 2, ..., m+1\}$.

Then, we can write

$$\Pi_i(N, v, P_{-j}) = \sum_{R \subseteq M' \backslash k} \sum_{T \subseteq P'_k \backslash i} \frac{1}{2^m} \frac{1}{p_k - 1} \frac{1}{\binom{p_k - 2}{t}} [v(Q \cup T \cup i) - v(Q \cup T)]$$

$$= \sum_{R \subseteq M \backslash k} \sum_{T \subseteq P_k \backslash \{i,j\}} \frac{1}{2^m} \frac{1}{p_k - 1} \frac{1}{\binom{p_k - 2}{t}} [v(Q \cup T \cup i) - v(Q \cup T)]$$

$$+\sum_{R\subseteq M\backslash k}\sum_{T\subseteq P_k\backslash\{i,j\}}\frac{1}{2^m}\frac{1}{p_k-1}\frac{1}{\binom{p_k-2}{t}}\left[v\left(Q\cup T\cup j\cup i\right)-v\left(Q\cup T\cup j\right)\right].$$

So:

$$\Pi_i\left(N,v,P\right)-\Pi_i\left(N,v,P_{-j}\right)=$$

$$\sum_{R\subseteq M\backslash k}\sum_{T\subseteq P_k\backslash\{i,j\}}A_1\left[v\left(Q\cup T\cup i\right)-v\left(Q\cup T\right)\right]$$

$$+\sum_{R\subseteq M\backslash k}\sum_{T\subseteq P_k\backslash\{i,j\}}A_2\left[v\left(Q\cup T\cup j\cup i\right)-v\left(Q\cup T\cup j\right)\right],$$

where

$$A_1=\frac{1}{2^{m-1}}\frac{1}{p_k}\frac{1}{\binom{p_k-1}{t}}-\frac{1}{2^m}\frac{1}{p_k-1}\frac{1}{\binom{p_k-2}{t}}$$

and

$$A_2=\frac{1}{2^{m-1}}\frac{1}{p_k}\frac{1}{\binom{p_k-1}{t+1}}-\frac{1}{2^m}\frac{1}{p_k-1}\frac{1}{\binom{p_k-2}{t}}.$$

It is straightforward to check that $A_1+A_2=0$.

Then,

$$\Pi_i\left(N,v,P\right)-\Pi_i\left(N,v,P_{-j}\right)=$$

$$\sum_{R\subseteq M\backslash k}\sum_{T\subseteq P_k\backslash\{i,j\}}A_1\left[v\left(Q\cup T\cup i\right)-v\left(Q\cup T\right)-v\left(Q\cup T\cup j\cup i\right)+v\left(Q\cup T\cup j\right)\right].$$

Since the last expression depends on i in the same way as it depends on j,

$$\Pi_i\left(N,v,P\right)-\Pi_i\left(N,v,P_{-j}\right)=\Pi_j\left(N,v,P\right)-\Pi_j\left(N,v,P_{-i}\right).$$

and the proof is finished. ■

Theorem 4. *The symmetric coalitional Banzhaf value* (Π) *is the unique coalitional value of Banzhaf satisfying balanced contributions for the unions and the quotient game property.*

Proof. (1) Existence. It follows from Lemmas 2 and 3 that the symmetric coalitional Banzhaf value satisfies the mentioned properties.

(2) Uniqueness. The proof immediately follows from similar reasoning to that found in [14], where it is shown that there is a unique coalitional value of

Shapley that satisfies the properties of quotient game and balanced contributions for the unions. ■

An important subclass of TU games are simple games; this is due to the fact that simple games have a lot of applications, specially in the field of political sciences for modelling voting systems. Usually, the restriction of a solution to the family of simple games is called a power index. A great deal of work concerning to the game theory is centered in the study of power indices.

With only a few changes, we propose a characterization for the symmetric coalitional Banzhaf value in the class of simple games. These changes came from the fact that in the class of simple games the additivity property is useless because the sum of two simple games is not a simple game. Dubey [4] proposed the transfer property as a substitute of the additivity property to characterize the Shapley value in this class of games. We denote by $S(N)$ the set of simple games with player set N and by $SU(N)$ the set of simple games with *a priori* unions and player set N.

A solution (power index) defined on $S(N)$ satisfies the transfer property (TR) if for every $v, w \in S(N)$, $f(v \wedge w) + f(v \vee w) = f(v) + f(w)$ where $(v \wedge w)(S) = \min\{v(S), w(S)\}$ and $(v \vee w)(S) = \max\{v(S), w(S)\}$.

Next, we summarized the characterizations of the Shapley value [4] and the Banzhaf value [5] on the family of simple games and the Owen value [16] on the family of simple games with *a priori* unions.

- The unique solution f defined on $S(N)$ that satisfies TR, NP, SYM and EFF is the Shapley value.
- The unique solution f defined on $S(N)$ that satisfies TR, NP, SYM and TP is the Banzhaf value.
- The unique solution f defined on $SU(N)$ that satisfies TR, NP, SU, SQ and EFF is the Owen value.

In a similar way, we obtain a characterization for the symmetric coalitional Banzhaf value on the family of simple games with *a priori* unions.

Corollary 5. *The unique solution f defined on $SU(N)$ that satisfies TR, NP, SU, SQ and CTP is the symmetric coalitional Banzhaf value.*

4. Applications

4.1. The Parliament of Balearic Islands.

The Parliament of Balearic Islands, one of the Spain's seventeen autonomous communities, is made up of 59 members. Following elections on 13 June, 1999, the Parliament was composed of 28 members of the conservative party PP, 16 members of the socialist party $PSOE$, 5 members of the socialist regional party

$PSM-EN$, 4 members of $EU-EV$, a coalition of communist and other left-wing parties, 3 members of the middle-of-the-road regional party UM, and 3 members of $PROG$, a coalition of progressive parties that presented candidature only on some island.

Analyzing this Parliament as a weighted majority game, the quota is 30 and all the minimal winning coalitions are: $\{PP, PSOE\}$, $\{PP, EU-EV\}$, $\{PSOE, PSM-EN, UM, EU-EV, PROG\}$, $\{PP, PROG\}$, $\{PP, UM\}$ and $\{PP, PSM-EN\}$.

We can expect the Presidency of the government to be occupied by a member of PP. The power indices computed by means of the Shapley value (φ) and the Banzhaf value (β), that appear in Table 1, would confirm this hypothesis:

Table 1

Party	φ	β
PP	2/3	15/16
PSOE	1/15	1/16
EU-EV	1/15	1/16
PROG	1/15	1/16
PSM-EN	1/15	1/16
UM	1/15	1/16

We may presume the existence of agreements among the left-wing parties. Moreover, from the regionalist ideas of $PSM-EN$ and UM we may assume that both parties will form a union. Then, we consider the following *a priori* unions structure:

$$P = \{\{PP\}, \{PSOE, EU-EV, PROG\}, \{PSM-EN, UM\}\},$$

where P_1 has 28 seats, P_2 has 23 seats and P_3 has 8 seats.

With this system of unions, we can recalculate the power of each party represented in this Parliament by means of values for games with *a priori* unions. In Table 2, we present three power indices: the Owen value (Φ), the Banzhaf-Owen value (Ψ) and the symmetric coalitional Banzhaf value (Π).

Table 2

Party	Φ	Ψ	Π
PP	1/3	1/2	1/2
PSOE	1/9	1/8	1/6
EU-EV	1/9	1/8	1/6
PROG	1/9	1/8	1/6
PSM-EN	1/6	1/4	1/4
UM	1/6	1/4	1/4

Except for PP, the total power corresponding to each party is greater. The ruling coalition is formed by left-wing parties and progressive parties. In this example, Π and Φ reflect the existing symmetry among unions in the quotient game. The value Ψ does not respect this property (by example, the second and the third union are symmetric, but the second union receives 3/8 and the third one receives 1/2). As we mentioned before, the main difference between Π and Φ is the sum of its components.

4.2. The Council of the European Union

At the ministerial level the Council of the European Union is composed of one representative from each Member State, at present 15. Each representative is empowered to take a decision for his Government. Which Ministers attend each Council meeting varies according to the subject discussed, yet its institutional unity remains intact.

One of the responsibilities of the Council is to define and implement common foreign and security policy. Some decisions about these issues have to be taken unanimously.

When we represent this decision rule by a simple game, all members in the Council are symmetric. The allocation that the Shapley value assigns to each member is 1/15. The Banzhaf value gives $1/2^{14}$ to each representative.

But, on several occasions there is an *a priori* cooperation structure that we add to the simple game through a system of unions. For instance, considering three unions with five members each, it is clear that every pair of unions is symmetric and, inside a union, each pair of representatives is also symmetric. In this context, the allocations given by the Owen value, the Banzhaf-Owen value and the symmetric coalitional Banzhaf value are $1/15, 1/2^6$ and 1/20, respectively.

Notice that even this situation is qualitatively different from the previous one since an *a priori* coalition structure is given, the Owen value makes no distinction.

The allocation given by the Banzhaf-Owen value is the same for each union ($5/2^6$), but this amount differs from the Banzhaf value of a union (1/4) in the quotient game. Nevertheless, the Banzhaf-Owen value as well as the symmetric coalitional Banzhaf value assign a considerably higher index of power to each representative and to each union. This is due to the fact that in a first stage only the decisions of representatives of the three unions are important, thus, it is easier to achieve an agreement than in the first situation where the positive vote of 15 members is necessary.

5. Final Remarks

During the realization of this paper, two characterizations of the Banzhaf-Owen value appear (see [1] and [2]).

It is easy to prove that the symmetric coalitional Banzhaf value is an extension of the Shapley value if the system of unions is formed by one coalition, *i.e.*, if $P = \{N\}$ then $\varphi(N, v) = \Pi(N, v, P)$.

Two possible extensions of the above work seem worthwhile. One deals with the extension of other values, *i.e.*, coalitional versions of values defined on $G(N)$ by means of properties such as the quotient game, symmetry in the unions and so on.

A second interesting extension deals with other possibilities of modification of the Banzhaf value, for example, in games with graph-restricted communication or in games with incompatible players.

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