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FROM SET-VALUED SOLUTIONS TO SINGLE-VALUED SOLUTIONS: THE CENTROID AND THE CORE-CENTER

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From set-valued solutions to single-valued solutions: The Centroid and the Core-Center*

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Abstract

This paper establishes a direct connection between a set-valued solution and a particular singled-valued solution obtained from it. Considering all the allocations provided by the set-valued solution from a probabilistic point of view, we define a single-valued allocation, namely, the centroid of the distribution. Based on that, we present a new solution concept for balanced games, the core-center, with at least two good properties: it is in the core of the game and responds to a principle of fairness that leads to an axiomatic characterization.

Keywords. cooperative games, TU games, set-valued solutions, single-valued solutions, Shapley value

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Introduction

In the framework of cooperative game theory there are several solution concepts which gave rise to different ways of dividing the worth of the grand coalition $v(N)$ among the players. We work on the set of transferable utility games, shortly, TU games. Although solution concepts admit different classifications, we divide them into two groups: set-valued solutions and single-valued solutions. Roughly speaking, set-valued solutions provide a set of outcomes that can be finite, infinite, or even empty. The way to determine a set-valued solution can be seen as a procedure in which the set of all initial assignments is gradually reduced, according to specific criteria based on the desirable properties that a solution should possess, until the final solution (not necessarily a singleton) is reached. Examples of this type are the stable sets (von Neumann and Morgenstern (1944)), the core (Gillies (1953)), the kernel (Davis and Maschler (1965)), the bargaining sets (Aumann and Maschler (1964)) etc. On the other hand, one can establish some properties or axioms that determine a unique solution for each game, this is known as an allocation rule (single-valued solution). The Shapley value (Shapley (1953)), the nucleolus (Schmeidler (1969)), and the τ-value (Tijs (1981)) are solutions of this type.

Each solution concept has its interpretation and attends to specific principles (fairness, stability...) and all of them enrich the theory of cooperative games. Besides, there have been many papers discussing on relations between single-valued solutions and set-valued solutions. Let us just mention a couple of them: when the core is non-empty, the nucleolus selects an imputation in it; and for the class of convex games, the Shapley value is in the core.

This work establishes a direct connection between a set-valued solution and a particular singled-valued solution obtained from it. This is made from a new angle: given a nonempty set-valued solution and a probability distribution defined over it, we choose a single-valued solution that inherits a large number of properties of the set. Precisely, we give a procedure that selects a unique outcome from a set-valued solution with, at least, two good features: it is in the convex hull of the set and it responds to some principle of fairness if players agree on the probability measure. In particular, if we consider as the set-valued solution the set of all marginal contributions vectors with their corresponding weights we obtain the Shapley value. Besides, recently González-Díaz et al. (2003) proved that the τ-value (Tijs (1981)) corresponds to the mean of a distribution defined over the core cover boundary (the weighted average of the edges of the core cover).

Our first result is a characterization of this single-valued solution in a general framework, i.e. given a set S, endowed with a probability measure we select a specific point in the convex hull of S. Next, we pay special attention when the set-valued solution is the core of a TU game. A new solution concept that selects a unique outcome in the core, called the core-center, is obtained. Easily, one can observe that it satisfies many well known properties in game theory.

One of our purposes was to introduce a new solution concept for balanced games that values fairly all the stable allocations selected by the core. In Maschler et al. (1979) it is showed that the nucleolus can be characterized as a "lexicographic center". With that in mind, we decided to study the real center of the core and we wonder if it has nice game theory interpretations. From our point of view, the core-center has really interesting properties, at the one side *within* the core and at the other one *with respect to* the core. For instance, one can say that the core-center is the "point" at which all the points in the core are balanced. So, it is like an equilibrium point: if one of the players demands more than the allocation provided by the core-center, then some player in the game has incentives to say that the allocation is not being fair with him.

One of the main properties satisfied by the core-center is what we call the fair additivity property. In the case that we allow to split the original set in subsets, and then calculate the solution for these subsets in order to calculate the solution of the original one, our allocation should weigh them appropriately. There are antecedents on game theory that look for this kind of fairness. One of the solutions for two persons bargaining problems which depends on the whole feasible set is the Equal Area Solution. This solution picks the Pareto optimal point where the area of the individually rational part of the feasible set above the solution point is equal to the area to the right of that point. Anbarci and Bigelow (1994) interpreted equal area as equal concessions. Later Calvo and Peters (2000) looked at the underlying dynamic process.

The fair additivity property jointly with other more standard properties in game theory leads to an axiomatic characterization of the core-center. We would like to point out that there is a certain parallelism with the characterization of the Shapley value using properties of efficiency, dummy players, symmetry and additivity. First, we prove that there is a unique core allocation that satisfies core-dependency, anonymity and translation invariance for simplicial cores. Secondly, we formulate the characterization for elemental cores by means of a decomposition on simplicial cores, and then we finish the proof tackling arbitrary cores by making use of the continuity property.

This paper has been organized as follows: in Section 1 we introduce the preliminary concepts on game theory, Section 2 is devoted to the definitions of probabilistic solutions, point choice rules and the centroid; In Section 3 we pay special attention to the centroid of the core (the core-center) jointly with its interpretations. Finally, in Section 4 we state the results that characterize the core-center.

1 Game Theory Background

A transferable utility or TU game G is a pair (N, v), where $N = \{1, \dots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a function assigning to every coalition $S \subset N$ a payoff $v(S)$. By convention, $v(\emptyset) = 0$.

Let G^N denote the set of all TU games with set of players N.

Definition 1. *A* solution concept *is a function (multifunction) which, given a game G, selects a subset of* $\mathbb{R}^n$

$$\begin{array}{rlcl} \psi : & \Omega \subset G^N & \longrightarrow & P(\mathbb{R}^n) \\ & G & \longmapsto & \psi(G) \end{array}$$

Definition 2. *An* allocation rule *is a function which, given a game G, selects a vector of* $\mathbb{R}^n$

$$\begin{array}{rlcl} \varphi : & \Omega \subset G^N & \longrightarrow & \mathbb{R}^n \\ & G & \longmapsto & \varphi(G) \end{array}$$

Next, the main properties of allocation rules and solution concepts are going to be introduced. But before starting with the properties let us introduce a pair of concepts that are going to be handled during this work.

Given a set $B \subset \mathbb{R}^n$, a real number r and a vector α we define the set $rB + \alpha$ in the following way: $rB + \alpha := \{x \in \mathbb{R}^n | x = ry + \alpha \text{ for some } y \in B\}$

Definition 3. *Given a permutation $\sigma \in \pi(N)$ we can define permutations of the following concepts:*

Coalition: $\sigma(S) := \{i \in N \mid \exists j \in S \text{ such that } \sigma(j) = i\}$ *for all* $S \subset N$.

Game: *Given a game $G \in G^N$, $\sigma(G) \equiv G^\sigma$ denotes the game (N, v^σ) where $v^\sigma(S) = v(\sigma^{-1}(S))$ for all $S \in 2^N$.*

Point: *Given a point $x = (x_1, \dots, x_n)$, $\sigma(x) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$.*

Set: $\sigma(B) := \{y \in \mathbb{R}^n \mid \exists x \in B \text{ such that } \sigma(x) = y\}$.

A game can be seen as a vector in $\mathbb{R}^{2^n}$, since it assigns a real value to each subset of N. An allocation rule is a function which selects a vector in $\mathbb{R}^n$ for each game G. So from now on we will say that an allocation rule φ is *continuous* if the corresponding function $\varphi : \mathbb{R}^{2^n} \to \mathbb{R}^n$ is continuous.

Definition 4. *A player i is said to be* dummy *in a game $G \in G^N$ if and only if*

$$v(S \cup \{i\}) - v(S) = v(\{i\}) \quad \forall S \subset N \backslash \{i\}$$

Definition 5. *A game $G \in G^N$ is* zero-normalized *if and only if*

$$v(\{i\}) = 0, \quad i = 1, 2, \ldots, n$$

Definition 6. *Two n-person games $G_1 = (N, v)$ and $G_2 = (N, w)$ are said to be Strategic Equivalent, SEQ, if there exist a positive real number r and n real constants $\alpha_1, \ldots, \alpha_n$ such that, for all $S \subset N$,*

$$w(S) = rv(S) + \sum_{i \in S} \alpha_i$$

An important consequence of the strategic equivalence is that concepts which are well behaved with regard to this property, can be studied in the class of zero-normalized games.

Definition 7. *An outcome $x \in \mathbb{R}^n$ is said to be* efficient *if and only if*

$$\sum_{i=1}^{n} x_i = v(N)$$

Definition 8. *An outcome $x \in \mathbb{R}^n$ is said to satisfy* individual rationality *if and only if*

$$x_i \geq v(\{i\}), \quad \forall i \in N$$

i.e. no player gets less than the profit he can obtain by staying alone.

Definition 9. *An outcome $x \in \mathbb{R}^n$ is said to be* reasonable *if and only if*

$$x_i \leq r_i, \quad \forall i \in N$$

where

$$r_i = \max_{S \,|\, i \in S} (v(S) - v(S \backslash \{i\}))$$

i.e. no player gets more than his maximum contribution to any coalition.

Definition 10. *An outcome x is* stable *if and only if*

$$\sum_{i \in S} x_i \geq v(S), \quad \forall S \subsetneq N$$

i.e. there is no coalition interested in leaving the great coalition

Definition 11. *An allocation rule* φ *satisfies* anonymity *if given a game* G *and a permutation* $\sigma \in \pi(N)$ *then,* $\varphi(G^\sigma) = \sigma(\varphi(G))$.

Definition 12. *An allocation rule* φ *satisfies* translation invariance *if given two games* $G_1 = (N, v)$, $G_2 = (N, w)$ *and* n *real constants* $\alpha_1, \ldots, \alpha_n$ *such that, for all* $S \subset N$,

$$w(S) = v(S) + \sum_{i \in S} \alpha_i$$

then $\varphi(G_2) = \varphi(G_1) + \alpha$.

Definition 13. *An allocation rule* φ *satisfies* scale invariance *if given two games* $G_1 = (N, v)$, $G_2 = (N, w)$ *and real number* r *such that, for all* $S \subset N$,

$$w(S) = rv(S)$$

then $\varphi(G_2) = r\varphi(G_1)$.

Definition 14. *An allocation rule* φ *satisfies* invariance with regard to strategic equivalence *if it satisfies both* translation and scale invariances.

Definition 15. *A game* $G = (N, v)$ *is* convex *if and only if*

$$v(S \cup \{i\}) - v(S) \leq v(T \cup i) - v(T), \quad \forall S \subset T \subset N \backslash \{i\}$$

The amount $v(S \cup \{i\}) - v(S)$ is called the $i's$ marginal contribution to a coalition S. Convexity says that for all $i \in N$, the $i's$ marginal contribution does not decrease as the coalition becomes larger. The set of all $n-$person convex games will be denoted by CG^n.

Definition 16. *Given a game* $G = (N, v)$, *the* preimputations *set is defined by,*

$$I^*(G) = \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N)\}$$

i.e. the set of all efficient outcomes

2 The Model

This section is devoted to introduce the main concepts we are to use along this work. They are not standard in the Game Theory literature.

2.1 Probabilistic-Solutions

Given a TU game G, we can impose some desirable properties that an allocation rule should satisfy, these properties can be based on stability, fairness... After this selection procedure we have a set $B \subset \mathbb{R}^n$, such that its points are those who satisfy the chosen properties. Besides, different points may have distinct weights within B, so we are going to refer to a probabilistic solution as a subset B of $\mathbb{R}^n$, endowed with a probability distribution.

$\mathcal{P}(\mathbb{R}^n)$ will denote the set of all probability distributions defined over $\mathbb{R}^n$. The support of such a probability distribution is defined as the smaller closed set whose complement has probability zero.

Definition 17. *A probabilistic-solution on a non empty collection of games $\Omega \subset G^N$ is a function:*

$$\begin{array}{rlcl} \mathbb{P}: & \Omega & \longrightarrow & \mathcal{P}(\mathbb{R}^n) \\ & G = (N, v) & \longmapsto & \mathbb{P}(G) \equiv \mathcal{P}_{\mathbb{P}_G} \end{array}$$

This definition is a generalization of the usual definition of solution concept. For instance the core of a game can be seen as the uniform distribution defined over the core itself. An allocation-rule can be seen as a distribution which assigns probability 1 to a concrete value. Let us denote the set of all probabilistic solutions with Λ.

Let $\mathbb{P}_G$ denote the support of $\mathbb{P}(G)$. Let us define some elemental properties that a probabilistic-solution $\mathbb{P}$ can satisfy:

Dummy player property: If player i is dummy in game G, then $y_i = v(\{i\}) \; \forall \, y = (y_1, \ldots, y_n) \in \mathbb{P}_G$.

Anonymity: A probabilistic solution $\mathbb{P} \in \Lambda$ satisfies the anonymity property iff $\mathcal{P}_{\mathbb{P}_{G^\sigma}}$ satisfies $\mathcal{P}_{\mathbb{P}_{G^\sigma}}(B) = \mathcal{P}_{\mathbb{P}_G}(\sigma^{-1}(B)) \quad \forall \sigma \in \pi(N) \quad \forall B \subset \mathbb{R}^n$

Efficiency: A probabilistic solution $\mathbb{P} \in \Lambda$ has the efficiency property iff

$\sum y_i = v(N) \quad \forall \, y \in \mathbb{P}_G$

S-Equivalence A probabilistic solution $\mathbb{P} \in \Lambda$ is relative invariant with regard to S-equivalence iff given two games $G_1 = (N, v)$ and $G_2 = (N, w)$ which are S-equivalent through the real number r and the vector α we have:

$$\mathcal{P}_{\mathbb{P}_{G_1}}(B) = \mathcal{P}_{\mathbb{P}_{G_2}}(rB + \alpha) \quad \forall B \subset \mathbb{R}^n$$

As a consequence of this definition $\mathbb{P}_{G_2} = r\mathbb{P}_{G_1} + \alpha$.

All these properties are natural extensions of the properties of the solution concepts.

2.2 Point-choice rules

Definition 18. *A point-choice rule on a nonempty collection $\Upsilon \subset \Lambda$ of probabilistic-solutions is a function which, given a probabilistic-solution of Υ and a game G, selects a single allocation:*

$$\begin{array}{rccc} X: & \Upsilon \times \Omega & \longrightarrow & \mathbb{R}^n \\ & (\mathbb{P}, G) & \longmapsto & X(\mathbb{P}(G)) \end{array}$$

Let us enumerate some basic properties a point-choice rule X could satisfy with regard to a probabilistic-solution $\mathbb{P}$:

Dummy player property: *If player i is dummy in $\mathbb{P}_G$, then:*

$X_i(\mathbb{P}(G)) = q_i$

(i is dummy in $B \subset \mathbb{R}^n$ if there exists q_i such that $y_i = q_i\, \forall y \in B$)

Anonymity: *A point-choice rule X satisfies the anonymity property with regard to a probabilistic-solution $\mathbb{P}$ iff*

$X(\sigma(\mathbb{P}(G))) = \sigma(X(\mathbb{P}(G))) \quad \forall \sigma \in \pi(N)$

Set consistency: *A point-choice rule X satisfies set-consistency with regard to a probabilistic-solution $\mathbb{P}$ iff*

$X(\mathbb{P}(G)) \in Conv(\mathbb{P}_G)$

Translation and scale invariance (TSI) *A point-choice rule X satisfies TSI with regard to a probabilistic-solution $\mathbb{P}$ iff for a given game G, for every real number r and for every vector α:*

$X(r\mathbb{P}(G) + \alpha) = rX(\mathbb{P}(G)) + \alpha$

Lemma 1. *The composition of a probabilistic-solution and a point-choice rule is an allocation rule.*

Now we are going to see the properties an allocation rule $\varphi = X \circ \mathbb{P}$ obtained in this way satisfies depending on the properties of the probabilistic-solution $\mathbb{P}$ and point-choice rule X which originate it.

$\mathbb{P}$		X		φ
Dummy P. prop.	$+$	*Dummy P. prop.*	$\Rightarrow$	*Dummy P. prop.*
Anonymity	$+$	*Anonymity*	$\Rightarrow$	*Anonymity*
Efficiency	$+$	*Set consistency*	$\Rightarrow$	*Efficiency*
S-Equivalence	$+$	*TSI*	$\Rightarrow$	*S-Equivalence*

2.3 The Centroid

This section proposes a natural procedure to continue with that started in the previous section. Once we have a probability distribution defined on $\mathbb{R}^n$, what to do if we want to select a single allocation? Our proposal in this section is to choose the center of gravity of the distribution.

Definition 19. *Given a probabilistic-solution $\mathbb{P}$, and a game G,* the Centroid Θ *is the point-choice rule defined in the following way:*

$$\Theta(\mathbb{P}(G)) = E(\mathbb{P}(G)) = E(\mathcal{P}_{\mathbb{P}_G})$$

where $E(\mathcal{P})$ denotes the expectation of the probability distribution $\mathcal{P}$. The centroid corresponds with the center of mass or center of gravity of $\mathbb{P}_G$.

Lemma 2. *Given a probabilistic-solution $\mathbb{P}$, the centroid satisfies dummy player property, anonymity, set consistency and TSI with regard to $\mathbb{P}$.*

Proof. All these properties are a consequence of the properties of the expectation of a probability distribution. □

Now we are going to introduce a new property which reinforces the election of the centroid as point-choice rule.

Given an allocation x and a player i, let $B_i(x)$ denote the set of all allocations which are worse for him than x, and $A_i(x)$ is the set of all allocations which are preferred for i to x:

- $B_i(x) := \{y \in \mathbb{R}^n \mid y_i < x_i\}$
- $A_i(x) := \{y \in \mathbb{R}^n \mid y_i > x_i\}$
- We also define $E_i(x) := \{y \in \mathbb{R}^n \mid y_i = x_i\}$

We can also weigh how "good" or "bad" these allocations are for a player i by means of the distance between the corresponding values on the ith coordinates, i.e. this weight takes in account how bad or good the point is according to the profits it involves for player i. For the sake of notational commodity, the following concepts are going to be introduced considering that the probability distributions $\mathbb{P}$ are continuous, but everything could be rewritten for arbitrary distributions.

Let $\mathbb{P}$ be a probabilistic solution, given an outcome x and a player i, we are going to consider that the relevance (weight) for player i of a point $y \in \mathbb{R}^n$ with regard to x, is the weight of the point according to $\mathbb{P}$ but re-scaled proportionally to $|x_i - y_i|$.

Definition 20. *Given a probability distribution $\mathbb{P}$ with density function f, an outcome x is said to be* fair *for player i with regard to $\mathbb{P}$ if and only if*

$$\int_{B_i(x)}(x_i - y_i)f(y)dy = \int_{A_i(x)}(y_i - x_i)f(y)dy$$

An allocation x is fair for a player i if, the total weight of the points in $A_i(x)$ coincides with that of the points in $B_i(x)$.

Definition 21. *Given a probabilistic solution $\mathbb{P}$, an outcome x is said to be* admissible *for player i with regard to $\mathbb{P}$ if and only if*

$$\int_{B_i(x)}(x_i - y_i)f(y)dy \geq \int_{A_i(x)}(y_i - x_i)f(y)dy$$

The idea of these concepts is that a player can object to an allocation such that he feels that the relevance of $A_i(x)$ is more than the one of $B_i(x)$.

Definition 22. *An allocation rule defined on a subset $\Omega \subset G^N$ is said to be fair (admissible) with regard to a probabilistic solution $\mathbb{P}$ if it is fair (admissible) for all players with regard to $\mathbb{P}$ for every game in Ω.*

Lemma 3. *Given a probabilistic solution $\mathbb{P}$ the unique allocation rule which is fair (admisible) with regard to $\mathbb{P}$ is the Centroid of $\mathbb{P}$.*

Proof. This Lemma is an immediate consequence of the properties of the center of gravity. Let us see how to prove it in probabilistic terms. Let us denote $E(\mathbb{P}(G))$ by $\bar{y}$ and let x be an allocation in $\mathbb{R}^n$, we want to show that, for every game G

$$\int_{B_i(x)}(x_i - y_i)f(y)dy = \int_{A_i(x)}(y_i - x_i)f(y)dy \Leftrightarrow x_i = \bar{y}_i \quad \forall i \in N$$

note that

$$\int_{\mathbb{R}^n}(x_i - y_i)f(y)dy = x_i \int_{\mathbb{R}^n} f(y)dy - \int_{\mathbb{R}^n} y_i f(y)dy = x_i - \bar{y}_i$$

but also

$$\int_{\mathbb{R}^n}(x_i - y_i)f(y)dy = \int_{B_i(x)}(x_i - y_i)f(y)dy - \int_{A_i(x)}(y_i - x_i)f(y)dy$$

since $\int_{E_i(x)}(x_i - y_i)f(y)dy = 0$, so

$$\int_{B_i(x)}(x_i - y_i)f(y)dy = \int_{A_i(x)}(y_i - x_i)f(y)dy, \forall i \in N \Leftrightarrow x_i = \bar{y}_i$$

□

3 The Core-Center

3.1 The core and its relatives

We introduce now the notion of core a game (Gillies (1953)) as well as the notion of strong ε-core; both of them are based on efficiency and stability. A point $x \in \mathbb{R}^n$ is said to be stable if there is no coalition S such that $\sum_{i \in S} x_i < v(S)$, analogously it is ε-stable if there is no coalition S such that $\sum_{i \in S} x_i < v(S) - \varepsilon$.

Definition 23. *The core of a game G, denoted by $\mathcal{C}(G)$ is the set of all efficient and stable outcomes*

$$\mathcal{C}(G) := \{x \in \mathbb{R}^n | \sum_{i \in S} x_i \geq v(S) \; \forall \, S \subsetneq N, \; \sum_{i \in N} x_i = v(N)\}$$

Definition 24. *Let ε be a real number. The strong ε-core of a game G, denoted by $\mathcal{C}_\varepsilon(G)$ is the set of all efficient and ε-stable outcomes:*[1]

$$\mathcal{C}_\varepsilon(G) := \{x \in \mathbb{R}^n | \sum_{i \in S} x_i \geq v(S) - \varepsilon \; \forall \, S \subsetneq N, \; \sum_{i \in N} x_i = v(N)\}$$

By definition, when $\varepsilon = 0$, $\mathcal{C}_0(G) \equiv \mathcal{C}(G)$.

Definition 25. *The least core of the game $G = (N, v)$, denoted by $\mathcal{LC}(G)$ is the intersection of all nonempty strong ε-cores. Equivalently, let $\varepsilon_0(G)$ be the smallest ε such that $\mathcal{C}_\varepsilon(G) \neq \emptyset$, then $\mathcal{LC}(G) = \mathcal{C}_{\varepsilon_0(G)}(G)$.*[2]

3.2 The ε-cores as probabilistic solutions

If we have chosen an ε-core in accordance with the ε-stability and the efficiency, there is no reason to put different weights to its points so from now on we are going to denote by $\mathcal{C}$, $\mathcal{LC}$, and C_ε the core, least core, and ε-core respectively but endowed with the uniform distribution, i.e. in what follows they are going to be treated as probabilistic solutions.

Lemma 4. *Given $\varepsilon \in \mathbb{R}$, the probabilistic-solution $\mathcal{C}_\varepsilon$ satisfies anonymity, efficiency and S-Equivalence. Besides, when $\varepsilon = 0$, $\mathcal{C}$ also satisfies dummy player property.*

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[1] Weak ε-cores are not going to be used, so there will be no ambiguity with this notation.

[2] In Maschler et al. (1979) the $\varepsilon_0(G)$ is introduced in such a way that it is straightforward to check that it is well defined, i.e. it exists and it is unique.

Proof. All these properties are an immediate consequence of the definitions. □

The strategic equivalence property is quite useful, since it allows us to work with 0-normalized games whenever we need to.

Once we have introduced the core we can define some other related concepts:

Definition 26. *An allocation rule φ is said to be core-dependent iff $core(G) = core(G') \Rightarrow \varphi(G) = \varphi(G')$.*

This property, even though not being very strong, is quite meaningful. For instance the nucleolus, which is always in the core when this is nonempty, does not satisfy core-dependency. Different games with the same core may have different nucleolus. The point is that redundant restrictions are useless for core-dependent allocation rules, but not necessarily for the remaining ones.

For any game $G = (N, v)$ let us introduce the family of "shifted" games[3] $G_\varepsilon = (N, v_\varepsilon)$ defined by:

$$v_\varepsilon(S) = \begin{cases} v(S) & \text{if } S = \emptyset, N \\ v(S) - \varepsilon & \text{if } S \neq \emptyset, N \end{cases}$$

3.3 Some Geometrical Considerations

We need to introduce some notation and make some considerations regarding the underlying geometry of a TU game. We denote the efficient hyperplane with H^E, so $H^E \equiv \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N)\}$. All the sets we consider in this paper are contained in H^E and this implies that all our framework is going to be developed in an $n-1$ dimensional euclidean space.

Definition 27. *A* convex polytope *P is the convex hull $conv(V)$ of a set of points $V = \{x^1, \ldots, x^s\}$ in $\mathbb{R}^n$, equivalently it is a bounded subset of $\mathbb{R}^n$ which is the intersection of a finite set of halfspaces.*

It is straightforward to check that the core of a game, when non empty, is a convex polytope (it is the intersection of halfspaces in H^E). From now on we are going to omit the word "convex" because we are only going to deal with such polytopes.

Definition 28. *An n-polytope is a polytope which lies in an n-dimensional space but there is no $(n-1)$-dimensional space containing it.*

[3] This concept has also been taken from Maschler et al. (1979)

Lemma 5. *Except for the least core, all nonempty strong ε-cores are n-1-polytopes. The least core is always an m-polytope with $1 \leq m \leq n-2$.*

Proof. The statement in this lemma has been taken from Maschler et al. (1979) so we are not going to prove it. Anyway, not being a completely straightforward result it is quite intuitive. $\square$

3.4 The Core-Center

Definition 29. *Given $\varepsilon \in \mathbb{R}$, the composition of the probabilistic solution $\mathcal{C}_\varepsilon$ and the centroid Θ is an allocation rule called the* ε-core-center *and it is denoted by μ_ε.*

We are going to pay special attention to the core-center μ, (i.e. the 0-core-center, μ_0). The class of games with nonempty core is the class of balanced games BG:

$$\begin{array}{rlcl} \mu: & BG & \longrightarrow & \mathbb{R}^n \\ & G=(N,V) & \longmapsto & \Theta(\mathcal{C}(G)) = \mu(G) \end{array}$$

Before going on, let us give a pair of interpretations for this new allocation rule. The core of a game can be thought as a system of particles. Attending to the law of mass conservation, the mass of a body is $\int dm = M$. In physics, the center of mass is a fundamental concept because it allows to simplify the study of a complex system just by taking its center of mass. The movement of a body can be analyzed by describing the movement of its center of mass. In practice, the center of mass behaves just like a single particle. Furthermore, to know the position of the center of gravity one can study the symmetries of the particles distribution. If there are axis or planes of symmetry, the center of mass will be on that axis or plane, and if there is a center of symmetry, that point will be exactly the center of mass. Other interesting property of the center of mass or center of gravity is the independence of the chosen reference system.

If one assumes that the body is homogeneous, that is, there is a uniform density ρ over all the points, then $dm = dV$. Informally, we can say that dV is the volume of an element of mass and

$$\overrightarrow{c_G} = \frac{\int \rho \overrightarrow{r} dV}{\int \rho dV} = \frac{\int \overrightarrow{r} dV}{V}$$

where V is the volume of the body. Roughly speaking, the core-center μ is the unique point in the core such that all the allocations in the core are balanced with respect to it.

Another interpretation comes from statistics. The core of a game (assuming that it is nonempty) provides all the allocations such that no subcoalition has incentives to go away and form a separate group. So if we define a uniform distribution over the points in the core, then,

$$\mu(N, v) = E[\mathcal{C}] \text{ where } \mathcal{C} \sim U(C(N, v))$$

So, the core-center is the expectation of a uniform distribution on the core.

Now we are going to introduce a new property, the "fair additivity". This property arises from the following idea; let us consider a game G and a player i. What happens to the core of the game if we allow player i to change, no matter how, the value v({i}) for a new one up to his choice within the interval $[v(\{i\}), \infty)$? In most of the cases this selection will lead to a new core, contained in the previous one (the restriction $x_i \geq v(\{i\})$ will be replaced by a stronger one). Once this player has made his choice, he has divided the original game G in two new games $\overline{G}_1$ and $\underline{G}_1$ such that the union of its cores is the core of the original game and their intersection has null measure. We say that fair additivity holds for an allocation rule φ if the solution of the original game is a weighted average of the solutions of the two subgames, where the weights depend on the size of the respective cores. We assert that this property has a taste of fairness as far as neither profit nor loss could be obtained by any player through this changes.

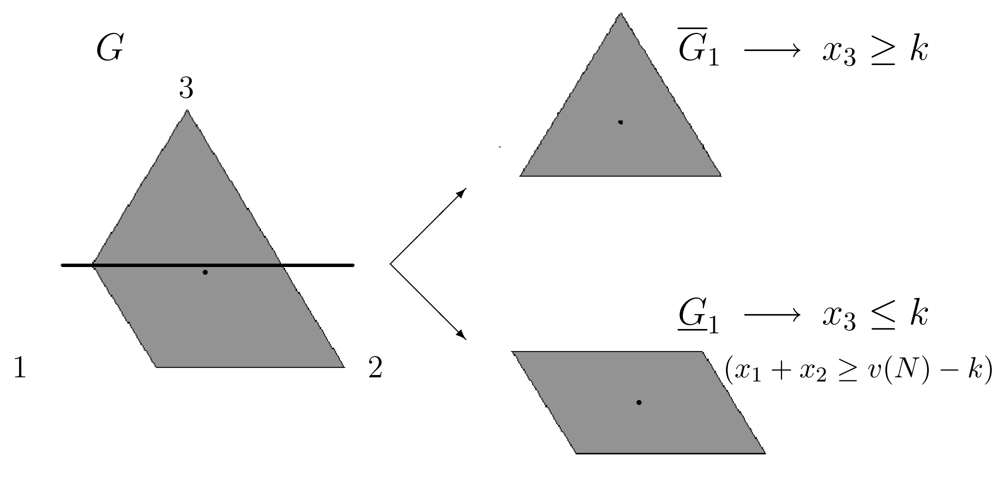

Figure 1: Example of a cut in a three players game

Definition 30. *Given a game $G = (N, v)$, a player i, and a real number $k \geq v(\{i\})$. A* cut *on the game G for player i at height k is denoted by $\chi^i_k(G)$ and defined as a pair of games $\{\overline{G}_1, \underline{G}_1\}$ such that:*

$$\overline{G}_1 = (N, \overline{v}_1), \quad \overline{v}_1(S) = \begin{cases} v(S) & \text{if } S \neq \{i\} \\ k & \text{if } S = \{i\} \end{cases}$$

$$\underline{G}_1 = (N, \underline{v}_1), \quad \underline{v}_1(S) = \begin{cases} v(S) & \text{if } S \neq N\backslash\{i\} \\ \max\{v(S), v(N) - k\} & \text{if } S = N\backslash\{i\} \end{cases}$$

The following lemma explains why this is called a "cut".

Proposition 1. *Let G be balanced game, a cut $\chi^i_k(G) = \{\overline{G}_1, \underline{G}_1\}$ has the following properties:*

i. $core(\overline{G}_1) \cup core(\underline{G}_1) = core(G)$.

ii. If core(G) is a s-polytope and $core(\overline{G}_1) \cap core(\underline{G}_1) \neq \emptyset$ then it lies in a m-dimensional space with $m < s$ (it has volume 0 in core(G)).

Proof. A cut $\chi^i_k(G)$ on the core of a game consists in taking the hyperplane $x_i = k$ which cuts it in two pieces (one of them can be empty if the hyperplane does not intersect the core). Once this consideration has be made the result has nothing to prove. □

So note that a cut of a game G defines a unique cut on its core, therefore the expression cut is going to be used to refer to both cuts on games and cuts on cores. Given a cut χ^i_k and a collection $\mathcal{G}$ of games, $\chi^i_k(\mathcal{G})$ consists in cutting successively all the cores of the games in $\mathcal{G}$ with the hyperplane $x_i = k$. So $\chi^i_k(\{G, G', G'', \dots\}) = \{\overline{G}_1, \underline{G}_1, \overline{G'}_1, \underline{G'}_1, \overline{G''}_1, \underline{G''}_1, \dots\}$.

Definition 31. *Given the probabilistic-solution $\mathcal{C}$, and a point-choice rule X. The allocation rule $\varphi = X \circ \mathcal{C}$ satisfies* Fair Additivity with regard to the core *iff, for every game G, and for every cut $\chi^i_k(G) = \{\overline{G}_1, \underline{G}_1\}$ we have:*

$$\varphi(\overline{G}_1)\mathcal{P}_{\mathcal{C}_G}(\mathcal{C}(\overline{G}_1)) + \varphi(\underline{G}_1)\mathcal{P}_{\mathcal{C}_G}(\mathcal{C}(\underline{G}_1)) = \varphi(G)$$

Lemma 6. *The core-center satisfies the following properties*

- *Dummy player property*
- *Individual rationality*
- *Reasonability*
- *Stability*
- *Anonymity*
- *Efficiency*
- *Core Dependency*
- *Invariance under Strategic Equivalence*
- *Fair additivity with regard to the core*

Proof. All of them are straightforward either because they are inherited from core properties or because they are a consequence of the properties of the center of gravity. □

Some other properties could have been stated, for instance, one could miss some monotonicity property (core-center will also have good monotonicity properties). They are not going to be used in the characterizations so we prefer to avoid introducing more notation.

4 The characterization

Now a characterization is provided for the core-center using standard properties in the literature of TU games.

- **Remark:** As a direct consequence of Lemma 3 it is straightforward to check that the unique allocation rule which is fair (admissible) with regard to $\mathcal{C}$ is the core-center.

Now let us state the main theorem in this paper;

Theorem 1. *Given a balanced game G, there is a unique allocation rule φ which lies inside the core satisfying:*

- *Continuity*
- *Core-dependency*
- *Anonymity*
- *Translation Invariance*
- *Fair Additivity with regard to the core*

And this allocation rule is the core-center.

Remark: Note that this solution is also going to satisfy efficiency and stability because it is inside the core.

From now on we assume (without loss of generality) that there are no dummy players. If there is a dummy player i, we know (because of stability and efficiency) that he will receive $v(\{i\})$ in all the points of the core, now using the core dependency of the allocation rule we know that $\varphi_i(G) = v(\{i\})$. So we can forget about the player i and work with the core of the game $G' = (N\backslash\{i\}, v')$ where v' is v restricted to $N\backslash\{i\}$. So we consider a game G with no dummy players.

Let TG denote the subclass of games in which the five properties of Theorem 1 characterize the core-center. The proof of Theorem 1 will be focused in show that $TG = BG$.

Now we provide an outline of the proof with the main steps in which we are going to divide it:

Step 1 This first phase consists in showing that the core-dependency, the anonymity and the translation invariance characterize the core-center when the core is simple enough.

Step 2 Next we show that the previous properties along with the fair additivity characterize the core-center for a wider class of games.

Step 3 Finally, making use of the previous results and the continuity property we show that the core of a balanced game can be approximated as much as we want with cores of games of the former class, and the result is derived. In the first two steps we assume that the core is an $(n-1)$-polytope and we only deal with the degenerate case in the last step.

4.1 A simplicial core

This subsection is devoted to show that when the core is simple enough its centroid, the core-center can be characterized using all the properties of theorem 1 but the continuity and the fair additivity.

Definition 32. *A set $\{a^0, a^1, \ldots, a^n\}$ in $\mathbb{R}^n$ is said to be geometrically independent if for any scalars $t_i \in \mathbb{R}$, the equations*

$$\sum_{i=1}^{n} t_i = 0 \quad \textit{and} \quad \sum_{i=1}^{n} t_i a^i = 0$$

imply that $t_0 = t_1 = \cdots = t_n = 0$. Note that $\{a^0, a^1, \ldots, a^n\}$ is geometrically independent if and only if the vectors $a^1 - a^0, \ldots, a^n - a^0$ are linearly independent.

Definition 33. *Let $\{a^0, a^1, \ldots, a^n\}$ be a geometrically independent set in $\mathbb{R}^n$. The n-simplex S_n spanned by $a^0, a^1, \ldots, a^n$ is the set of all points x of $\mathbb{R}^n$ such that*

$$x = \sum_{i=1}^{n} t_i a^i \quad \textit{where} \quad \sum_{i=1}^{n} t_i = 1 \quad \textit{and} \quad t_i \geq 0, \, \forall i$$

Each a^i is a vertex of the n-simplex. The numbers t_i are the barycentric coordinates for x of S_n with respect to $a^0, a^1, \ldots, a^n$. The subscript of S_n is the dimension of the simplex. An n-simplex is regular if the distance between any two vertices is constant.

Definition 34. *The centroid, or barycentre, of an n-simplex S_n spanned by $a^0, a^1, \ldots, a^n$ is*

$$\Theta(S_n) = \sum_{i=0}^{n} \frac{a^i}{n+1}$$

When working with TU games, there is a regular simplex which play an important role. Given an n-player's game G:

- $I(G) = \{x \in \mathbb{R}^n \mid \sum x_i = v(N), x_i \geq v(\{i\})\, \forall i \in N\}$ denotes the set of imputations of the game G.

It is a well known result that $I(G)$ is indeed a regular $(n-1)$-simplex. When working with the core, the restrictions play a very important role, so we are going to introduce some notation regarding them. All the restrictions are going to be of the following type, given a coalition $S \subset N$: $R_S \equiv \{x \in \mathbb{R}^n \mid \sum_{i \in S} x_i \geq v(S)\}$. We say a player i is *involved* in a restriction R_S iff $i \in S$. Let R_S be a restriction, and $q = |S|$, the number of players of coalition S, then we say that R_S is a q-restriction. There are two special types of restrictions; the 1-restrictions and the $(n-1)$-restrictions which are going to be called *elemental restrictions.* We say a restriction is *redundant* in the core if removing it does not affect the core. Conversely, the restrictions which are not redundant ones are going to be called *active* restrictions. If all the active restrictions of the core of a game are elemental restrictions, we refer to it as an *elemental core.* Note that these restrictions are going to be studied in the $n-1$ dimensional space H^E obtained from applying the efficiency condition $\sum_{i \in N} x_i = v(N)$. Let H_S denote the hyperplane corresponding to the restriction R_S, i.e. $x \in H_S \Leftrightarrow \sum_{i \in S} x_i = v(S)$. Note that because of the efficiency condition, the hyperplane H_S has dimension $n-2$.

Lemma 7. *Given a non empty coalition $S \neq N$ the hyperplanes H_S and $H_{N \setminus S}$ are parallel in H^E.*

Proof. We are going to show that given a hyperplane $H := \sum_{i \in S} x_i = k_1$ in H^E, then there exists k_2 such that H can be also expressed as $\sum_{i \in N \setminus S} x_i = k_2$. This is stronger than the statement of this lemma, so once this is proved the result is obtained. We are working in H^E so we know:

$\sum_{i\in N} x_i = v(N) \Rightarrow \sum_{i\in S} x_i + \sum_{i\in N\setminus S} x_i = v(N)$. If we impose the condition $\sum_{i\in S} x_i = k_1$ we have $k_1 + \sum_{i\in N\setminus S} x_i = v(N) \Rightarrow \sum_{i\in N\setminus S} x_i = v(N) - k_1 = k_2$. □

Remark: As a consequence of the last lemma we can write an $(n-1)$-restriction $\sum_{j\in N\setminus\{i\}} x_j \geq k_i$ in the following way: $x_i \leq v(N) - k_i$. So these kind of restrictions are also going to be referred to as $(n-1)$-restrictions.

Definition 35. *Given a game G, if A is the set of active restrictions we have two special cases:*

- $A = \{x_i \geq v(\{i\}) \mid i \in N\}$ =\{*all 1-restrictions*\} *In this case the core is called* up-simplex.
- $A = \{\sum_{j\neq i} x_j \geq v(\{N\setminus\{i\}\}) \mid i \in N\}$ =\{*all* $(n-1)$*-restrictions*\} *The core is called* down-simplex.

Proposition 2. *If the core $\mathcal{C}$ of a game is either an up-simplex or a down-simplex then it is a regular simplex.*

Proof. We are going to study the two cases separately:

Up-simplex: When the core is an up-simplex it coincides with the set of imputations, and it is a well known result that $I(G)$ is a regular simplex. The n vertices of $I(G)$ are the points

$$u^i = (v(\{1\}), v(\{2\}), \ldots, \overbrace{v(N) - \sum_{j\neq i} v(\{j\})}^{i}, \ldots, v(\{n\})) \quad i \in N$$

If we consider the translation $\mathcal{C}' = \mathcal{C} + t$ where t is the vector $(-v(\{1\}), \ldots, -v(\{n\}))$. The vertices of $\mathcal{C}'$ are the points

$$\hat{u}^i = (0, 0, \ldots, \overbrace{v(N) - \sum_{j\in N} v(\{j\})}^{i}, 0) \quad i \in N$$

Down-simplex:[4] We know that the core is a $(n-1)$-polytope, so it is going to have at least n points.[5] We also know that a vertex of the core must

[4]Once the case of the up-simplex has been proved, we could proceed in a more geometrical way, using the parallelism relations between the faces of the up-simplex and those in the down-simplex, we can apply *similarity* results to obtain that the down-simplex is indeed a simplex which besides is regular.

[5]If it has only k vertices with $k < n$, the affine hull of these k points lies in a k-1 dimensional space and contains the core (the convex hull), and this contradicts the fact that the core is a $(n-1)$-polytope.

be the intersection of $n-1$ hyperplanes, as we have n hyperplanes we can consider n different subsets of A, each of them containing $n-1$ hyperplanes. So the core has, at most, n vertices. This consideration along with the former one leads to a total number of n vertices in the core. Let us calculate these points when we have 4 players:

$d^1 = H_{1,2,3} \cap H_{1,2,4} \cap H_{1,3,4} \cap H^E$

$$\begin{array}{rlcl} H_{1,2,3}: & x_1+x_2+x_3 = v(\{1,2,3\}) & \Rightarrow & x_1 = v(\{1,2,3\}) - x_2 - x_3 \\ H_{1,2,4}: & x_1+x_2+x_4 = v(\{1,2,4\}) & \Rightarrow & x_1 = v(\{1,2,4\}) - x_2 - x_4 \\ H_{1,3,4}: & x_1+x_3+x_4 = v(\{1,3,4\}) & \Rightarrow & x_1 = v(\{1,3,4\}) - x_3 - x_4 \\ H^E: & x_1+x_2+x_3+x_4 = v(N) & & \end{array}$$

If we combine the equation $H_{1,2,3}$ with H^E we obtain:

$$v(\{1,2,3\}) - x_2 - x_3 + x_2 + x_3 + x_4 = v(N) \Rightarrow x_4 = v(N) - v(\{1,2,3\})$$

If we use the rest of the equations we obtain $x_i = v(N) - v(N\backslash\{i\})$ when $i \neq 1$ and $x_1 = \sum_{i\neq 1} v(N\backslash\{i\}) - 2v(N)$. When we have n players, the procedure is the same and we obtain the following extreme points:

$$d^i = (d^i_1, d^i_2, \ldots, \overbrace{\sum_{j\neq i} v(N\backslash\{j\}) - (n-2)v(N)}^{i}, \ldots, d^i_n) \quad i \in N$$

And $d^i_j = v(N) - v(N\backslash\{j\})$ for all $j \neq i$.

If we consider the translation $\mathcal{C}' = \mathcal{C} + t$ where t is the vector $(v(N\backslash\{1\}) - v(N), v(N\backslash\{2\}) - v(N), \ldots, v(N\backslash\{n\}) - v(N))$. The vertices of $\mathcal{C}'$ are the points:

$$\hat{d}^i = (0, 0, \ldots, \overbrace{\sum_{j\in N} v(N\backslash\{j\}) - (n-1)v(N)}^{i}, 0) \quad i \in N$$

And this clearly constitutes a regular simplex.

□

Proposition 3. *If the core $\mathcal{C}$ of a game G is an* up-simplex *and φ is an allocation rule which lies in the core and satisfies core-dependency, anonymity and translation invariance then $\varphi(G)$ is the centroid of the simplex.*

Proof. Let us calculate the value this allocation rule selects in the game $G' = G + t$ where $t = (-v(\{1\}), \ldots, -v(\{n\}))$.

The core of the game G' is the simplex $\mathcal{C}'$, and its vertices are the n points $\hat{u}^i$ calculated in Proposition 2, and this game G' satisfies that $core(\sigma(G')) = \sigma(\mathcal{C}') = \mathcal{C}'$ for all $\sigma \in \pi(N)$ i.e. all the players are symmetric. This property of the game G' along with the anonymity property and the core-dependency implies that $\varphi(G') = \sigma(\varphi(G'))$ for all $\sigma \in \pi(N)$ and this leads to the fact that there exists a constant k such that $\varphi(G') = (k, \ldots, k)$. The value for this k is:

$$\frac{v(N) - \sum_{j \in N} v(\{j\})}{n}, \quad \text{so } \varphi(G') = \sum_{i=1}^{n} \frac{\hat{u}^i}{n},$$

the centroid of the simplex $\mathcal{C}'$.

If we use the translation invariance, we obtain that $\varphi(G) = \varphi(G') - t$, the centroid of the simplex $\mathcal{C}$. $\square$

Proposition 4. *If the core $\mathcal{C}$ of a game G is a* down-simplex *and φ is an allocation rule which lies in the core and satisfies core-dependency, anonymity and translation invariance then $\varphi(G)$ is the centroid of the simplex.*

Proof. This proof is completely analogous, we only need to use the vector $t = (v(N \backslash \{1\}) - v(N), v(N \backslash \{2\}) - v(N), \ldots, v(N \backslash \{n\}) - v(N))$ for the translation. $\square$

Corollary 1. *If G is a game such that $core(G)$ is either an up or down-simplex then $G \in TG$.*

4.2 An elemental core

Now we are going to combine the results in propositions 3 and 4 with the fair additivity property to show that a game G with an elemental core belongs to TG.

Let us illustrate what we are going to do with an example when the game has only three players, all the restrictions are therefore elemental restrictions (figure 2).

I the game in this example we see that the core is inside the imputations set, which is a simplex. Besides, it divides $I(G)$ into the sets T^i, which are also simplices. If we consider the game $G' = (N, v')$ which differs from G in the fact that $v'(12) = v'(13) = v'(23) = 0$, it is straightforward to check that $core(G') = I(G)$, if we consider now the cut $\chi^1_{v(23)}(G')$ we obtain two games G_1, G_2 such that $core(G_2) = T^1$. We can continue now cutting the

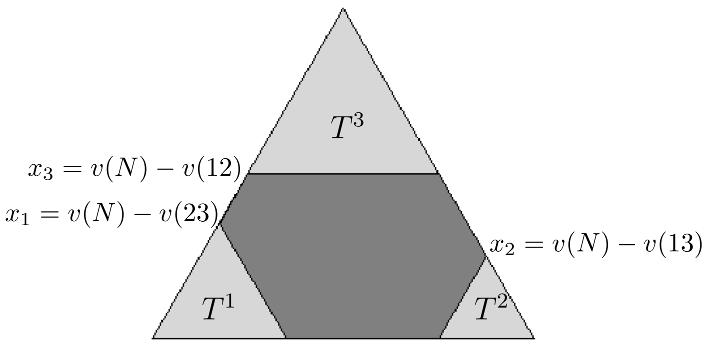

Figure 2: Example of a three players game

game G_1 with $\chi^2_{v(13)}(G_1)$ obtaining G_3, G_4 such that $core(G_4) = T^2$. Finally cutting G_3 with $\chi^3_{v(12)}(G_3)$ we obtain G_5, G_6 in such a way that $core(G_5) = core(G)$ (furthermore $G_5 = G$) and $core(G_6) = T^3$. Suppose now that φ is an allocation rule fulfilling the assumptions in Theorem 1. Using the fair additivity property we can write:

$$\left.\begin{array}{rcl} \varphi(G') & = & \varphi(G_1)\frac{Vol(G_1)}{Vol(G')} + \varphi(G_2)\frac{Vol(G_2)}{Vol(G')} \\ \\ \varphi(G_1) & = & \varphi(G_3)\frac{Vol(G_3)}{Vol(G_1)} + \varphi(G_4)\frac{Vol(G_4)}{Vol(G_1)} \\ \\ \varphi(G_3) & = & \varphi(G_5)\frac{Vol(G_5)}{Vol(G_3)} + \varphi(G_6)\frac{Vol(G_6)}{Vol(G_3)} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \varphi(G') = \varphi(G_2)\frac{Vol(G_2)}{Vol(G')} + \varphi(G_4)\frac{Vol(G_4)}{Vol(G')} + \varphi(G_5)\frac{Vol(G_5)}{Vol(G')} + \varphi(G_6)\frac{Vol(G_6)}{Vol(G')}$$

$$\Rightarrow \varphi(G_5) = \varphi(G')\frac{Vol(G')}{Vol(G_5)} - \varphi(G_2)\frac{Vol(G_2)}{Vol(G_5)} - \varphi(G_4)\frac{Vol(G_4)}{Vol(G_5)} - \varphi(G_6)\frac{Vol(G_6)}{Vol(G_5)}$$

Now, as the cores of the games G', G_2, G_4, and G_6 are simplices, using propositions 3 and 4 we have completely determined $\varphi(G_5)$; $G_5 = G \Rightarrow \varphi(G_5) = \varphi(G)$. So for this game G, an allocation rule is completely determined if it satisfies the five properties of Theorem 1.

The next part of this subsection is devoted to formalize and generalize the procedure described in the previous example. From now on, in order to make the proof more readable we going to refer indifferently to a games and cores and we are going to cut either the cores or the games. Having been assumed core-dependency, this change of notation is not a problem; if an allocation rule has been characterized for a game G, and the game G' has the same core than the game G, then the former allocation rule is also characterized for this new game. Besides, we are also going to say that a core $\mathcal{C}$ belongs to TG if a game such that $\mathcal{C}$ is its core does. Note that we also use Vol to refer to the measure of the core of a given game, this leads to a more bearable notation. This change of notation is not a problem, in fact, fixed an initial game G and its core $\mathcal{C}$ we can write $\mathcal{P}_{\mathcal{C}_G}(\mathcal{C}') = \frac{vol(\mathcal{C}')}{vol(\mathcal{C}_G)}$ for a given $\mathcal{C}' \subset \mathcal{C}$.

Definition 36. *Let $\mathcal{C}^0$ be the core of a game G^0, we define a* cuts decomposition *of the game G^0 and its core $\mathcal{C}^0$, $\Delta_{G^0} = \{\mathcal{G}, C, \mathcal{X}\}$ as follows:*

- *A sequence $\mathcal{G} = (G^1, \ldots, G^m)$ of games.*
- *A sequence of cores: $C = (\mathcal{C}^1, \ldots, \mathcal{C}^m)$ such that $\mathcal{C}^j = core(G^j)$ for all $j \in \{1 \ldots m\}$.*
- *A sequence $\mathcal{X} = (\chi^1, \ldots, \chi^{m-1})$ of cuts defined in the following way:*
  - *χ^1 defines a cut $\{\overline{G}^0, \underline{G}^0\}$ in the game G^0, such that $\overline{G}^0 \equiv G^1$.*
  - *If χ^i defines a cut $\{\overline{G}, \underline{G}\}$ then $\overline{G} \equiv G^i$ and χ^{i+1} defines a cut in $\underline{G}$.*
  - *Finally if χ^{m-1} defines a cut $\{\overline{G}, \underline{G}\}$ then $\overline{G} \equiv G^{m-1}$ and $\underline{G} \equiv G^m$.*

It is straightforward to check that this defines a decomposition of the game G^0 and its corresponding core $\mathcal{C}^0$, i.e. $\bigcup_{i=1}^m \mathcal{C}^i = \mathcal{C}^0$ and $Vol(\bigcap_{i=1}^m \mathcal{C}^i) = 0$.

Lemma 8. *If φ is an allocation rule satisfying fair additivity with regard to the core and $\Delta_G = \{\mathcal{G}, C, \mathcal{X}\}$ is a decomposition of a game G then:*

$$\varphi(G) = \frac{\sum_{G^j \in C} \varphi(G^j) Vol(\mathcal{C}^j)}{Vol(\mathcal{C})}$$

Proof. Trivial from the definition of the fair additivity property and the definition of decomposition. □

Lemma 9. *Given a game $G = (N, v)$ and a player $i \in N$:*

$$x_i \geq \bar{v}(\{i\}) := \sum_{j \neq i} v(N \backslash \{j\}) - (n-2)v(N) \quad \textit{for all } x \in core(G).$$

Proof. Let us take a point $x \in core(G)$. Because of the core restrictions we have $\sum_{k\neq j} x_k \geq v(N\backslash\{j\})$, this condition along with the efficiency condition $\sum_{k\neq j} x_j = v(N) - x_j$ leads to $x_j \leq v(N) - v(N\backslash\{j\})$ for all $j \neq i$, and this implies that

$$\sum_{j\neq i} x_j \leq (n-1)v(N) - \sum_{j\neq i} v(N\backslash\{j\}),$$

using again the efficiency condition $\sum_{j\neq i} x_j = v(N) - x_i$ we obtain:

$$x_i \geq \sum_{j\neq i} v(N\backslash\{j\}) - (n-2)v(N)$$

□

This lemma allows us to define $m_i := \max\{v(\{i\}), \bar{v}(\{i\})\}$ as a kind of minimum right for every player i in a game $G = (N, v)$. Based on this vector m we can also define a restricted set of imputations in the following way: $\bar{I}(G) = \{x \in \mathbb{R}^n \mid \sum x_i = v(N), x_i \geq m_i \ \forall i \in N\}$. By definition of the m_i, $core(G) \subset \bar{I}(G)$.

The following lemma formalizes the result which was implicit in the three players example of figure 2.

Lemma 10. *Let G be a game and $\mathcal{C}$ its core. If there exists a decomposition $\Delta_{\bar{I}(G)} = \{\mathcal{G}, C, \mathcal{X}\}$ of $\bar{I}(G)$ such that $\mathcal{C} \in C$ and for all $\mathcal{C}^j \in C$ different from $\mathcal{C}$, $\mathcal{C}^j \in TG$ then it implies that $\mathcal{C} \in TG$.*

Proof. Using lemma 8, $\varphi(G)Vol(\mathcal{C}) = \sum_{G^j \in C} \varphi(G^j)Vol(\mathcal{C}^j)$. Let $\mathcal{C}^k$ be the element of C such that $\mathcal{C}^k = \mathcal{C}$ now, isolating $\varphi(G^k)$ we obtain:

$$\varphi(G^k) = \frac{\varphi(\bar{I}(G))Vol(\bar{I}(G)) - \sum_{G^j \in C, j\neq k} \varphi(G^j)Vol(\mathcal{C}^j)}{Vol(\mathcal{C}^k)}$$

where $\varphi(\bar{I}(G))$ denotes the value that φ selects for any game such that its core coincides with $\bar{I}(G)$.

Provided that $\bar{I}(G)$ and all $\mathcal{C}^j \in C$ different from $\mathcal{C}$ belong to TG, we can conclude that $\mathcal{C} \in TG$. □

Before going on we need to introduce a little more notation. Given a game $G = (N, v)$ let us define the following sets:

$$\begin{array}{ll}
T & := \{x \in H^E \mid x_k \geq m_k \ \forall k \in N\} \\
T^i & := \{x \in H^E \mid x_i \geq v(N) - v(N\backslash\{i\}) \ ; \ x_k \geq m_k \ \forall k \in N\backslash\{i\}\} \\
T^{ij} & := \{x \in H^E \mid x_l \geq v(N) - v(N\backslash\{l\}), \ l = i, j \ ; \ x_k \geq m_k \ \forall k \in N\backslash\{i, j\}\} \\
\vdots & \\
T^I & := \{x \in H^E \mid x_i \geq v(N) - v(N\backslash\{i\}) \ \forall i \in I \ ; \ x_k \geq m_k \ \forall k \in N\backslash I\}
\end{array}$$

Lemma 11. *These sets satisfy the following properties:*

- $T \equiv \bar{I}(G)$.
- *Given a set* $I \subset N$ *and two sets* A, B *such that* $A \cup B = I$ *it verifies that* $T^I \equiv T^A \cap T^B$, *and consequently* $T^I \equiv \bigcap_{i \in I} T^i$.
- *All these* T^I *are up-simplices, so if the core of a game coincides with some of these sets then it belongs to* TG.
- $T^{N \backslash \{i\}}$ *if nonempty coincides with* $\{y\}$, *being* y *the point such that* $y_i = m_i$ *and* $y_j = v(N) - v(N \backslash \{j\})$ *for all* $j \neq i$. *Furthermore, it is only going to be empty when* $m_i > v(\{i\})$. *This property is just a consequence of the definition of* m_i.
- $T^N = \emptyset$, *this is derived from the previous property.*

Proof. All these properties are very easy to check using the restrictions which originate each T^I. $\square$

Lemma 12. *If the game* G *has an elemental core* $\mathcal{C}$, *then* $\mathcal{C} = \bar{I}(G) \backslash \bigcup_{i=1}^{n} T^i$.

Proof. Removing T^i from $\bar{I}(G)$ is equivalent to introduce in it the restriction $x_i \leq v(N) - v(N \backslash \{i\})$ which is also equivalent to the restriction $\sum_{j \neq i} x_j \geq v(N \backslash \{i\})$. Once this remark has been done, there is nothing else to proof. $\square$

Now one could try to combine lemmas 10 and 12 to obtain, given a game G with an elemental core, an appropriate decomposition the set $\bar{I}(G)$. This is indeed the idea of what remains to be done to show that all the elemental cores are in TG. But there is a question which still needs to be solved, the pairwise intersections of the T^i are not necessarily empty and this will make the proof of the next proposition a little bit harder.

Proposition 5. *Let* $G = (N, v)$ *be a game and* $\mathcal{C}$ *its core. If* $\mathcal{C}$ *is an elemental core, then there exists a decomposition* $\Delta_{\bar{I}(G)} = \{\mathcal{G}, C, \mathcal{X}\}$ *of* $\bar{I}(G)$ *such that* $\mathcal{C} \in C$ *and for all* $\mathcal{C}^j \in C$ *different from* $\mathcal{C}$, $\mathcal{C}^j \in TG$.

Proof. Let m be the vector with components $m_i := \max\{v(\{i\}), \bar{v}(\{i\})\}$. Let G_0 be the game with characteristic function:

$$v_0(S) = \begin{cases} m_i & S = \{i\},\ i \in N \\ 0 & 1 < |S| < n \\ v(N) & S = N \end{cases}$$

By definition $core(G_0) = \bar{I}(G)$. Let $\Delta_{\bar{I}(G)} = \{\mathcal{G}, C, \mathcal{X}\}$ be a decomposition such that:

- $\mathcal{G} = (\overline{G}^1, \overline{G}^2, \dots, \overline{G}^n, \underline{G}^n)$
- $C = (\overline{\mathcal{C}}^1, \overline{\mathcal{C}}^2, \dots, \overline{\mathcal{C}}^n, \underline{\mathcal{C}}^n)$
- $\mathcal{X} = (\chi^1_{k_1}(G^0), \chi^2_{k_2}(\underline{G}^1), \dots, \chi^i_{k_i}(\underline{G}^{i-1}), \dots, \chi^n_{k_n}(\underline{G}^{n-1}))$, and for each player i, $k_i = v(N) - v(N\backslash\{i\})$ and $\chi^i_{k_i}(G^{i-1}) = \{\overline{G}^i, \underline{G}^i\}$

It is straightforward to check that $\underline{\mathcal{C}}^n = \mathcal{C}$ so we need to show that $\overline{G}^i \in TG$ for all $i \in N$.

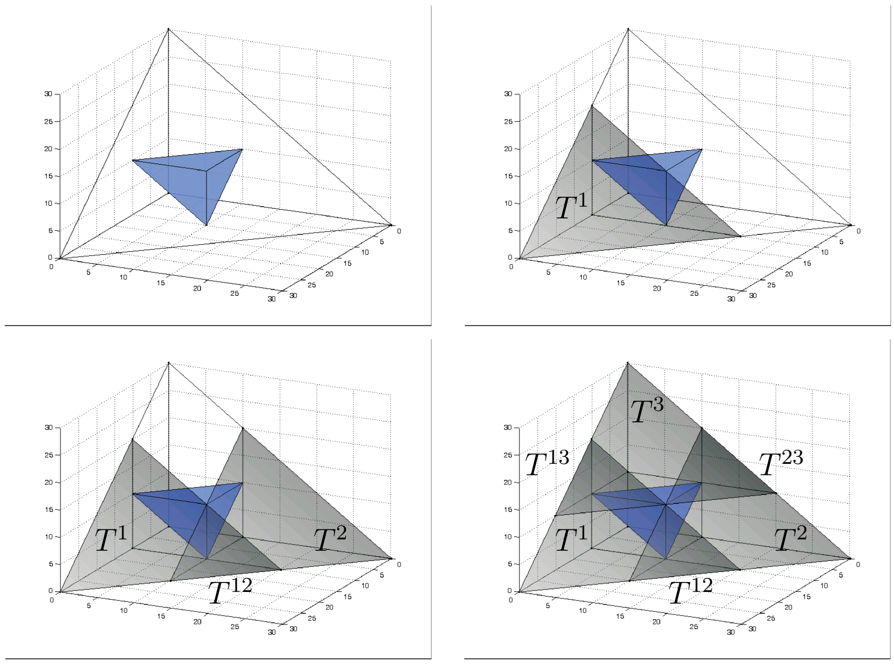

Figure 3: First steps of the decomposition with 4 players

Let us explain the idea of this decomposition; We begin with the set $\bar{I}(G)$, after the cut $\chi^1_{k_1}(G^0)$ we have divided it in T^1 and $\bar{I}(G)\backslash T^1 = \underline{\mathcal{C}}^1$, the core of the game $\underline{G}^1$. Now we introduce a second cut $\chi^2_{k_2}(\underline{G}^1)$, splitting $\bar{I}(G)\backslash T^1$ in two new sets, $T^2\backslash T^{12}$ and $\underline{\mathcal{C}}^1\backslash T^2$, note that the first of these last two sets is not going to be T^2 because $T^1 \cap T^2$ was removed with the first cut. Now we need to make the third cut and so on, but at every step we need to take care of the common parts of the T^i which can have already been drawn from $\bar{I}(G)$. These comments allow us to write down the relationship between the sets $\mathcal{C}^j$ of the decomposition and the T^i; $\mathcal{C}^i = T^i \backslash \bigcup_{i<j} T^{ij}$. So if we want to prove that $\mathcal{C}^i \in TG$ we should find an appropriate decomposition of T^i in elements of TG. This is nearly the same question we are trying

to solve in this proposition, but a little bit easier. Let us explain what is going on; initially we were trying to decompose $\bar{I}(G)$ using the T^i, and the problem was that these simplices, even though all of them belong to TG, not necessarily are going to have empty intersections. In order to solve that issue we have found a new one which, once solved will imply the solution of the previous one. In this new problem we have again that the intersection of the T^{ij} may be nonempty, so we need to use the T^{kij} and we obtain $\mathcal{C}^{ij} = T^{ij} \setminus \bigcup_{k<i} T^{kij}$. So we obtain again a new problem similar to the first one. Continuing with this procedure, there will be a moment in which we need to show that $\mathcal{C}^I = T^I \setminus \bigcup_{j<\min\{i \,|\, i \in I\}} T^{I \cup \{j\}}$ where $|I| = n-1$ belongs to TG, but this step is straightforward, using lemma 11, as $|I \cup \{j\}| = n$ we know that $T^{I \cup \{j\}}$ is empty, i.e. $\mathcal{C}^I = T^I$, and $T^I \in TG$ because it is a point[6]. Now using that $\mathcal{C}^I \in TG$ for all I such that $|I| = n-1$ we have that $\mathcal{C}^J \in TG$ if $|J| = n-2$. Now, walking backwards till the first step we obtain that the $\mathcal{C}^i$ belong to TG for all $i \in N$, so $\mathcal{C} \in TG$. □

Corollary 2. *Let G be a game and $\mathcal{C}$ its core. If $\mathcal{C}$ is an elemental core then $G \in TG$.*

Proof. This corollary is just a consequence of combining Proposition 5 with Lemma 10. □

4.3 The general case: An arbitrary core

Now we are ready make the proof of the whole result:

Proof of Theorem 1.

Let us consider a balanced game $G \in G^N$ with core $\mathcal{C}$. We are going to distinguish two cases:

i. The core is an $n-1$-polytope

ii. The core is an m polytope with $1 \leq m \leq n-2$

i. The core is an $n-1$-polytope

Given the set $\bar{I}(G)$, its faces are determined by the hyperplanes $x_i = m_i$. Given a player i, the hyperplane $x_i = m_i$ determines the face in which i obtains the lower value he can get in $\bar{I}(G)$: m_i, and opposite to that face we have the points where i obtains his maximum profit: $M_i = v(N) - \sum_{j \neq i} m_j$. Let us denote $L = M_i - m_i = v(N) - \sum_{j \in N} m_j$, note that L does not depend

[6]Although it has not been proved, it is evident that those games whose core is a singleton belong to TG.

on the player. Given an integer q (for commodity we assume $q > 2$), we obtain $\delta = L/q$. For each face in $\bar{I}(G)$ we make q cuts on it, parallel to the hyperplane in which that face lies. So we partition $\bar{I}(G)$ using the following hyperplanes:

$$H^i_k \equiv x_i = m_i + k\delta \quad \forall\, i \in N,\ k = \{0, \dots, q\}$$

These hyperplanes are also going to partition $\mathcal{C}$, and this is the partition we are interested in; let us see how to define properly these sequence of cuts, but with regard to the game G:

Stage 0 We begin with the set of games $\mathcal{G}^0 = G$

$\vdots$

Stage i, $i \in N$ Now we define the cuts for player i

> **Step** $i.0$ We cut $\bar{I}(G)$ with $x_i = m_i$; $\mathcal{G}^{i,0} = \chi^i_{m_i}(\mathcal{G}^{i-1,q})$
>
> **Step** $i.1$ $\mathcal{G}^{i,1} = \chi^i_{m_i+\delta}(\mathcal{G}^{i,0})$
>
> $\vdots$
>
> **Step** $i.k$ $\mathcal{G}^{i,k} = \chi^i_{m_i+k\delta}(\mathcal{G}^{i-1,k-1})$
>
> $\vdots$
>
> **Step** $i.q$ $\mathcal{G}^{i,q} = \chi^i_{m_i+q\delta}(\mathcal{G}^{i-1,q-1})$

Let us denote the set $\mathcal{G}^{n,q}$ by $\mathcal{G}^\delta$. Note that many of these cuts are not going to define proper cuts on many of the cores of the corresponding games in $\mathcal{G}^{i-1,k-1}$, i.e. they are going to cut a given core onto an empty set and the proper core itself. Saving notation let $\mathcal{C}'$ denote $Core(G')$, it is also straightforward to check that $\bigcup_{G' \in \mathcal{G}^\delta} \mathcal{C}' = \mathcal{C}$ and given two games $G_1, G_2 \in \mathcal{G}^\delta$ $Vol(Core(G_1) \cap Core(G_2)) = 0$.

At this point we have divided $\mathcal{C}$ in many cores. Now we are going to prove that given $\varepsilon > 0$ we can find δ small enough such that the volume of the cores of games in $\mathcal{G}^\delta$ which are not elemental is at most $\varepsilon Vol(\mathcal{C})$.

The situation we are going to have is similar to that in Figure 4, most of the cores of games in $\mathcal{G}^\delta$ are strictly contained in $\mathcal{C}$, the non elemental restrictions are redundant for such these cores (note that they are not necessarily going to be all of them simplicial cores when $n > 3$). Given a nonempty core $\bar{\mathcal{C}}$ of a game in $\mathcal{G}^\delta$ we know that there exist restrictions

$$x_i \geq m_i + k_i\delta \quad x_i \leq m_i + (k_i+1)\delta \equiv \sum_{j \neq i} x_j \geq m_i + (k_i+1)\delta \quad \forall i \in N$$

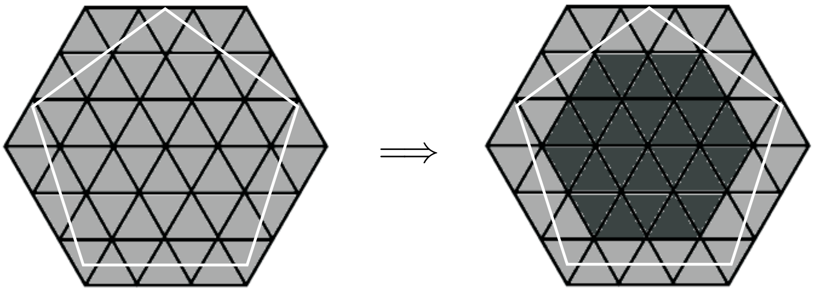

Figure 4: Approximation of a non elemental core via elemental cores

such that they determine a polytope $\bar{P}$ containing $\bar{\mathcal{C}}$. The maximum euclidean distance between two points in $\bar{P}$ is $\sqrt{n}\delta$. Note that each face of $\mathcal{C}$ is determined by a restriction, likewise, this restriction depends on a coalition S. So the maximum number of faces of the core of a game with n players is $f_n = \sum_{i=1}^{n-1} \binom{n}{i}$. Let $\mathcal{F}(\mathcal{C})$ denote the set of all faces of a core $\mathcal{C}$.

If we take a point y inside $\mathcal{C}$ such that the distance from this point to any of the faces of $\mathcal{C}$ is more than $\sqrt{n}\delta$ it is straightforward to check that y is inside the core of a game in $\mathcal{G}$ such that all non elemental restrictions are redundant, i.e. y is inside an elemental core. Therefore we can find an upper bound for the volume of the points $y \in \mathcal{C}$ which are not in an elemental core:

Given a face F of $\mathcal{C}$, consider the subset $B(F,\delta) = \{x \in \mathbb{R}^n \mid d(x,F) < \sqrt{n}\delta\}$. Clearly $\lim_{\delta\to 0} B(\delta) = F$, and the volume of F in H^E is 0, because it is a lies in an $n-2$ dimensional space. Now, using that all the $B(F,\delta)$ are bounded sets we can conclude that $Vol(B(F,\delta))$ goes to 0 when δ does. Therefore, given $\varepsilon > 0$ we can find δ small enough such that $Vol(B(F,\delta)) < \frac{\varepsilon}{f_n}$ for all the faces in $\mathcal{C}$. Once one such δ has been chosen y is a point of $\mathcal{C}$ which is not in an elemental core then it must lie in $B(F,\delta)$ for some face F of $\mathcal{C}$. So the total volume of this points is bounded from above by $\sum_{F\in\mathcal{F}(\mathcal{C})} Vol(B(F,\delta) < \sum_{F\in\mathcal{F}(\mathcal{C})} \frac{\varepsilon}{f_n} = f_n\frac{\varepsilon}{f_n} = \varepsilon$. Let us denote by $\mathcal{EG}^\delta$ the games in $\mathcal{G}^\delta$ with an elemental core.

Now, using the fair additivity property of the solution φ:

$$\varphi(G) = \frac{1}{Vol(\mathcal{C})} \sum_{G'\in\mathcal{G}^\delta} Vol(\mathcal{C}')\varphi(G') =$$

$$= \frac{1}{Vol(\mathcal{C})}\Bigg(\sum_{G'\in\mathcal{EG}^\delta} Vol(\mathcal{C}')\varphi(G') + \sum_{G'\in\mathcal{G}\setminus\mathcal{EG}^\delta} Vol(\mathcal{C}')\varphi(G')\Bigg) \tag{1}$$

Due to Corollary 2 we know that φ has already been characterized for all the games in the first addend of Equation 1. It is also known that the second addend tends to 0 as long as δ does. And now

$$\varphi(G) = \lim_{\delta\to 0}\varphi(G) = \lim_{\delta\to 0}\frac{1}{Vol(\mathcal{C})}\sum_{G'\in\mathcal{G}^\delta} Vol(\mathcal{C}')\varphi(G') =$$

$$= \frac{1}{Vol(\mathcal{C})}\left(\lim_{\delta\to 0}\sum_{G'\in\mathcal{E}\mathcal{G}^\delta} Vol(\mathcal{C}')\varphi(G') + \lim_{\delta\to 0}\sum_{G'\in\mathcal{G}\setminus\mathcal{E}\mathcal{G}^\delta} Vol(\mathcal{C}')\varphi(G')\right) =$$

$$= \lim_{\delta\to 0}\frac{1}{Vol(\mathcal{C})}\sum_{G'\in\mathcal{E}\mathcal{G}^\delta} Vol(\mathcal{C}')\varphi(G').$$

So we have expressed $\varphi(G)$ as the limit of a weighted sum of solutions of elemental cores, and this implies that $G \in TG$.

ii. The core is an m polytope with $1 \leq m \leq n-2$

In this case, as a consequence of Lemma 5, $\mathcal{C}$ is the least core $\mathcal{LC}$ of the game G. Now let us consider the sequence of shifted games $\{G_{1/n}\}_{n=1}^{\infty}$, it is straightforward to check that $\lim_{n\to\infty} G_{1/n} = G$. The core of the game $G_{1/n}$ coincides with the $\frac{1}{n}$-core of the game G. By Lemma 5, all these $\frac{1}{n}$-cores are going to be $(n-1)$-polytopes, and now, by the previous part of this proof we know that these games have already been characterized. Finally using the continuity property $\varphi(G) = lim_{n\to\infty}\varphi(G_{1/n})$.

So at this point we have that properties in the statement of this theorem characterize an allocation rule, besides it is straightforward to check that the core-center satisfies all these properties. So the result has been proved.

□

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